UNIVERSITY OF DEBRECEN

An answer to the question "What is the essential difference between algebra and topology?" of Shukur Al-Aeashi

Árpád Száz

Preprints No. 420
(Technical Reports No. 2017/2)

INSTITUTE OF MATHEMATICS

2017

AN ANSWER TO THE QUESTION ”WHAT IS THE ESSENTIAL DIFFERENCE BETWEEN ALGEBRA AND TOPOLOGY?” OF SHUKUR AL-AEASHI

ÁRPÁD SZÁZ

In the present form, this is not a good question. Namely, topology, in a narrower sense, is the extensive theory of topological spaces $X(\mathcal{T}) = (X, \mathcal{T})$ consisting of a set X and a family $\mathcal{T}$ of subsets of X which is closed under finite intersections and arbitrary unions.

In my opinion, the use of the concept of open sets as a starting point, is the greatest mistake in mathematics. It was first suggested by Tietze (1923), and later standardized by Bourbaki (1940), Kelley (1955) and Engelking (1977).

Hausdorff (1914), Kuratowski (1922), Weil (1937), Tukey (1940), Efremovič (1952), Császár (1960), Doičinov (1964) and several further mathematicians offered more convenient tools such as neighbourhoods, closures, uniformities, covers, proximities and convergences for instance.

Neighbourhoods of points or sets, and convergence of one net of points or sets to another are very powerful tools. However, having in mind Weil's uniformities, we prefer to use families of relations and corelations, and several natural closure and projection operations on them.

A subset R of a product set $X \times Y$ is called a relation on X to Y. And, a family $\mathcal{R}$ of relations on X to Y is called a relator on X to Y. They have been studied in several papers by the present author and his students.

While, a function U on one power set $\mathcal{P}(X)$ to another $\mathcal{P}(Y)$ is called a corelation on X to Y. And, a family $\mathcal{U}$ of corelations on X to Y is called a corelator on X to Y. Thus, complement and closure operations are corelations.

Since functions are very particular relations, corelations and corelators are very particular cases of relations and relators. However, if the ground sets X and Y are fixed, then the former ones are important generalizations of the latter ones.

Therefore, corelations and corelators have to be studied before relations and relators. We note that relators have already led to some substantial generalizations of the usual algebraic and topological structures such as Galois connections and proximity relations for instance.

A better question than that of Al-aeashi could be: ”What is the essential difference between Algebra and Analysis?”. In my opinion, the only essential difference is in the cardinality of the family of relations or corelations considered on the corresponding ground set. The following two examples will clarify our imaginations.

The work of the author on relators and corelators has been supported by the Hungarian Scientific Research Fund (OTKA) Grant K-111651.

An ordered vector space over $\mathbb{R}$ is a pair $X(\leq) = (X, \leq)$ consisting of a vector space X over $\mathbb{R}$ and an order relation $\leq$ on X which is compatible with the linear operations to the extent that, for any $r \in \mathbb{R}$ and $x, y, z \in X$:

(1) $x \leq y$ implies $x + z \leq y + z$;

(2) $0 \leq r$ and $x \leq y$ imply $r x \leq r y$.

The relation $\leq$ can be identified with the function defined by

$$f(x) = \leq (x) = \{ y \in X : \quad x \leq y \}$$

for all $x \in X$. And, the above compatibility properties can be expressed in the form that:

(1) $f(x) = x + f(0)$ for all $x \in X$;

(2) $f(0) + f(0) \subseteq f(0)$; (3) $r f(0) \subseteq f(0)$ for all $r \geq 0$.

However, instead of the relation $\leq$ and the function f, it is more convenient to consider the corelation defined by

$$F(A) = f[A] = \bigcup_{x \in A} f(x)$$

for all $A \subseteq X$. And, to establish the corresponding properties of F.

A vector relator space over $\mathbb{R}$ is a pair $X(\mathcal{R}) = (X, \mathcal{R})$ consisting a vector space X over $\mathbb{R}$ and and a relator $\mathcal{R}$ on X which is compatible with the linear operations to the extent that:

(1) $R(x) = x + R(0)$ for all $X \in X$ and $R \in \mathcal{R}$;

(2) $R(0)$ is an absorbing, balanced subset of X for all $R \in \mathcal{R}$;

(3) for each $R \in \mathcal{R}$ there exists $S \in \mathcal{R}$ such that $S(0) + S(0) \subseteq R(0)$.

