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CHARACTERIZATIONS OF TOPOLOGICALLY SEMI-OPEN AND QUASI-OPEN SETS IN RELATOR SPACES

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ABSTRACT. Following a basic terminology of the second author, a family $\mathcal{R}$ of relations on a set X will be called a relator on X, and the ordered pair $X(\mathcal{R}) = (X, \mathcal{R})$ will be called a relator space.

For a subset A of the relator space $X(\mathcal{R})$, we may briefly define

$$\mathrm{cl}_{\mathcal{R}}(A) = \bigcap \{ R^{-1}[A] : \quad R \in \mathcal{R} \} \qquad \text{and} \qquad \mathrm{int}_{\mathcal{R}}(A) = \mathrm{cl}_{\mathcal{R}}(A^c)^c,$$

with $A^c = X \setminus A$, and moreover

$$\mathrm{res}_{\mathcal{R}}(A) = \mathrm{cl}_{\mathcal{R}}(A) \setminus A \qquad \text{and} \qquad \mathrm{bnd}_{\mathcal{R}}(A) = \mathrm{cl}_{\mathcal{R}}(A) \setminus \mathrm{int}_{\mathcal{R}}(A).$$

Now, for instance, we may also naturally define

$$\mathcal{T}_{\mathcal{R}} = \{ A \subseteq X : \quad A \subseteq \mathrm{int}_{\mathcal{R}}(A) \} \quad \text{and} \quad \mathcal{E}_{\mathcal{R}} = \{ A \subseteq X : \quad \mathrm{int}_{\mathcal{R}}(A) \neq \emptyset \}.$$

Moreover, following a theorem and a definition of Norman Levine (1963), on semi-open sets in topological spaces, a subset A of the relator space $X(\mathcal{R})$ may be naturally called

(a) topologically semi-open if $A \subseteq \mathrm{cl}_{\mathcal{R}}\big(\mathrm{int}_{\mathcal{R}}(A)\big)$;

(b) topologically quasi-open if $V \subseteq A \subseteq \mathrm{cl}_{\mathcal{R}}(V)$ for some $V \in \mathcal{T}_{\mathcal{R}}$.

Thus, as a generalizations of a theorem of N. Levine, we can prove the following Hyers-Ulam type stability result.

(1) If A is a topologically semi-open subset of a topologically filtered, topological relator space $X(\mathcal{R})$, then there exist $V \in \mathcal{T}_{\mathcal{R}}$ and a nowhere dense subset B of $X(\mathcal{R})$ such that

$$A = V \cup B \qquad \text{and} \qquad V \cap B = \emptyset.$$

Moreover, by making use of some observations of K. Dlaska, N. Ergun and M. Ganster (1994) and Z. Duszyński and T. Noiri (2012), we can also prove the following more general statements.

(2) A subset A of a topological relator space $X(\mathcal{R})$ is topologically semi-open if and only if there exist $V \in \mathcal{T}_{\mathcal{R}}$ and $B \subseteq X$ such that

$$A = V \cup B \qquad \text{and} \qquad B \subseteq \mathrm{bnd}_{\mathcal{R}}(V).$$

(3) A subset A of a reflexive relator space $X(\mathcal{R})$ is topologically semi-open if and only if there exists $B \subseteq X$ such that

$$A = \mathrm{int}_{\mathcal{R}}(A) \cup B \qquad \text{and} \qquad B \subseteq \mathrm{res}_{\mathcal{R}}\big(\mathrm{int}_{\mathcal{R}}(A)\big).$$

The topologically quasi-open sets are always topologically semi-open. Moreover, in a topological relator space the two notions coincide. While, in a reflexive relator space, we can only state that the topologically open sets are topologically quasi-open, and thus also topologically semi-open.

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Contents

1. Introduction

The following basic definition and the subsequent statements were first established by Levine [60]. They have attracted the interest of a surprisingly great number of mathematicians.

By [9], D. E. Cameron considered semi-open sets to be the most important contribution of Levine to Topology. And, by the Google Scholar, the paper of Levine has been cited by 2693 works.

Definition 1.1. A subset A of a topological space $X(\mathcal{T})$ is called semi-open if there exists $V \in \mathcal{T}$ such that

$$V \subseteq A \subseteq V^{-}.$$

Remark 1.2. This means that A is, in a reasonable sense, near to, or can be approximated by an open set V.

Concerning the family $SO(X, \mathcal{T})$ of all semi-open subsets of $X(\mathcal{T})$, Levine, for instance, proved the following two theorems.

Theorem 1.3. *Under the above notations, we have*

(1) $\mathcal{T} \subseteq SO(X, \mathcal{T})$ *(2)* $\mathcal{T} = \{ A^{\circ} : \ A \in SO(X, \mathcal{T}) \}$.

Remark 1.4. By taking $X = \mathbb{R}$ and $A = [0, 1]$, one can at once see that, in general, $\mathcal{T}$ is a proper subset of $SO(X, \mathcal{T})$.

Theorem 1.5. *For any* $A \subseteq X(\mathcal{T})$, *the following assertions are equivalent:*

(1) $A \in SO(X, \mathcal{T})$; *(2)* $A \subseteq A^{\circ -}$.

Remark 1.6. The latter inclusion, which means that A is open with respect to the composite operation $\circ -$, was perhaps first proved to be equivalent to the equality $A^- = A^{\circ -}$ by Njåstad [73] who, being not aware of the results of Levine, used the term "β–set" instead of "semi-open set".

