UNIVERSITY OF DEBRECEN

Semi-inner products and parapreseminorms on groups and a generalization of a theorem of Maksa and Volkmann on additive functions

Árpád Száz

INSTITUTE OF MATHEMATICS

2018

SEMI-INNER PRODUCTS AND PARAPRESEMINORMS ON GROUPS AND A GENERALIZATION OF A THEOREM OF MAKSA AND VOLKMANN ON ADDITIVE FUNCTIONS

ÁRPÁD SZÁZ

ABSTRACT. By using inner products and paraprenorms on groups, we prove a natural generalization of a basic theorem of Gyula Maksa and Peter Volkmann on additive functions.

1. INTRODUCTION

In this paper, by using inner products and paraprenorms on groups, we shall prove a natural generalization of the following basic theorem of Maksa and Volkmann $[81]$.

Theorem 1.1. *For functions* $f : G \to E$ *from a group* G *to a real or complex inner product space* E, *the inequality*

$$\| f(x\,y) \| \geq \| f(x) + f(y) \| \qquad \big(x,\, y \in G \big)$$

implies

$$f(x\,y) = f(x) + f(y) \qquad \big(x,\, y \in G \big).$$

Remark 1.2. For the origins of this striking theorem, see Volkmann $[133]$, Maksa $[80]$ and Kurepa $[73]$. The latter author also studied the converse inequality and provided two illustrating examples.

The $G(\cdot) = \mathbb{R}(+)$ and $E(+) = \mathbb{R}(+)$ particular case of Theorem 1.1 was later also proved, in a completely different way, by Kwon at al. $[74]$ without citing the works of the above mentioned authors.

Remark 1.3. Before the inequalities

$$| f(x) + f(y) | \leq | f(x + y) | \qquad \text{and} \qquad \| f(x) + f(y) \| \leq \| f(x + y) \|,$$

the squared and normed Cauchy equations

$$f(x + y)^2 = \big(f(x) + f(y) \big)^2 \qquad \text{and} \qquad \| f(x + y) \| = \| f(x) + f(y) \|$$

were also intensively investigated by a great number of mathematicians.

See, for instance, Robinson $[100]$, Hosszú $[55]$, Vincze $[130, 131, 132]$, Fischer and Muszély $[34, 35]$, Haruki $[53, 54]$, Dhombres and Aczél $[22, 1]$, Swiatak and Hosszú $[114, 115]$, Kuczma $[67]$, Skof $[109, 110, 111]$, Ger $[39, 40, 41, 43]$, Schöpf $[104]$, Piejko $[96]$, Batko and Tabor $[8, 9]$, Ger and Koclega $[44]$, Tabor