Note that, if $\mathcal{P}$ is a family of preseminorms on a vector space X, then the relator

$$\mathcal{R} = \{ B_r^p : \quad p \in \mathcal{P}, \quad r > 0 \},$$

where

$$B_r^p = \{ (x, y) \in X^2 : \quad p(-x + y) < r \},$$

can be proved to have only the above three properties.

Of course, if the family $\mathcal{P}$ is directed upstairs in the sense that for any $p, q \in \mathcal{P}$ there exists $\rho \in \mathcal{P}$ such that $p \leq \rho$ and $q \leq \rho$, then the relator $\mathcal{R}$ has the useful addition property that:

(4) for any $R, S \in \mathcal{R}$ there exists $T \in \mathcal{R}$ such that $T(0) \subseteq R(0) \cap S(0)$.

Note that, because of the translation property (1), properties (2), (3) and (4) have several useful consequences.

Instead of a relator $\mathcal{R}$ on X to Y, it is also more convenient to consider the corelator

$$\mathcal{R}^{\triangleright} = \{ R^{\triangleright} : \quad R \in \mathcal{R} \},$$

where $R^{\triangleright}$ is the corelation defined by

$$R^{\triangleright}(A) = R[A]$$

for all $A \subseteq X$.

Conversely, if $\mathcal{U}$ is corelation on X to Y, then as a helpful tool we may also naturally consider the relator

$$\mathcal{U}^{\triangleleft} = \left\{ U^{\triangleleft} : \quad U \in \mathcal{U} \right\},$$

where $U^{\triangleleft}$ the relation defined such that

$$U^{\triangleleft}(x) = U\left(\{x\}\right)$$

for all $x \in X$.

Thus, the maps $\triangleright$ and $\triangleleft$ establish an increasing Galois connection between relations and corelatation on X to Y. Moreover, relations can be identified with union-preserving corelations.

Therefore, the fact that corelators can generate more general structures than relators can be demonstrated by taking a corelation U on X which is not union-preserving.

The best such example is when U is just the complementation operation on X. Namely, in this case, it can be easily shown that, for any relator $\mathcal{R}$ on X, we have not only $\mathrm{Cl}_U \neq \mathrm{Cl}_{\mathcal{R}}$, but also $\mathrm{Cl}_U(B) \neq \mathrm{Cl}_{\mathcal{R}}(B)$ for all $B \subseteq X$ with $B \neq \emptyset$.

Here, in contrast to the standard notation $\delta_{\mathcal{U}}$, for any corelator $\mathcal{U}$ on X to Y and $A \subseteq X$ and $B \subseteq Y$, we write $A \in \mathrm{Cl}_{\mathcal{U}}(B)$ if $U(A) \cap B \neq \emptyset$ for all $U \in \mathcal{U}$. And, for any $x \in X$, we write $x \in \mathrm{cl}_{\mathcal{U}}(B)$ if $\{x\} \in \mathrm{Cl}_{\mathcal{U}}(B)$.

Now, for any relator $\mathcal{R}$ on X to Y, we may also naturally define $\mathrm{Cl}_{\mathcal{R}} = \mathrm{Cl}_{\mathcal{R}^{\triangleright}}$ and $\mathrm{cl}_{\mathcal{R}} = \mathrm{cl}_{\mathcal{R}^{\triangleright}}$. Thus, for any $A \subseteq X$ and $B \subseteq Y$, we have $A \in \mathrm{Cl}_{\mathcal{R}}(B)$ if and only if $R[A] \cap B \neq \emptyset$ for all $R \in \mathcal{R}$. And, we can also easily note that $\mathrm{cl}_{\mathcal{R}}(B) = \bigcap_{R \in \mathcal{R}} R^{-1}[B]$.

Moreover, for any corelator $\mathcal{U}$ on X to Y, we have $\mathrm{cl}_{\mathcal{U}} = \mathrm{cl}_{\mathcal{U}^{\triangleleft}}$. And, for any corelation U on X to Y, $A \subseteq X$ and $B \subseteq Y$, we have $A \in \mathrm{Cl}_{U^{\triangleleft}}(B)$ if and only if $A \cap \mathrm{cl}_U(B) \neq \emptyset$.

