UNIVERSITY OF DEBRECEN

Solvability relations in groupoids

Árpád Száz

Preprints No. 428
(Technical Reports No. 2019/2)

INSTITUTE OF MATHEMATICS

2019

SOLVABILITY RELATIONS IN GROUPOIDS

ÁRPÁD SZÁZ

Abstract. If X is a groupoid, then for any $x, y \in X$ we define

$$\Phi(x, y) = \{ u \in X : \ u x = y \} \qquad \text{and} \qquad \Psi(x, y) = \{ v \in X : \ x v = y \},$$

and moreover $\varphi(x) = \Phi(x, x)$ and $\psi(x) = \Psi(x, x)$.

Here, φ, ψ and Φ, Ψ will be considered as relations on X and X^2 to X, respectively. And, they will be called the main solvability relations in X. Note that, more precisely, the groupoid X should also be indicated in the above notations. For instance, instead of Φ we should write Φ_X.

To feel the importance of these relations, note that if $u \in \varphi(x)$ and $v \in \Psi(x, u)$, then $u x = x$ and $x v = u$. Therefore, u is a left unit for x and v is a right inverse of x relative to u. Thus, the above solvability relations can be used to classify and investigate groupoids.

For instance, the groupoid X may be called prefunctional if the restrictions of the relations φ and ψ to the set

$$X_0 = \begin{cases} X \setminus \{0\} & \text{if } X \text{ has a zero } 0, \\ X & \text{if } X \text{ does not have a zero} \end{cases}$$

are functions of X_0 to X. That is, the sets $\varphi(x)$ and $\psi(x)$ are singletons for all $x \in X_0$.

If X is a prefunctional groupoid, then by identifying singletons with their elements we may also define

$$\sigma(x) = \Phi\big(x, \psi(x)\big) \qquad \text{and} \qquad \rho(x) = \Psi\big(x, \varphi(x)\big)$$

for all $x \in X_0$.

Moreover, we may call the prefunctional groupoid X to be semifunctional if the relations σ and ρ are also functions of X_0 to X. Surprisingly, if X is a prefunctional semigroup, then $\sigma = \rho$, and thus ρ is not needed.

If X is a semifunctional semigroup with zero 0, then X may be called a Brand–Clifford semigroup and X_0 may be called a Brand partial groupoid. Thus, the difficult definitions and properties of Brandt partial groupoids can be briefly expressed in terms of the solvability relations.

The principal task here is to determine the solvability relations in a given semigroup X. Unfortunately, this can be done only in some very particular cases. Of course, if X is a group, then the solvability relations in X can be easily computed, and they are functions.

In this respect it is also worth mentioning that a groupoid X may be called a quasi-group if it is functional in the sense that the relations Φ and Ψ are functions of X^2 to X. Moreover, the famous Green relations $\mathcal{L}$ and $\mathcal{R}$ can also be nicely defined in terms of the relations Φ and Ψ considered in the groupoid obtained from X by adjoining a unit if necessary.

2010 *Mathematics Subject Classification.* Primary 20N02; Secondary 20L05.
Key words and phrases. Groupoids, equations in groupoids, solvability relations.

Contents

Introduction

If X is a groupoid, then for any $x, y \in X$ we define

$$\Phi(x, y) = \{ u \in X : \quad u x = y \} \qquad \text{and} \qquad \Psi(x, y) = \{ v \in X : \quad x v = y \},$$

and moreover

$$\varphi(x) = \Phi(x, x) \qquad \text{and} \qquad \psi(x) = \Psi(x, x).$$

Here, φ, ψ and Φ, Ψ will be considered as relations on X and X^2 to X, respectively. And, they will be called the main solvability relations in X. Note that, more precisely, the groupoid X should also be indicated in the above notations. For instance, instead of Φ we should write Φ_X.

