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Four general continuity properties, for pairs of functions, relations and relators, whose particular cases could be investigated by hundreds of mathematicians

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FOUR GENERAL CONTINUITY PROPERTIES, FOR PAIRS OF FUNCTIONS, RELATIONS AND RELATORS, WHOSE PARTICULAR CASES COULD BE INVESTIGATED BY HUNDREDS OF MATHEMATICIANS

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Abstract. This is a research proposal for those who are interested the in the unification of several continuity-like properties of functions and relations in the framework of relator spaces. For this, motivated by Galois connections, we shall use a pair of relators instead of a single function or relation.

A family $\mathcal{R}$ of relations on one set X to another Y is called a relator on X to Y. All reasonable generalizations of the usual topological structures (such as proximities, closures, topologies, filters and convergences, for instance) can be derived from relators. Therefore, they should not be studied separately.

From the various topological and algebraic stuctures (such as lower bounds, minimum and infimum, for instance) derived from relators, by using Pataki connections, we obtain several closure and modification operations for relators. Each of them leads to four reasonable continuity or increasingness properties.

1. Relations and relators

A subset R of a product set $X \times Y$ is called a *relation* on X to Y. In particular, a relation R on X to itself is simply called a relation on X. And, $\Delta_X = \{(x, x) : x \in X\}$ is called the identity relation on X.

If R is a relation on X to Y, then for any $x \in X$ and $A \subseteq X$ the sets $R(x) = \{y \in Y : (x, y) \in R\}$ and $R[A] = \bigcup_{a \in A} R(a)$ are called the *images* of x and A under R, respectively.

Moreover, the sets $D_F = \{x \in X : F(x) \neq \emptyset\}$ and $R_F = F[X]$ are called the *domain* and *range* of F. If in particular $D_F = X$, then we say that F is a relation of X to Y, or that F is a *total relation* on X to Y.

In particular, a relation f on X to Y is called a *function* if for each $x \in D_f$ there exists $y \in Y$ such that $f(x) = \{y\}$. In this case, by identifying singletons with their elements, we may simply write $f(x) = y$ in place of $f(x) = \{y\}$.

Moreover, a function $\star$ of X to itself is called a *unary operation* on X. While, a function $*$ of X^2 to X is called a *binary operation* on X. And, for any $x, y \in X$, we usually write $x^\star$ and $x * y$ instead of $\star(x)$ and $*(x, y)$.

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If R is a relation on X to Y, then we have $R=\bigcup_{x\in X}\{x\}\times R(x)$. Therefore, the values $R(x)$, where $x\in X$, uniquely determine R. Thus, a relation R on X to Y can be naturally defined by specifying $R(x)$ for all $x\in X$.

For instance, the *complement relation* F^c can be defined such that $F^c(x)=F(x)^c=Y\setminus F(x)$ for all $x\in X$. Thus, we also have $F^c=X\times Y\setminus F$. Moreover, we can note that $F^c[A]^c=\bigcap_{a\in A}F(a)$ for all $A\subseteq X$. (See [51].)

While, the *inverse relation* R^{-1} can be defined such that $R^{-1}(y)=\{x\in X:\ y\in R(x)\}$ for all $y\in Y$. Thus, we also have $R^{-1}=\{(y,x):\ (x,y)\in R\}$. Moreover, we can note that $F^{-1}[B]=\{x\in X:\ F(x)\cap B\neq\emptyset\}$ for all $B\subseteq Y$.

Moreover, if in addition S is a relation on Y to Z, then the *composition relation* $S\circ R$ can be defined such that $(S\circ R)(x)=S[R(x)]$ for all $x\in X$. Thus, we also have $(S\circ R)[A]=S\big[R[A]\big]$ for all $A\subseteq X$.

While, if S is a relation on Z to W, then the *box product relation* $R\boxtimes S$ can be defined such that $(R\boxtimes S)(x,z)=R(x)\times S(z)$ for all $x\in X$ and $z\in Z$. Thus, we have $(R\boxtimes S)[A]=S\circ A\circ R^{-1}$ for all $A\subseteq X\times Z$. (See [51].)

Hence, by taking $A=\{(x,z)\}$, and $A=\Delta_Y$ if $Y=Z$, one can at once see that the box and composition products are actually equivalent tools. However, the box product can be immediately defined for an arbitrary family of relations too.

Now, a relation R on X may be defined to be *reflexive* if $\Delta_X\subseteq R$, and *transitive* if $R\circ R\subseteq R$. Moreover, R may be defined to be *symmetric* if $R^{-1}\subseteq R$, and *antisymmetric* if $R\cap R^{-1}\subseteq\Delta_X$.

Thus, a reflexive and transitive (symmetric) relation may be called a *preorder (tolerance) relation*. And, a symmetric (antisymmetric) preorder relation may be called an *equivalence (partial order) relation*.

According to algebra, for any relation R on X, we may naturally define $R^0=\Delta_X$, and $R^n=R\circ R^{n-1}$ if $n\in\mathbb{N}$. Moreover, we may also define $R^\infty=\bigcup_{n=0}^{\infty}R^n$. Thus, R^∞ is the smallest preorder relation containing R [8].

Now, in contrast to $(F^c)^c=F$ and $(F^{-1})^{-1}=F$, we have $(R^\infty)^\infty=R^\infty$. Moreover, analogously to $(F^c)^{-1}=(F^{-1})^c$, we also have $(R^\infty)^{-1}=(R^{-1})^\infty$. Thus, in particular R^{-1} is also a preorder on X if R is a preorder on X.

A family $\mathcal{R}$ of relations on one set X to another Y is called a *relator* on X to Y. And, the ordered pair $(X,Y)(\mathcal{R})=\big((X,Y),\ \mathcal{R}\big)$ is called a *relator space*. (For the origins and motivations, see [25, 31, 39, 42] and the references in [25].)

If in particular $\mathcal{R}$ is a relator on X to itself, then we may simply say that $\mathcal{R}$ is a relator on X. And, by identifying singletons with their elements, we may simply write $X(\mathcal{R})$ in place of $(X,X)(\mathcal{R})$, since $(X,X)=\{\{X\}\}$.

Relator spaces of this simpler type are already substantial generalizations of the various *ordered sets* [2] and *uniform spaces* [6]. However, they are insufficient for some important purposes. (See, for instance, [7] and [38].)

A relator $\mathcal{R}$ on X to Y, or a relator space $(X,Y)(\mathcal{R})$, is called *simple* if there exists a relation R on X to Y such that $\mathcal{R}=\{R\}$. In this case, by identifying singletons with their elements, we may write $(X,Y)(R)$ in place of $(X,Y)\big(\{R\}\big)$.

According to Száz [41], a simple relator space $X(R)$ may be called a *goset* (generalized ordered set). Moreover, by Ganter and Wille [7, p. 17], a simple relator space $\big(X,Y\big)(R)$ may be called a *formal context* or *context space*.