To show that the map $\triangleleft$ may also have some useful applications, we can note that, for any corelator $\mathcal{U}$ on X, we may also naturally define

$$\mathcal{U}^{\circ} = \left\{ U^{\circ} : \quad U \in \mathcal{U} \right\} \qquad \text{and} \qquad \mathcal{U}^{\mathbf{-1}} = \left\{ U^{\mathbf{-1}} : \quad U \in \mathcal{U} \right\},$$

where

$$U^{\circ} = U^{\triangleleft \triangleright} \qquad \text{and} \qquad U^{\mathbf{-1}} = U^{\triangleleft -1 \triangleright}.$$

Moreover, for any two corelators $\mathcal{U}$ on X to Y and $\mathcal{V}$ on Y to Z, we may also naturally define

$$\mathcal{V} \bullet \mathcal{U} = \left\{ V \bullet U : \quad U \in \mathcal{U}, \ \ V \in \mathcal{V} \right\}, \qquad \text{where} \qquad V \bullet U = \left(V^{\triangleleft} \circ U^{\triangleleft} \right)^{\triangleright}.$$

References

[1] Birkhoff, G., *Lattice Theory*, Amer. Math. Soc., Providence, RI, 1967.

[2] Bourbaki, N., *General Topology, Chapters 1–4*, Springer-Verlag, Berlin, 1989.

[3] Buglyó, S. and Száz, Á., *A more important Galois connection between distance functions and inequality relations*, Sci. Ser. A Math. Sci. (N.S.), **18** (2009), 17–38.

[4] Čech, E., *Topological Spaces*, Academia, Prague, 1966.

[5] Császár, Á. *Foundations of General Topology*, Pergamon, London, 1963.

[6] Curtis, D. W. and Mathews, J. C., *Generalized uniformities for pairs of spaces*, Topology Conference, Arizona State University, Tempe, Arizona, 1967, 212–246.

