**UNIVERSITY OF DEBRECEN**

# Generalizations of some ordinary and extreme connectedness properties of topological spaces to relator spaces

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# GENERALIZATIONS OF SOME ORDINARY AND EXTREME CONNECTEDNESS PROPERTIES OF TOPOLOGICAL SPACES TO RELATOR SPACES

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*Dedicated to the memory of János Kurdics*
*who was the first to note that connectedness is a particular case of well-chainedness*

ABSTRACT. Motivated by some ordinary and extreme connectedness properties of topologies, we introduce several reasonable connectedness properties of relators (families of relations). Moreover, we establish some immediate connections among these properties.

More concretely, we investigate relationships among various minimalness (well-chainedness), connectedness, hyper- and ultra-connectedness, door, superset, submaximality and resolvability properties.

Since most generalized topologies and all proper stacks can be derived from preorder relators, the results obtained greatly extends some former results on topologies. Moreover, they are also closely related to some former results on well-chained and connected uniformities.

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## 1. Connectedness properties of topologies

By Thron [212, p. 18], topological spaces were first suggested by Tietze [213] and Alexandroff [4]. They were later standardized by Bourbaki [18], Kelley [81] and Engelking [53]. (For some historical facts, see also Folland [57].)

If $\mathcal{T}$ is a family of subsets of a set $X$ such that $\mathcal{T}$ is closed under finite intersections and arbitrary unions, then the family $\mathcal{T}$ is called a *topology* on $X$, and the ordered pair $X(\mathcal{T}) = (X, \mathcal{T})$ is called a *topological space*.

The members of $\mathcal{T}$ are called the *open subsets* of $X$. While, the members of $\mathcal{F} = \mathcal{T}^c = \{A^c \subseteq X : \ A \in \mathcal{T}\}$ are called the *closed subsets* of $X$. And, the members of $\mathcal{T} \cap \mathcal{F}$ are called the *clopen subsets* of $X$.

Note that $\emptyset \subseteq \mathcal{T}$ such that $\emptyset = \bigcup \emptyset$ and $X = \bigcap \emptyset$. Therefore, we necessarily have $\{\emptyset, X\} \subseteq \mathcal{T}$, and thus also $\{\emptyset, X\} \subseteq \mathcal{F}$. Consequently, $\{\emptyset, X\} \subseteq \mathcal{T} \cap \mathcal{F}$ is always true. That is, $\emptyset$ and $X$ are always clopen subsets of $X$.

According to Száz [166, 169, 178], the members of the family

$$\mathcal{E} = \{ A \subseteq X : \quad \exists \ U \in \mathcal{T} \setminus \{\emptyset\} : \quad U \subseteq A \}$$

may be naturally called the *fat subsets* of $X$.

Hence, it is clear that $\mathcal{E} \neq \emptyset$ if and only if $X \neq \emptyset$. Moreover, $\mathcal{E}$ is a *proper stack* on $X$ in the sense that $\emptyset \notin \mathcal{E}$ and $\mathcal{E}$ is *ascending* in $X$. That is, if $A \in \mathcal{E}$ and $A \subseteq B \subseteq X$, then $B \in \mathcal{E}$ also holds.