A relator $\mathcal{R}$ on X, or a relator space $X(\mathcal{R})$, may, for instance, be naturally called *reflexive* if each member of $\mathcal{R}$ is reflexive. Thus, we may also naturally speak of *preorder, tolerance*, and *equivalence relators*.

For instance, for a family $\mathcal{A}$ of subsets of X, the family $\mathcal{R}_{\mathcal{A}} = \{ R_A : \ A \in \mathcal{A} \}$, where $R_A = A^2 \cup A^c \times X$, is a preorder relator on X. Such relators were first used by Davis [3] and Pervin [21].

While, for a family $\mathcal{D}$ of pseudo-metrics on X, the family $\mathcal{R}_{\mathcal{D}} = \{ B_r^d : \ r > 0, d \in \mathcal{D} \}$, where $B_r^d = \{ (x, y) \in X^2 : \ d(x, y) < r \}$, is a tolerance relator on X. Such relators were first considered by Weil [63].

Moreover, if $\mathfrak{S}$ is a family of partitions of X, then the family $\mathcal{R}_{\mathfrak{S}} = \{ S_{\mathcal{A}} : \ \mathcal{A} \in \mathfrak{S} \}$, where $S_{\mathcal{A}} = \bigcup_{A \in \mathcal{A}} A^2$, is an equivalence relator on X. Such practically important relators were first studied by Levine [12].

2. Structures derived from relators

If $\mathcal{R}$ is a relator on X to Y, then for any $A \subseteq X$, $B \subseteq Y$ and $x \in X$ we write:

(1) $A \in \operatorname{Int}_{\mathcal{R}}(B)$ if $R[A] \subseteq B$ for some $R \in \mathcal{R}$,

(2) $A \in \operatorname{Cl}_{\mathcal{R}}(B)$ if $R[A] \cap B \neq \emptyset$ for all $R \in \mathcal{R}$,

(3) $x \in \operatorname{int}_{\mathcal{R}}(B)$ if $\{x\} \in \operatorname{Int}_{\mathcal{R}}(B)$, (4) $x \in \operatorname{cl}_{\mathcal{R}}(B)$ if $\{x\} \in \operatorname{Cl}_{\mathcal{R}}(B)$,

(5) $B \in \mathcal{E}_{\mathcal{R}}$ if $\operatorname{int}_{\mathcal{R}}(B) \neq \emptyset$, (6) $B \in \mathcal{D}_{\mathcal{R}}$ if $\operatorname{cl}_{\mathcal{R}}(B) = X$.

Moreover, if in particular $\mathcal{R}$ is a relator on X, then for any $A \subseteq X$ we also write:

(7) $A \in \tau_{\mathcal{R}}$ if $A \in \operatorname{Int}_{\mathcal{R}}(A)$, (8) $A \in \tau\!\!\!\!-_{\mathcal{R}}$ if $A^c \notin \operatorname{Cl}_{\mathcal{R}}(A)$,

(9) $A \in \mathcal{T}_{\mathcal{R}}$ if $A \subseteq \operatorname{int}_{\mathcal{R}}(A)$, (10) $A \in \mathcal{F}_{\mathcal{R}}$ if $\operatorname{cl}_{\mathcal{R}}(A) \subseteq A$.

The relations $\operatorname{Int}_{\mathcal{R}}$ and $\operatorname{int}_{\mathcal{R}}$ are called *the proximal and topological interiors* generated by $\mathcal{R}$, respectively. While, the members of the families, $\tau_{\mathcal{R}}$, $\mathcal{T}_{\mathcal{R}}$ and $\mathcal{E}_{\mathcal{R}}$ are called the *proximally open, topologically open, and fat subsets* of the relator spaces $X(\mathcal{R})$ and $(X, Y)(\mathcal{R})$, respectively.

The origins of the relations Cl_R and $\operatorname{Int}_{\mathcal{R}}$ go back to Efremović's proximity δ [4] and Smirnov's strong inclusion $\Subset$ [23], respectively. The families $\tau_{\mathcal{R}}$ and $\mathcal{E}_{\mathcal{R}}$ were first explicitly used by the first author [31]. In particular, the practical notation $\tau\!\!\!\!-_{\mathcal{R}}$ has been suggested by János Kurdics.

Because of the above definitions, for any relator $\mathcal{R}$ on X to Y and $B \subseteq Y$, we have

$$\operatorname{Cl}_{\mathcal{R}}(B) = \mathcal{P}(X) \setminus \operatorname{Int}_{\mathcal{R}}\big(B^c\big) \qquad \text{and} \qquad \operatorname{cl}_{\mathcal{R}}(B) = X \setminus \operatorname{int}_{\mathcal{R}}\big(B^c\big),$$

and

$$\mathcal{D}_{\mathcal{R}} = \big\{ D \subseteq Y : \quad D^c \notin \mathcal{E}_{\mathcal{R}} \big\} = \big\{ D \subseteq Y : \quad \forall \ E \in \mathcal{E}_{\mathcal{R}} : \quad E \cap D \neq \emptyset \big\}.$$

Moreover, if in particular, $\mathcal{R}$ is a relator on X, then we also have

$$\tau\!\!\!\!-_{\mathcal{R}} = \big\{ A \subseteq X : \quad A^c \in \tau_{\mathcal{R}} \big\} \qquad \text{and} \qquad \mathcal{F}_{\mathcal{R}} = \big\{ A \subseteq X : \quad A^c \in \mathcal{T}_{\mathcal{R}} \big\}.$$

In this respect, it is also worth mentioning that, for any relator $\mathcal{R}$ on X to Y we have

$$\operatorname{Cl}_{\mathcal{R}^{-1}} = \operatorname{Cl}_{\mathcal{R}}^{-1} \qquad \text{and} \qquad \operatorname{Int}_{\mathcal{R}^{-1}} = \mathcal{C}_Y \circ \operatorname{Int}_{\mathcal{R}}^{-1} \circ \mathcal{C}_X,$$

where $\mathcal{C}_X(A) = X \setminus A$ for all $A \subseteq X$. Moreover, in particular, for any relator $\mathcal{R}$ on X, we have $\overline{\tau}_{R} = \tau_{R^{-1}}$. Therefore, the proximal closures and proximally open sets are usually more convenient tools than the topological closures (proximal interiors) and topologically open sets, respectively.

The fat sets are frequently also more convenient tools than the topologically open sets [29]. For instance, if $\leq$ is a certain order relation on X, then $\mathcal{T}_{\leq}$ and $\mathcal{E}_{\leq}$ are just the families of all *ascending and residual subsets* of the ordered set $X(\leq)$, respectively.