[7] Davey, B. A. and Priestley, H. A., *Introduction to Lattices and Order*, Cambridge University Press, Cambridge, 2002.
[8] Davis, A. S., *Indexed systems of neighbordoods for general topological spaces*, Amer. Math. Monthly **68** (1961), 886–893.
[9] Deák, J., *A counterexample on completeness in relator spaces*, Publ. Math. Debrecen **41** (1992), 307–309.
[10] Efremovič, V. A., *The geometry of proximity*, Mat. Sb. **31** (1952), 189–200. (Russian.)
[11] Efremović, V. A. and Švarc, A. S., *A new definition of uniform spaces, Metrization of proximity spaces.* Dokl. Acad. Nauk. SSSR **89** (1953), 393–396. (Russian)
[12] Engelking, R., *General Topology*, Polish Scientific Publishers, Warszawa, 1977.
[13] Fletcher, P. and Lindgren, W. F., *Quasi-Uniform Spaces*, Marcel Dekker, New York, 1982.
[14] Ganter, B. and Wille, R., *Formal Concept Analysis*, Springer-Verlag, Berlin, 1999.
[15] Hausdorff, F., *Grundzüge der Mengenlehre*, De Gruyter, Berlin, 1914.
[16] Höhle, U. and Kubiak, T., *On regularity of sup-preserving maps: generalizing Zareckiĭ's theoremark*, Semigroup Forum, **83** (2011), 313–319.
[17] Isbell, J. R., *Uniform Spaces*, Amer. Math. Soc., Providence, 1964.
[18] Kelley, J. L., *General Topology*, Van Nostrand Reinhold Company, New York, 1955.
[19] Kuratowski, C., *Topologie I, II*, Polish Scientific Publishers, Warszawa, 1966, 1968.
[20] Kurdics, J. and Száz, Á., *Well-chained relator spaces*, Kyungpook Math. J. **32** (1992), 263–271.
[21] Mala, J. and Száz, Á., *Properly topologically conjugated relators*, Pure Math. Appl., Ser. B **3** (1992), 119–136.
[22] Mala, J. and Száz, Á., *Modifications of relators*, Acta Math. Hungar. **77** (1997), 69–81.
[23] Nachbin, L., *Topology and Order*, D. Van Nostrand, Princeton, 1965.
[24] Nakano, H. and Nakano, K., *Connector theory*, Pacific J. Math. **56** (1975), 195–213.
[25] Pataki, G., *Supplementary notes to the theory of simple relators*, Radovi Mat. **9** (1999), 101–118.
[26] Pataki, G., *On the extensions, refinements and modifications of relators*, Math. Balk. **15** (2001), 155–186.
[27] Pataki, G. and Száz, Á., *A unified treatment of well-chainedness and connectedness properties*, Acta Math. Acad. Paedagog. Nyházi. (N.S.) **19** (2003), 101–165.
[28] Pervin, W. J., *Quasi-uniformization of topological spaces*, Math. Ann. **147** (1962), 316–317.
[29] Rakaczki, Cs. and Száz, Á., *Semicontinuity and closedness properties of relations in relator spaces*, Mathematica (Cluj) **45** (2003), 73–92.
[30] Smirnov, Yu. M., *On proximity spaces*, Math. Sb. **31** (1952), 543–574. (Russian)
[31] Szabó, Gy. and Á. Száz, *Defining nets for integration*, Publ. Math. Debrecen **36** (1989), 237–252.
[32] Száz, Á., *Preseminormed spaces*, Publ. Math. Debrecen **30** (1983), 217–224.
[33] Száz, Á., *Basic tools and mild continuities in relator spaces*, Acta Math. Hungar. **50** (1987), 177–201.
[34] Száz, Á., *Directed, topological and transitive relators*, Publ. Math. Debrecen **35** (1988), 179–196.
[35] Száz, Á., *Projective and inductive generations of relator spaces*, Acta Math. Hungar. **53** (1989), 407–430.
[36] Száz, Á., *Lebesgue relators*, Monatsh. Math. **110** (1990), 315–319.
[37] Száz, Á., *The fat and dense sets are more important than the open and closed ones*, Abstracts of the Seventh Prague Topological Symposium, 1991, Inst. Math. Czechoslovak Acad. Sci., p. 106.
[38] Száz, Á., *Inverse and symmetric relators*, Acta Math. Hungar. **60** (1992), 157–176.
[39] Száz, Á., *Structures derivable from relators*, Singularité **3**(1992), 14–30.
[40] Száz, Á., *Cauchy nets and completeness in relator spaces*, Colloq. Math. Soc. János Bolyai **55** (1993), 479–489.
[41] Száz, Á., *Refinements of relators*, Tech. Rep., Inst. Math., Univ. Debrecen **76**(1993), 19 pp.
[42] Száz, Á., *Neighbourhood relators*, Bolyai Soc. Math. Stud. **4** (1995), 449–465.
[43] Száz, Á., *Relations refining and dividing each other*, Pure Math. Appl., Ser. B **6** (1995), 385–394.

[44] Száz, Á., *Topological characterizations of relational properties*, Grazer Math. Ber. **327** (1996), 37–52.

[45] Száz, Á., *Uniformly, proximally and topologically compact relators*, Math. Pannon. **8** (1997), 103–116.

[46] Száz, Á., *An extension of Kelley's closed relation theorem to relator spaces*, Filomat **14** (2000), 49–71.

[47] Száz, Á., *Preseminorm generating relations and their Minkowski functionals*, Publ. Elektrotehn. Fak., Univ. Beogr., Ser. Mat. **12** (2001), 16–34.

[48] Száz, Á. *A Galois connection between distance functions and inequality relations*, Math. Bohem., **127** (2002), 437–448.

[49] Száz, Á., *Somewhat continuity in a unified framework for continuities of relations*, Tatra Mt. Math. Publ. **24** (2002), 41–56.

[50] Száz, Á., *Upper and lower bounds in relator spaces*, Serdica Math. J. **29** (2003), 239–270.

[51] Száz, Á., *An extension of Baire's category theorem to relator spaces*, Math. Morav. **7** (2003), 73–89.

[52] Száz, Á., *Rare and meager sets in relator spaces*, Tatra Mt. Math. Publ. **28** (2004), 75–95.

[53] Száz, Á., *Galois-type connections on power sets and their applications to relators*, Tech. Rep., Inst. Math., Univ. Debrecen **2005/2**, 38 pp.

[54] Száz, Á., *Supremum properties of Galois–type connections*, Comment. Math. Univ. Carolin. **47** (2006), 569–583.