Njåstad's observation was later rediscovered by Isomichi [50, Theorem 2] and Noiri [74, Lemma 2]. Isomichi, being not aware the results of Levine and Njåstad, used the term "subcondensed set" instead of "semi-open set" and "β–set".

However, it is now more interesting to note that Levine also proved a close analogue of a stability theorem of Hyers [48] which, in a relevant formulation, says that an ε–approximately additive function of one Banach space to another is the sum of an additive function and an ε–small function.

Hyers' stability theorem has been generalized by a great number of mathematicians including the second author [113, 116, 117]. It has led to an enormous theory of the stability of functional equations and inequalities [49, 11, 51].

By the Google Scholar, the paper of Hyers has been cited by 3588 works. While, a similar statement of Pólya and Szegő [84, Aufgabe 99, pp. 17, 171], having almost the same power, was first cited by Kuczma [55, p. 424] at the suggestion of R. Ger with reference to a talk with M. Laczkovich. (See [116] for the details.)

More concretely, Levine's [60, Theorem 7] states the following stability property of open sets in topological spaces.

Theorem 1.7. *If $A \in SO(X, \mathcal{T})$, then there exist $V \in \mathcal{T}$ and a nowhere dense subset B of $X(\mathcal{T})$ such that*

$$A = V \cup B \qquad \text{and} \qquad V \cap B = \emptyset .$$

Remark 1.8. By taking $X = \mathbb{R}$, $V =]\,0, 1\,[$ and $B = \{2\}$ Levine also noted that the converse of this theorem is false.

Levine's theorem was later improved by Dlaska, Ergun and Ganster [33] who established the following

Theorem 1.9. *For any $A \subseteq X(\mathcal{T})$, the following assertions are equivalent:*

(1) $A \in SO(X, \mathcal{T})$;

(2) there exist $V \in \mathcal{T}$ and a nowhere dense subset B of $X(\mathcal{T})$ such that

$$A = V \cup B \qquad \text{and} \qquad B \subseteq \operatorname{bnd}(V) .$$

Remark 1.10. Here, $\operatorname{bnd}(V)$ denotes the boundary of V which is defined by

$$\operatorname{bnd}(V) = V^- \setminus V^\circ .$$

This theorem was later reformulated by Duszyński and Noiri [35] in the following more convenient form.

Theorem 1.11. *For any $A \subseteq X(\mathcal{T})$, the following assertions are equivalent:*

(1) $A \in SO(X, \mathcal{T})$;

(2) there exists $B \subseteq X$ such that

$$A = A^\circ \cup B \qquad \text{and} \qquad B \subseteq \operatorname{bnd}(A^\circ) .$$

Remark 1.12. Note that
$$\operatorname{bnd}(A^\circ) = A^{\circ -} \setminus A^{\circ\circ} = A^{\circ -} \setminus A^\circ .$$
Thus, by using the residue
$$\operatorname{res}(V) = V^- \setminus V$$
of a subset V of $X(\mathcal{T})$, we can simply write $\operatorname{res}(A^\circ)$ instead of $\operatorname{bnd}(A^\circ)$.

As an interesting stability-like property of semi-open sets, Levine also proved the following

Theorem 1.13. *If* $A \in SO(X, \mathcal{T})$ *and*
$$A \subseteq B \subseteq A^- ,$$
then $B \in SO(X, \mathcal{T})$ *also holds.*

Remark 1.14. Therefore, if $A \in SO(X, \mathcal{T})$, then we also have $A^- \in SO(X, \mathcal{T})$. Thus, in particular, by Theorem 1.3, we also have $V^- \in SO(X, \mathcal{T})$ for all $V \in \mathcal{T}$.

By using Theorems 1.1 and 1.13, Levine also proved the following

Theorem 1.15. $\mathcal{A} = SO(X, \mathcal{T})$ *is the smallest family of subsets of* $X(\mathcal{T})$ *such that*

(1) $\mathcal{T}_{\mathcal{R}} \subseteq \mathcal{A}$*;* *(2)* $A \in \mathcal{A}$ *and* $A \subseteq B \subseteq A^-$ *imply* $B \in \mathcal{A}$.

In the present paper, we shall show that Levine's definition and the above theorems on semi-open sets in topological spaces can be naturally extended not only to generalized topological and closure spaces, but also to relator spaces which are common generalizations not only of topological, closure and proximity spaces, but also those of ordered sets, context spaces and uniform spaces.

The necessary prerequisites on relators, which are certainly unfamiliar to the reader, will be briefly laid out in the subsequent preparatory sections which will also contain some new results on relators. These sections may also be useful for all those readers who are not very much interested in the various generalizations of open sets having been studied recently by a great number of authors.

2. A few basic facts on relations

A subset F of a product set $X \times Y$ is called a *relation on* X *to* Y. In particular, a relation on X to itself is called a *relation on* X. And, $\Delta_X = \{ (x, x) : x \in X \}$ is called the *identity relation of* X.

If F is a relation on X to Y, then by the above definitions we can also state that F is a relation on $X \cup Y$. However, for several purposes, the latter view of the relation F would be quite unnatural.

If F is a relation on X to Y, then for any $x \in X$ and $A \subseteq X$ the sets $F(x) = \{ y \in Y : (x, y) \in F \}$ and $F[A] = \bigcup \{ F(x) : x \in A \}$ are called the *images of* x *and* A *under* F, respectively.

If $(x, y) \in F$, then instead of $y \in F(x)$, we may also write $x\,F\,y$. However, instead of $F[A]$, we cannot write $F(A)$. Namely, it may occur that, in addition to $A \subseteq X$, we also have $A \in X$.