To clarify the advantage of fat sets over the open ones, we can also note that if in particular $X = \mathbb{R}$, and R is a relation on X such that

$$R(x) = \{ x-1 \} \cup [x , +\infty [$$

for all $x \in X$, then $\mathcal{T}_R = \{ \emptyset, X \}$, but $\mathcal{E}_R$ is quite large family. Namely, the supersets of each $R(x)$, with $x \in X$, are also in $\mathcal{E}_R$.

If $\mathcal{R}$ is a relator on X to Y, and Φ and Ψ are relations on a relator space $\Gamma(\mathcal{U})$ to X and Y, respectively, then by using the relation $(\Phi \otimes \Psi)$, defined such that

$$(\Phi \otimes \Psi)(\gamma) = \Phi(\gamma) \times \Psi(\gamma)$$

for all $\gamma \in \Gamma$, we may also define

(11) $\Phi \in \operatorname{Lim}_{\mathcal{R}}(\Psi)$ if $(\Phi \otimes \Psi)^{-1}[R] \in \mathcal{E}_{\mathcal{U}}$ for all $R \in \mathcal{R}$,

(12) $\Phi \in \operatorname{Adh}_{\mathcal{R}}(\Psi)$ if $(\Phi \otimes \Psi)^{-1}[R] \in \mathcal{D}_{\mathcal{U}}$ for all $R \in \mathcal{R}$.

Now, for any $A \subseteq X$, we may also naturally write:

(13) $A \in \operatorname{lim}_{\mathcal{R}}(\Psi)$ if $A_\Gamma \in \operatorname{Lim}_{\mathcal{R}}(\Psi)$, (14) $A \in \operatorname{adh}_{\mathcal{R}}(\Psi)$ if $A_\Gamma \in \operatorname{Adh}_{\mathcal{R}}(\Psi)$,

where A_Γ is a relation on Γ to X such that $A_\Gamma(\gamma) = A$ for all $\gamma \in \Gamma$.

The *big limit relation* $\operatorname{Lim}_{\mathcal{R}}$, suggested by Efremović and Švarc [5], is, in general, a much stronger tool in the relator space $(X, Y)(\mathcal{R})$ than the *big closure and interior relations* $\operatorname{Cl}_{\mathcal{R}}$ and $\operatorname{Int}_{\mathcal{R}}$ suggested by Efremović [4] and Smirnov [23].

Namely, it can be shown that, for any $A \subseteq X$ and $B \subseteq Y$, we have $A \in \operatorname{Cl}_{\mathcal{R}}(B)$ if and only if there exist a preordered set $\Gamma(\leq)$ and functions φ and ψ of Γ to A and B, respectively, such that $\varphi \in \operatorname{Lim}_{\mathcal{R}}(\psi)$ $\big(\varphi \in \operatorname{Adh}_{\mathcal{R}}(\psi)\big)$.

Finally, we note that if $\mathcal{R}$ is a relator on X to Y, then according to [39] for any $A \subseteq X$, $B \subseteq Y$, $x \in X$, and $y \in Y$ we may also naturally write:

(a) $B \in \operatorname{Ub}_{\mathcal{R}}(A)$ and $A \in \operatorname{Lb}_{\mathcal{R}}(B)$ if $A \times B \subseteq R$ for some $R \in \mathcal{R}$,

(b) $y \in \operatorname{ub}_{\mathcal{R}}(A)$ if $\{y\} \in \operatorname{Ub}_{\mathcal{R}}(B)$, (c) $x \in \operatorname{lb}_{\mathcal{R}}(B)$ if $\{x\} \in \operatorname{Lb}_{\mathcal{R}}(A)$,

(d) $A \in \mathcal{U}_{\mathcal{R}}$ if $\operatorname{ub}_{\mathcal{R}}(A) \neq \emptyset$, (e) $B \in \mathcal{L}_{\mathcal{R}}$ if $\operatorname{lb}_{\mathcal{R}}(B) \neq \emptyset$.

Moreover, in particular $\mathcal{R}$ is a relator on X, then for any $A \subseteq X$ we may also naturally define:

(f) $\max_{\mathcal{R}}(A) = A \cap \operatorname{ub}_{\mathcal{R}}(A)$, (g) $\min_{\mathcal{R}}(A) = A \cap \operatorname{lb}_{\mathcal{R}}(A)$,

(h) $\operatorname{Max}_{\mathcal{R}}(A) = \mathcal{P}(A) \cap \operatorname{Ub}_{\mathcal{R}}(A)$, (i) $\operatorname{Min}_{\mathcal{R}}(A) = \mathcal{P}(A) \cap \operatorname{Lb}_{\mathcal{R}}(A)$,

and thus also

(j) $\sup_{\mathcal{R}}(A) = \min_{\mathcal{R}}\big(\operatorname{ub}_{\mathcal{R}}(A)\big)$, (k) $\inf_{\mathcal{R}}(A) = \max_{\mathcal{R}}\big(\operatorname{lb}_{\mathcal{R}}(A)\big)$.

(l) $\operatorname{Sup}_{\mathcal{R}}(A)=\operatorname{Min}_{\mathcal{R}}\big[\operatorname{Ub}_{\mathcal{R}}(A)\big]$, (m) $\operatorname{Inf}_{\mathcal{R}}(A)=\operatorname{Max}_{\mathcal{R}}\big[\operatorname{Lb}_{\mathcal{R}}(A)\big]$.

Now, analogously to the families $\tau_{\mathcal{R}}$ and $\mathcal{T}_{\mathcal{R}}$, we may also naturally define:

(n) $A\in\mathfrak{u}_{\mathcal{R}}$ if $A\in\operatorname{Ub}_{\mathcal{R}}(A)$,

(o) $A\in\mathfrak{U}_{\mathcal{R}}$ if $A\subseteq\operatorname{ub}_{\mathcal{R}}(A)$, (p) $A\in\mathfrak{L}_{\mathcal{R}}$ if $A\subseteq\operatorname{lb}_{\mathcal{R}}(A)$.

Thus, for instance, it can be shown that

$$A\in\mathfrak{u}_{\mathcal{R}}\iff A\in\operatorname{Lb}_{\mathcal{R}}(A)\iff A\in\operatorname{Min}_{\mathcal{R}}(A)\iff A\in\operatorname{Inf}_{\mathcal{R}}(A),$$

and $\mathfrak{u}_{\mathcal{R}}=\operatorname{Min}_{\mathcal{R}}\big[\mathcal{P}(X)\big]=\operatorname{Max}_{\mathcal{R}}\big[\mathcal{P}(X)\big]$. Moreover, $\operatorname{Lb}_{\mathcal{R}}=\operatorname{Ub}_{\mathcal{R}^{-1}}=\operatorname{Ub}_{\mathcal{R}}^{-1}$.