[55] Száz, Á. and Túri, J., *Comparisons and compositions of Galois–type connections*, Miskolc Math. Notes, **7** (2006), 189–203.

[56] Száz, Á., *Minimal structures, generalized topologies, and ascending systems should not be studied without generalized uniformities*, Filomat **21** (2007), 87–97.

[57] Száz, Á., *Galois type connections and closure operations on preordered sets*, Acta Math. Univ. Comen. **78** (2009), 1–21.

[58] Száz, Á., *Applications of relations and relators in the extensions of stability theorems for homogeneous and additive functions*, Aust. J. Math. Anal. Appl. **6** (2009), 1–66.

[59] Száz, Á., *Foundations of the theory of vector relators*, Adv. Stud. Contemp. Math. **20** (2010), 139–195.

[60] Száz, Á., *The Hyers–Ulam and Hahn–Banach theorems and some elementary operations on relations motivated their set-valued generalizations*, In: P. M. Pardalos, P. G. Georgiev and H. M. Srivastava (Eds.), Stability, Approximations, and Inequalities, Springer Optimization and Its Applications **68** (2012), 631–705.

[61] Száz, Á., *Lower semicontinuity properties of relations in relator spaces*, Adv. Stud. Contemp. Math. (Kyungshang) **23** (2013), 107–158.

[62] Száz, Á., *Inclusions for compositions and box products of relations*, J. Int. Math. Virt. Inst. **3** (2013), 97–125.

[63] Száz, Á., *An extension of an additive selection theorem of Z. Gajda and R. Ger to vector relator spaces*, Sci. Ser. A Math. Sci. (N.S.) **24**, (2013), 33–54.

[64] Száz, Á., *A particular Galois connection between relations and set functions*, Acta Univ. Sapientiae, Math. **6** (2014), 73–91.

[65] Száz, Á., *Galois and Pataki connections on generalized ordered sets*, Tech. Rep., Inst. Math., Univ. Debrecen **2014/3**, 27 pp.

[66] Száz, Á., *Generalizations of Galois and Pataki connections to relator spaces*, J. Int. Math. Virt. Inst. **4** (2014), 43–75.

[67] Száz, Á., *A unifying framework for studying continuity, increasingness, and Galois connections*, manuscript, 2015.

[68] Száz, Á., *The closure-interior Galois connection and its applications to relational inclusions and equations*, Tech. Rep., Inst. Math., Univ. Debrecen **2015/2**, 40 pp.

[69] Száz, Á., *Basic tools, increasing functions, and closure operations in generalized ordered sets*, In: P. M. Pardalos and Th. M. Rassias (Eds.), Contributions in Mathematics and Engineering: In Honor of Constantin Caratheodory, Springer, 2016, 551–616.

[70] Száz, Á. and Zakaria, A., *Mild continuity properties of relations and relators in relator spaces*, In: P. M. Pardalos and Th. M. Rassias (Eds.), Essays in Mathematics and its Applications: In Honor of Vladimir Arnold, Springer, 2016, 439–511.

[71] Száz, Á., *A natural Galois connection between generalized norms and metrics*, Tech. Rep., Inst. Math., Univ. Debrecen **20016/4**, 8 pp.

[72] Száz, Á., *Four general continuity properties, for pairs of functions, relations and relators, whose particular cases could be investigated by hundreds of mathematicians*, Tech. Rep., Inst. Math., Univ. Debrecen **2017/1**, 17 pp.

[73] Thron, W. J., , *Topological Structures*, Holt, Rinehartans Winston, New York, 1966.

[74] Tietze, H., *Beiträge zur allgemeinen Topologie I. Axiome für verschiedene Fassungen des Umgebungsbegriffs*, Math. Ann. **88** (1923), 290–312.

[75] Tukey, J. W., *Convergence and Uniformity in Topology*, Princeton Univ. Press, Princeton, 1940.

[76] Weil, A., *Sur les espaces á structure uniforme et sur la topologie générale*, Actual. Sci. Ind. 551, Herman and Cie, Paris, 1937.

Árpád Száz, Department of Mathematics, University of Debrecen, H–4002 Debrecen, Pf. 400, Hungary

E-mail address: szaz@science.unideb.hu