To feel the importance of these relations, note that if $u \in \varphi(x)$ and $v \in \Psi(x, u)$, then $u x = x$ and $x v = u$. Therefore, u is a left unit for x and v is a right inverse of x relative to u. Thus, the above solvability relations can be used to classify and investigate groupoids.

For instance, the groupoid X may be naturally called *prefunctional* if the restrictions of the relations φ and ψ to the set

$$X_0 = \begin{cases} X \setminus \{0\} & \text{if } X \text{ has a zero } 0; \\ X & \text{if } X \text{ does not have a zero} \end{cases}$$

are functions of X_0 to X. That is, the sets $\varphi(x)$ and $\psi(x)$ are singletons for all $x \in X_0$. If the groupoid X has a zero 0, then $\varphi(0) = X$ and $\psi(0) = X$. Therefore, even the relations φ and ψ cannot, in general, be functions.

However, if X is a prefunctional groupoid, then by identifying singletons with their elements we may also naturally define

$$\sigma(x) = \Phi\big(x,\, \psi(x)\big) \qquad \text{and} \qquad \rho(x) = \Psi\big(x,\, \varphi(x)\big)$$

for all $x \in X_0$. Moreover, we may naturally call the prefunctional groupoid X to be *semifunctional* if the relations σ and ρ are also functions of X_0 to X. Surprisingly, if X is a prefunctional semigroup, then $\sigma = \rho$, and thus ρ is not needed.

If X is a semifunctional semigroup with zero 0, then X may be naturally called a *Brand–Clifford semigroup* [12, 13] and X_0 may be naturally called a *Brand partial groupoid* [3, 13]. Thus, the difficult definitions and properties of Brandt partial groupoids can be briefly expressed in terms of the solvability relations.

In this respect, it is also worth mentioning that if X is a groupoid, then following the ideas of Preston [45] and Clifford [11] we may also naturally define

$$\alpha(x) = \big\{\, u \in \varphi(x) : \quad \Psi(x,\, u) \neq \emptyset \big\}$$

and

$$\beta(x) = \big\{\, v \in \psi(x) : \quad \Phi(x,\, v) \neq \emptyset \,\big\}$$

for all $x \in X$, and

$$\gamma(u) = \big\{\, x \in X : \quad u \in \varphi(x) \cap \psi(x) \,\big\}$$

and

$$\theta(u) = \Big\{\, x \in \gamma(u) : \quad \Phi(x,\, u) \cap \Psi(x,\, u) \neq \emptyset \,\Big\}$$

for all $u \in X$.

Namely, for instance, if X is a semigroup, then it can be shown that $\theta(u) \neq \emptyset$ if and only if u is idempotent if and only if $\theta(u)$ is a subgroup of X with unit u. Moreover, $\theta(u) \cap \theta(v) \neq \emptyset$ implies $u = v$. Therefore, the semigroup may be called a *Clifford semigroup* [11] if the relation θ is onto X.

The principal task here is to determine the solvability relations in a given semigroup X. Unfortunately, this can be done only in some very particular cases. Of course, if X is a group, then the solvability relations in X can be easily computed, and they are functions.

In this respect it is also worth mentioning that a groupoid X may be called a quasi-group if it is *functional* in the sense that the relations Φ and Ψ are functions of X^2 to X. Moreover, the famous Green relations $\mathcal{L}$ and $\mathcal{R}$ [21, 13] can also be nicely defined in terms of the solvability relations Φ and Ψ.

For instance, if X is a groupoid, then by [24, Lemma1.2], for any $x, y \in X$, we may naturally write

$$x\,\mathcal{L}\,y \quad \Longleftrightarrow \quad \Phi_{X^*}(x,\, y) \neq \emptyset \quad \text{and} \quad \Phi_{X^*}(y,\, x) \neq \emptyset\,,$$

where $X^* = X$ if X has a unit and $X^* = X \cup \{1\}$, with an appropriate extension of the original multiplication, if X does not have a unit.