However, the above algebraic structures are not independent of the former topological ones. Namely, if R is a relation on X to Y, then for any $A\subseteq X$ and $B\subseteq Y$ we have

$$\begin{aligned}A\times B\subseteq R\iff\forall\,a\in A:\ B\subseteq R(a)\iff\forall\,a\in A:\ R(a)^{c}\subseteq B^{c}\\ \iff\forall\,a\in A:\ R^{c}(a)\subseteq B^{c}\iff R^{c}[A]\subseteq B^{c}.\end{aligned}$$

Therefore, if $\mathcal{R}$ is a relator on X to Y, then by the corresponding definitions, for any $A\subseteq X$ and $B\subseteq Y$, we also have

$$A\in\operatorname{Lb}_{\mathcal{R}}(B)\iff A\in\operatorname{Int}_{\mathcal{R}^{c}}(B^{c})\iff A\in\big(\operatorname{Int}_{\mathcal{R}^{c}}\circ\mathcal{C}_{Y}\big)(B).$$

Hence, we can already infer that

$$\operatorname{Lb}_{\mathcal{R}}=\operatorname{Int}_{\mathcal{R}^{c}}\circ\mathcal{C}_{Y},\qquad\text{and}\qquad\operatorname{Int}_{\mathcal{R}}=\operatorname{Lb}_{\mathcal{R}^{c}}\circ\mathcal{C}_{Y}.$$

Therefore, in contrast to a common belief, some algebraic and topological structures are just as closely related to each other by the above equalities, and their particular cases

$$\operatorname{lb}_{\mathcal{R}}=\operatorname{int}_{\mathcal{R}^{c}}\circ\mathcal{C}_{Y},\qquad\text{and}\qquad\operatorname{int}_{\mathcal{R}}=\operatorname{lb}_{\mathcal{R}^{c}}\circ\mathcal{C}_{Y},$$

as the exponential and the trigonometric functions are by the celebrated Euler formulas [24, p. 227].

Now, a function $\mathfrak{F}$ of the class of all relator spaces to some other class may be called a *structure for relators* if, for any relator $\mathcal{R}$ on X to Y, the value $\mathfrak{F}_{\mathcal{R}}=\mathfrak{F}_{\mathcal{R}}^{XY}=\mathfrak{F}\big((X,Y)(\mathcal{R})\big)$ is in a power set depending only on X and Y.

3. Some important operations for relators

In particular, a function $\square$ of the class of all relator spaces to the class of all relators may be called a *direct unary operation for relators* if, for any relator $\mathcal{R}$ on X to Y, the value $\mathcal{R}^{\square}=\mathcal{R}^{\square_{XY}}=\square\big((X,Y)(\mathcal{R})\big)$ is also a relator on X to Y.

An arbitrary unary operation $\square$ for relators is called *extensive, intensive, involutive and idempotent* if for any relator $\mathcal{R}$ on X to Y we have $\mathcal{R}\subseteq\mathcal{R}^{\square}$, $\mathcal{R}^{\square}\subseteq\mathcal{R}$, $\mathcal{R}^{\square\square}=\mathcal{R}$, and $\mathcal{R}^{\square\square}=\mathcal{R}^{\square}$, respectively.

In particular, an increasing idempotent operation for relators is called a *modification (or projection) operation.* While, an extensive (intensive) modification operation for relators is called a *closure (interior) operation.*

Moreover, an increasing extensive (intensive) operation is called a *preclosore (preinterior) operation.* And, an extensive (intensive) idempotent operation is called a *semiclosure (semiinterior)* operation.

For instance, the functions c and -1, defined by

$$\mathcal{R}^{c} = \big\{ R^{c} : \quad R \in \mathcal{R} \big\} \qquad \text{and} \qquad \mathcal{R}^{-1} = \big\{ R^{-1} : \quad R \in \mathcal{R} \big\}$$

for any relator $\mathcal{R}$ on X to Y, are increasing involution operations for relators such that $\big(\mathcal{R}^{c}\big)^{-1} = \big(\mathcal{R}^{-1}\big)^{c}$. Thus, the operation c is *inversion compatible.*

And, the functions ∞ and ∂, defined by

$$\mathcal{R}^{\infty} = \big\{ R^{\infty} : \quad R \in \mathcal{R} \big\} \qquad \text{and} \qquad \mathcal{R}^{\partial} = \big\{ S \subseteq X^{2} : \quad S^{\infty} \in \mathcal{R} \big\}$$

for any relator $\mathcal{R}$ on X, are modification operations for relators such that, for any two relators $\mathcal{R}$ and $\mathcal{S}$ on X, we have

$$\mathcal{R}^{\infty} \subseteq \mathcal{S} \quad \Longleftrightarrow \quad \mathcal{R} \subseteq \mathcal{S}^{\partial}.$$

Therefore, the operations ∞ and ∂ form a Galois connection [2, p. 155]. Thus, in particular $\infty\,\partial$ is a closure operation for relators such that $\infty = \infty\,\partial\,\infty$.

To investigate inclusions between generalized topologies derived from relations and relators, the operations ∞ and ∂ were first introduced by Mala [14] and Pataki [19], respectively. Moreover, by using several more powerful structures derived from relators, Száz [33] and Pataki [19] defined a great abundance of important closure (refinement) operations for relators. Some of them were already considered by Kenyon [10] and H. Nakano and K. Nakano [17].

For instance, for any relator $\mathcal{R}$ on X to Y, the relators

$$\begin{aligned}
\mathcal{R}^{*} &= \big\{ S \subseteq X \times Y : \quad \exists\ R \in \mathcal{R} : \quad R \subseteq S \big\}, \\
\mathcal{R}^{\#} &= \big\{ S \subseteq X \times Y : \quad \forall\ A \subseteq X : \quad \exists\ R \in \mathcal{R} : \quad R[A] \subseteq S[A] \big\}, \\
\mathcal{R}^{\wedge} &= \big\{ S \subseteq X \times Y : \quad \forall\ x \in X : \quad \exists\ R \in \mathcal{R} : \quad R(x) \subseteq S(x) \big\},
\end{aligned}$$

and

$$\mathcal{R}^{\vartriangle} = \big\{ S \subseteq X \times Y : \quad \forall\ x \in X : \quad \exists\ u \in X : \quad \exists\ R \in \mathcal{R} : \quad R(u) \subseteq S(x) \big\}$$

are called the *uniform, proximal, topological, and paratopological closures (refinements)* of the relator $\mathcal{R}$, respectively.

Thus, we evidently have $\mathcal{R} \subseteq \mathcal{R}^{*} \subseteq \mathcal{R}^{\#} \subseteq \mathcal{R}^{\wedge} \subseteq \mathcal{R}^{\vartriangle}$ for any relator R on X to Y. Moreover, if in particular $\mathcal{R}$ is a relator on X, then we can easily prove that $\mathcal{R}^{\infty} \subseteq \mathcal{R}^{*\infty} \subseteq \mathcal{R}^{\infty *} \subseteq \mathcal{R}^{*}$.

However, it is now more important to note that, because of the corresponding definitions of Section 2, we also have

$$\begin{aligned}
\mathcal{R}^{\#} &= \big\{ S \subseteq X \times Y : \quad \forall\ A \subseteq X : \quad A \in \operatorname{Int}_{R}\big(S[A] \big) \big\}, \\
\mathcal{R}^{\wedge} &= \big\{ S \subseteq X \times Y : \quad \forall\ x \in X : \quad x \in \operatorname{int}_{R}\big(S(x) \big) \big\}, \\
\mathcal{R}^{\vartriangle} &= \big\{ S \subseteq X \times Y : \quad \forall\ x \in X : \quad S(x) \in \mathcal{E}_{\mathcal{R}} \big\}.
\end{aligned}$$

Morover, by using a Pataki connections [19, 55], we can, for instance, prove the following theorems and their corollaries.

Theorem 3.1. $\#$, $\wedge$ *and* $\vartriangle$ *are closure operations for relators such that, for any two relators* $\mathcal{R}$ *and* $\mathcal{S}$ *on* X *to* Y, *we have*

(1) $\mathcal{S} \subseteq \mathcal{R}^{\#} \iff \mathcal{S}^{\#} \subseteq \mathcal{R}^{\#} \iff \operatorname{Int}_{\mathcal{S}} \subseteq \operatorname{Int}_{\mathcal{R}} \iff \operatorname{Cl}_{\mathcal{R}} \subseteq \operatorname{Cl}_{\mathcal{S}}$,

(2) $\mathcal{S} \subseteq \mathcal{R}^{\wedge} \iff \mathcal{S}^{\wedge} \subseteq \mathcal{R}^{\wedge} \iff \operatorname{int}_{\mathcal{S}} \subseteq \operatorname{int}_{\mathcal{R}} \iff \operatorname{cl}_{\mathcal{R}} \subseteq \operatorname{cl}_{\mathcal{S}}$,

(3) $\mathcal{S} \subseteq \mathcal{R}^{\Delta} \iff \mathcal{S}^{\Delta} \subseteq \mathcal{R}^{\Delta} \iff \mathcal{E}_{\mathcal{S}} \subseteq \mathcal{E}_{\mathcal{R}} \iff \mathcal{D}_{\mathcal{R}} \subseteq \mathcal{D}_{\mathcal{S}}$.

Corollary 3.2. *For any relator* $\mathcal{R}$ *on* X *to* Y *,*

(1) $\mathcal{S} = \mathcal{R}^{\#}$ *is the largest relator on* X *to* Y *such that* $\operatorname{Int}_{\mathcal{S}} = \operatorname{Int}_{\mathcal{R}}$ *, or equivalently* $\operatorname{Cl}_{\mathcal{S}} = \operatorname{Cl}_{\mathcal{R}}$ *;*

(2) $\mathcal{S} = \mathcal{R}^{\wedge}$ *is the largest relator on* X *to* Y *such that* $\operatorname{int}_{\mathcal{S}} = \operatorname{int}_{\mathcal{R}}$ *, or equivalently* $\operatorname{cl}_{\mathcal{S}} = \operatorname{cl}_{\mathcal{R}}$ *;*

(3) $\mathcal{S} = \mathcal{R}^{\Delta}$ *is the largest relator on* X *to* Y *such that* $\mathcal{E}_{\mathcal{S}} = \mathcal{E}_{\mathcal{R}}$ *, or equivalently* $\mathcal{D}_{\mathcal{S}} = \mathcal{D}_{\mathcal{R}}$ *.*

Theorem 3.3. $\#\partial$ *is a closure operation for relators such that for any two relators* $\mathcal{R}$ *and* $\mathcal{S}$ *on* X *, we have*

$$\mathcal{S} \subseteq \mathcal{R}^{\#\partial} \iff \mathcal{S}^{\#\partial} \subseteq \mathcal{R}^{\#\partial} \iff \tau_{\mathcal{S}} \subseteq \tau_{\mathcal{R}} \iff \bar{\tau}_{\mathcal{S}} \subseteq \bar{\tau}_{\mathcal{R}} .$$

Corollary 3.4. *For any relator* $\mathcal{R}$ *on* X *,* $\mathcal{S} = \mathcal{R}^{\#\partial}$ *is the largest relator on* X *such that* $\tau_{\mathcal{S}} = \tau_{\mathcal{R}}$ *or equivalently* $\bar{\tau}_{\mathcal{S}} = \bar{\tau}_{\mathcal{R}}$ *.*

Remark 3.5. $\wedge\partial$ is only a preclosure operation for relators. Moreover, if $\mathcal{R}$ is a relator on X, then in general there does not exist a largest relator $\mathcal{S}$ such that $\mathcal{T}_{\mathcal{S}} = \mathcal{T}_{\mathcal{R}}$. (See Mala [14, Example 5.3] and Pataki [19, Example 7.2] .)

In the light of this and other disadvantages of the structure $\mathcal{T}$, it is rather curious that most of the works in topology and analysis are based on open sets suggested by Tietze [62] and standardized by Bourbaki [1] and Kelley [9] .

Moreover, it also a striking fact that, despite the results of Pervin [21] , Fletcher and Lindgren [6] and Száz [45] , generalized topologies and minimal structures are still intensively investigated by a great number of mathematicians.

Concerning the structures $\mathcal{T}$ and $\mathcal{F}$, instead of an analogue of Theorem 3.3, we can only prove the following generalizations of the results of Mala and Száz [14, 16] .

Theorem 3.6. $\wedge\infty$ *is a modification operation for relators such that, for any two relators* $\mathcal{R}$ *and* $\mathcal{S}$ *on* X *, we have*

$$\mathcal{S}^{\wedge\infty} \subseteq \mathcal{R}^{\wedge} \iff \mathcal{S}^{\wedge\infty} \subseteq \mathcal{R}^{\wedge\infty} \iff \mathcal{T}_{\mathcal{S}} \subseteq \mathcal{T}_{\mathcal{R}} \iff \mathcal{F}_{\mathcal{S}} \subseteq \mathcal{F}_{\mathcal{R}} .$$

Corollary 3.7. *For any relator* $\mathcal{R}$ *on* X *,* $\mathcal{S} = \mathcal{R}^{\wedge\infty}$ *is the largest preorder relator on* X *such that* $\mathcal{T}_{\mathcal{S}} = \mathcal{T}_{\mathcal{R}}$ *or equivalently* $\mathcal{F}_{\mathcal{S}} \subseteq \mathcal{F}_{\mathcal{R}}$ *.*

Remark 3.8. Quite similar theorems can be proved concerning the modification operations $\#\infty$ and $\infty\#$.

Their advantage over the closure operation $\#\partial$ lies mainly in the fact that, in contrast to the letter one, they are still *stable* in the sense that they leave the relator $\{X^2\}$ fixed for any set X.

Finally, we note that, by using the notations $\circledast = c\#c$ and $\circland = c \wedge c$, we can also prove the following partial analogues of Theorem 3.1 and its corollary.

Theorem 3.9. $\circledast$ *and* $\circland$ *are closure operations for relators such that, for any two relators* $\mathcal{R}$ *and* $\mathcal{S}$ *on* X *to* Y *, we have*

(1) $\mathcal{S} \subseteq \mathcal{R}^{\textcircled{\#}} \iff \mathcal{S}^{\textcircled{\#}} \subseteq \mathcal{R}^{\textcircled{\#}} \iff \operatorname{Lb}_{\mathcal{S}} \subseteq \operatorname{Lb}_{\mathcal{R}}$,

(2) $\mathcal{S} \subseteq \mathcal{R}^{\textcircled{\wedge}} \iff \mathcal{S}^{\textcircled{\wedge}} \subseteq \mathcal{R}^{\textcircled{\wedge}} \iff \operatorname{lb}_{\mathcal{S}} \subseteq \operatorname{lb}_{\mathcal{R}}$.

Corollary 3.10. *For any relator $\mathcal{R}$ on X to Y,*

(1) $\mathcal{S} = \mathcal{R}^{\textcircled{\#}}$ *is the largest relator on X to Y such that* $\operatorname{Lb}_{\mathcal{S}} = \operatorname{Lb}_{\mathcal{R}}$ *;*

(2) $\mathcal{S} = \mathcal{R}^{\textcircled{\wedge}}$ *is the largest relator on X to Y such that* $\operatorname{lb}_{\mathcal{S}} = \operatorname{lb}_{\mathcal{R}}$ *.*

If $\square$ is an unary operation for relators, then a relator $\mathcal{R}$ on X on to Y is called $\square$*–fine* if $\mathcal{R}^{\square} = \mathcal{R}$. Moreover, two relators $\mathcal{R}$ and $\mathcal{S}$ on X to Y are called $\square$*–equivalent* if $\mathcal{R}^{\square} = \mathcal{S}^{\square}$.

In particular a relator $\mathcal{R}$ on X to Y is called $\square$*–simple* if it is $\square$–equivalent to a simple relator $\{R\}$ on X to Y. Moreover, for instance, a relator R on X is called $\square$*–well-chained* if $\mathcal{R}^{\square\infty} = \{X^2\}^{\square\infty}$.

Beside the above unary operations, we may also naturally introduce some useful binary operations for relators. For instance, for any two relators $\mathcal{R}$ on X to Y and $\mathcal{S}$ on Y to Z, we may naturally define

$$\mathcal{S} \circ \mathcal{R} = \big\{ S \circ R : \quad R \in \mathcal{R}, \quad S \in \mathcal{S} \big\}.$$

Hence, by using that $\big(S \circ R \big)^{-1} = R^{-1} \circ S^{-1}$ for all $R \in \mathcal{R}$ and $S \in \mathcal{S}$, we can easily see that $\big(\mathcal{S} \circ \mathcal{R} \big)^{-1} = \mathcal{R}^{-1} \circ \mathcal{S}^{-1}$. Moreover, it can also be easily seen that the composition of relators is also associative.

Thus, a unary operation $\square$ for relators, may be called *left (right) composition compatible* [49] if

$$\big(\mathcal{S} \circ \mathcal{R} \big)^{\square} = \Big(\mathcal{S} \circ \mathcal{R}^{\square} \Big)^{\square} \qquad \bigg(\big(\mathcal{S} \circ \mathcal{R} \big)^{\square} = \Big(\mathcal{S}^{\square} \circ \mathcal{R} \Big)^{\square} \bigg)$$

for any two relators $\mathcal{R}$ on X to Y and $\mathcal{S}$ on Y to Z.

Unfortunately, the operations $\wedge$ and $\triangle$ are only left composition compatible [55]. Moreover, they are not inversions compatible [30]. Therefore, we shall also need the notations $\vee = \wedge - 1$ and $\triangledown = \triangle - 1$.

4. Four basic continuity properties for pairs of relators

To motivate our forthcoming unifying definition for continuity properties, we shall start with some simple observations on increasing functions [57, 58].

For these, the reader may recall that a goset (generalized ordered sets) $X(\leq)$ is, by definition, a simply relator space of the form $X(R)$ with $\leq$ in place of R.

Definition 4.1. A function f of a simple relator space $X(R)$ to another $Y(S)$ is called *increasing* if for any $u, v \in X$

$$u\,R\,v \quad \Longrightarrow \quad f(u)\,S\,f(v).$$

Remark 4.2. Since $u\,R\,v$ is only another notation for $(u, v) \in R$, it is clear the following assertions are equivalent:

(1) f is increasing;

(2) $v \in R(u) \implies f(v) \in S\big(f(u)\big)$;

(3) $(u, v) \in R \implies \big(f(u), f(v)\big) \in S$

Moreover, concerning increasing increasing functions, we can also easily prove the following

Theorem 4.3. *For a function f of a simple relator space $X(R)$ to another $Y(S)$, the following assertions are equivalent:*

(1) f *is increasing;*

(2) $f \circ R \subseteq S \circ f$, *(3)* $R \subseteq f^{-1} \circ S \circ f$;

(4) $f \circ R \circ f^{-1} \subseteq S$, *(5)* $R \circ f^{-1} \subseteq f^{-1} \circ S$.

Proof. By the corresponding definitions, it is clear that, for any $u \in X$, the following assertions are equivalent:

$$(u, v) \in R \implies \big(f(u),\, f(v)\big) \in S\,;$$
$$v \in R(u) \implies f(v) \in S\big(f(u)\big)\,;$$
$$f\,[\,R(u)\,] \subseteq S\big(f(u)\big)\,;$$
$$(f \circ R)(u) \subseteq (S \circ f)(u)\,.$$

Therefore, by Remark 4.2, assertions (1) and (2) are also equivalent.

The proofs of the remaining equivalences depend on the increasingness and associativity of composition, and the inclusions

$$\Delta_X \subseteq f^{-1} \circ f \qquad \text{and} \qquad f \circ f^{-1} \subseteq \Delta_Y\,,$$

where Δ_X and Δ_Y are the identity functions of X and Y, respectively.

Remark 4.4. The latter inclusions indicate that assertions (2)–(5) need not be equivalent for an arbitrary relation f on $X(R)$ to $Y(S)$.

Therefore, they can be naturally used to define different increasingness properties of a relation f on $X(R)$ to $Y(S)$.

Remark 4.5. Having in mind set-valued functions, a relation F on a goset $X(\leq)$ to a set Y may be naturally called increasing if $u \leq v$ implies $F(u) \subseteq F(v)$ for all $u, v \in X$.

Thus, it can be easily shown that the relation F is increasing if and only if its inverse F^{-1} is *ascending-valued* in the sense that $F^{-1}(y)$ is an ascending subset of $X(\leq)$ for all $y \in Y$.

By using the more convenient notation $R = \leq$, the latter statement can be reformulated in the form that $R\,[\,F^{-1}(y)\,] \subseteq F^{-1}(y)$ for all $y \in Y$. That is, $R \circ F^{-1} \subseteq F^{-1}$, and thus $R \circ F^{-1} \subseteq F^{-1} \circ \Delta_Y$.

However, it is now more important to note that, by using our former operations on relators, Theorem 4.3 can be reformulated in the following instructive form.

Theorem 4.6. *If f is a function of one simple relator space $X(R)$ to another $Y(S)$, then under the notations*

$$\mathcal{F} = \{f\}\,, \qquad \mathcal{R} = \{R\} \qquad \textit{and} \qquad \mathcal{S} = \{S\}$$

the following assertions are equivalent:

(1) f *is increasing;*

(2) $\left(\mathcal{S}^* \circ \mathcal{F}^*\right)^* \subseteq \left(\mathcal{F}^* \circ \mathcal{R}^*\right)^*$, *(3)* $\left(\left(\mathcal{F}^*\right)^{-1} \circ \mathcal{S}^* \circ \mathcal{F}^*\right)^* \subseteq \mathcal{R}^{**}$,

(4) $\mathcal{S}^{**} \subseteq \Big(\mathcal{F}^* \circ \mathcal{R}^* \circ \big(\mathcal{F}^*\big)^{-1} \Big)^*$, *(5)* $\Big(\big(\mathcal{F}^*\big)^{-1} \circ \mathcal{S}^* \Big)^* \subseteq \Big(\mathcal{R}^* \circ \big(\mathcal{F}^*\big)^{-1} \Big)^*$.

Proof. The check the equivalences of the assertions (2)–(5) of this theorem to assertions (2)–(5) of Theorem 4.3 it is convenient to use that $*$ is an *inversion and composition compatible closure operation for relators.* Thus,

(a) $\big(\mathcal{R}^*\big)^{-1} = \big(\mathcal{R}^{-1}\big)^*$ for any relator $\mathcal{R}$ on X to Y;

(b) $\mathcal{R} \subseteq S^* \iff \mathcal{R}^* \subseteq \mathcal{S}^*$ for any relators $\mathcal{R}$ and $\mathcal{S}$ on X to Y;

(c) $\big(\mathcal{S} \circ \mathcal{R}\big)^* = \big(\mathcal{S}^* \circ \mathcal{R}^*\big)^*$ for any relators $\mathcal{R}$ on X to Y and $\mathcal{S}$ on Y to Z.

Remark 4.7. Note that in Remark 4.2 and Theorems 4.3 and 4.6, R and S may be thought of not only as certain order relations $\leq_X$ and $\leq_Y$, but also as some surroundings $B_\delta^{d_X}$ and $B_\varepsilon^{d_Y}$.

Therefore, instead of the term "increasing", we can equally well use the term "continous". Namely, if $R = B_\delta^{d_X}$ and $S = B_\varepsilon^{d_Y}$, then assertion (3) of Remark 4.2 means only that $d_X(u, v) < \delta$ implies $d_Y\big(f(u), f(v)\big) < \varepsilon$.

Now, by pexiderizing the inclusions (2)–(5) in Theorem 4.6, we may naturally introduce the following general definition whose origins go back to [25, 22, 38].

Definition 4.8. Let $(X, Y)(\mathcal{R})$ and $(Z, W)(\mathcal{S})$ be relator spaces. Moreover, let $\mathcal{F}$ be a relator on X to Z, and $\mathcal{G}$ be a relator on Y to W.

Then for a family $\square = (\square_i)_{i=1}^6$ of direct unary operations for relators, we say that the pair

(1) $(\mathcal{F}, \mathcal{G})$ is *upper* $\square$*–continuous* with respect to the relators $\mathcal{R}$ and $\mathcal{S}$ if

$$\Big(\mathcal{S}^{\square_1} \circ \mathcal{F}^{\square_2} \Big)^{\square_3} \subseteq \Big(\mathcal{G}^{\square_4} \circ \mathcal{R}^{\square_5} \Big)^{\square_6},$$

(2) $(\mathcal{F}, \mathcal{G})$ is *mildly* $\square$*–continuous* with respect to the relators $\mathcal{R}$ and $\mathcal{S}$ if

$$\bigg(\Big(\mathcal{G}^{\square_1} \Big)^{-1} \circ \mathcal{S}^{\square_2} \circ \mathcal{F}^{\square_3} \bigg)^{\square_4} \subseteq \mathcal{R}^{\square_5 \square_6},$$

(3) $(\mathcal{F}, \mathcal{G})$ is *vaguely* $\square$*–continuous* with respect to the relators $\mathcal{R}$ and $\mathcal{S}$ if

$$\mathcal{S}^{\square_1 \square_2} \subseteq \bigg(\mathcal{G}^{\square_3} \circ \mathcal{R}^{\square_4} \circ \Big(\mathcal{F}^{\square_5} \Big)^{-1} \bigg)^{\square_6},$$

(4) $(\mathcal{F}, \mathcal{G})$ is *lower* $\square$*–continuous* with respect to the relators $\mathcal{R}$ and $\mathcal{S}$ if

$$\bigg(\Big(\mathcal{G}^{\square_1} \Big)^{-1} \circ \mathcal{S}^{\square_2} \bigg)^{\square_3} \subseteq \bigg(\mathcal{R}^{\square_4} \circ \Big(\mathcal{F}^{\square_5} \Big)^{-1} \bigg)^{\square_6}.$$

Remark 4.9. To keep in mind the above assumptions, for any $R \in \mathcal{R}$, $S \in \mathcal{S}$, $F \in \mathcal{F}$ and $G \in \mathcal{G}$, one can use the diagram:

$$\begin{array}{ccc} X & \xrightarrow{F} & Z \\ {\scriptstyle R}\big\downarrow & & \big\downarrow{\scriptstyle S} \\ Y & \xrightarrow{G} & W \end{array}$$

Remark 4.10. Now, for any $F \in \mathcal{F}$ and $G \in \mathcal{G}$, the pair (F, G) may, for instance, be naturally called upper $\Box$–continuous, if the pair $\big(\{F\}, \{G\}\big)$ is upper $\Box$–continuous. That is,

$$\Big(\mathcal{S}^{\Box_1} \circ \{F\}^{\Box_2}\Big)^{\Box_3} \subseteq \Big(\{G\}^{\Box_4} \circ \mathcal{R}^{\Box_5}\Big)^{\Box_6} .$$

Unfortunately, this condition may greatly differ from the more natural requirement that $\big(\mathcal{S}^{\Box_1} \circ F\big)^{\Box_3} \subseteq \big(G \circ \mathcal{R}^{\Box_5}\big)^{\Box_6}$ which should also be given an appropriate name.

In this respect, it is worth noticing that, for instance, we have

$$\{F\}^{\#} = \{F\}^{*} \quad \text{and} \quad \{F\}^{\wedge} = \{F\}^{*}, \quad \text{but} \quad \{F\}^{\triangle} = \big\{F \circ X^{X}\big\}^{*}$$

for all $F \in \mathcal{F}$.

Remark 4.11. Thus, the the pair (F, G) may, for instance, be naturally called selectionally upper $\Box$–continuous if for any selection functions f of F and g of G the pair (f, g) is upper $\Box$–continuous.

Moreover, the pair $(\mathcal{F}, \mathcal{G})$ may, for instance, be naturally called *elementwise upper* $\Box$*–continuous* if for any $F \in \mathcal{F}$ and $G \in \mathcal{G}$, the pair (F, G) is upper $\Box$–continuous. This may greatly differ from property (1).

Remark 4.12. If in particular $\Box$ is a direct unary operation for relators, then the pair $(\mathcal{F}, \mathcal{G})$ may, for instance, be also naturally called upper $\Box$–continuous if it is upper $(\Box)_{i=1}^{6}$–continuous. That is,

$$\Big(\mathcal{S}^{\Box} \circ \mathcal{F}^{\Box}\Big)^{\Box} \subseteq \Big(\mathcal{G}^{\Box} \circ \mathcal{R}^{\Box}\Big)^{\Box} .$$

Remark 4.13. Thus, the pair $(\mathcal{F}, \mathcal{G})$ may, for instance, be naturally called *properly upper continuous* if it is upper $\Box$–continuous with $\Box$ being the identity operation for relators. That is, $\mathcal{S} \circ \mathcal{F} \subseteq \mathcal{G} \circ \mathcal{R}$.

Moreover, the pair $(\mathcal{F}, \mathcal{G})$ may, for instance, be also naturally called *uniformly, proximally, topologically and paratopologically upper continuous* if it is $\Box$–continuous with $\Box = *, \#, \wedge$ and $\triangle$, respectively.

Thus, by using the operations $\Box\infty$ and $\Box\partial$ instead of $\Box$, we can quite similarly speak of the corresponding quasi-countinuity and pseudo-continuity properties of $(\mathcal{F}, \mathcal{G})$.

Remark 4.14. Finally, we note that if in particular $X = Y$ and $Z = W$, then the relator $\mathcal{F}$ and a relation $F \in \mathcal{F}$ may, for instance, be naturally called upper $\Box$–continuous if the pairs $(\mathcal{F}, \mathcal{F})$ and (F, F) are upper $\Box$–continuous, respectively.

5. Relationshipps with Galois and Pataki connections

By [46, 54], we may naturally introduce the following

Definition 5.1. Let $X(R)$ and $Y(S)$ be simple relator spaces. Moreover, let f be a function of X to Y and g be a function of Y to X.

Then, we say that f is *increasingly right* g*–normal* if for any $x \in X$ and $y \in Y$

$$f(x)\, S\, y \quad \Longrightarrow \quad x\, R\, g(y) .$$

Remark 5.2. In this case, we may also say that the functions f and g form an *increasing right Galois connection* between $X(R)$ and $Y(S)$.

Now, analogously to Theorem 4.3, we can only prove the following

Theorem 5.3. *Under the notations of Definition 5.1, the following assertions are equivalent:*

(1) f *is increasingly right* g*–normal;*

(2) $S \circ f \subseteq g^{-1} \circ R$*;* *(3)* $g \circ S \circ f \subseteq R$.

Proof. To prove equivalence of (1) and (2), note that, for any $x \in X$, the following assertions are equivalent:

$$f(x) S y \implies x R g(y),$$
$$y \in S\big(f(x)\big) \implies g(y) \in R(x),$$
$$y \in S\big(f(x)\big) \implies y \in g^{-1}[R(x)],$$
$$S\big(f(x)\big) \subseteq g^{-1}[R(x)]$$
$$\big(S \circ f\big)(x) \subseteq \big(g^{-1} \circ R\big)(x).$$

Remark 5.4. From this theorem, by using the operation $*$, we can easily derive an analogue of Theorem 4.6.

However, it is now more important to note that, by using Theorem 5.3 and the operation $\circledast = c * c$, we can also prove the following

Theorem 5.5. *Under the notations*

$$\mathcal{F} = \{f\}, \qquad \mathcal{G} = \{g\}, \qquad \mathcal{R} = \{R\} \qquad \text{and} \qquad \mathcal{S} = \{S\},$$

the following assertions are equivalent:

(1) f *is increasingly right* g*–normal;*

(2) $\big(\mathcal{S}^{\circledast} \circ \mathcal{F}^{\circledast}\big)^{\circledast} \subseteq \Big(\big(\mathcal{G}^{\circledast}\big)^{-1} \circ \mathcal{R}^{\circledast}\Big)^{\circledast}$*;* *(3)* $\big(\mathcal{G}^{\circledast} \circ \mathcal{S}^{\circledast} \circ \mathcal{F}^{\circledast}\big)^{\circledast} \subseteq \mathcal{R}^{\circledast\circledast}$.

Remark 5.6. To check this, note that for any relator $\mathcal{R}$ on X to Y and relation S on X to Y we have

$$S \in \mathcal{R}^{\circledast} \iff S \in \mathcal{R}^{c*c} \iff S^c \in \mathcal{R}^{c*} \iff \exists\ U \in \mathcal{R}^c: \ U \subseteq S^c$$
$$\iff \exists R \in \mathcal{R}: \ R^c \subseteq S^c \iff \exists R \in \mathcal{R}: \ S \subseteq R.$$

Therefore,

$$\mathcal{R}^{\circledast} = \big\{ S \subseteq X \times Y : \ \exists\ R \in \mathcal{R}: \ S \subseteq R \big\}.$$

Hence, it can be easily seen that $\circledast$ is also an inversion and composition compatible closure operation for relators. Moreover, we can also note that

$$\mathcal{R}^{\circledast} = \bigcup_{R \in \mathcal{R}} \mathcal{P}(R).$$

Now, analogously to Definition 4.8, we may also naturally introduce the following

Definition 5.7. Let $(X, Y)(\mathcal{R})$ and $(Z, W)(\mathcal{S})$ be relator spaces. Moreover, let $\mathcal{F}$ be a relator on X to Z, and $\mathcal{G}$ be a relator on W to Y.

Then for a family $\Box = (\Box_i)_{i=1}^{6}$ of direct unary operations for relators, we say that:

(1) $\mathcal{F}$ is *increasingly upper right* $\Box$–$\mathcal{G}$–*normal* with respect to the relators $\mathcal{R}$ and $\mathcal{S}$ if

$$\left(\mathcal{S}^{\Box_1} \circ \mathcal{F}^{\Box_2}\right)^{\Box_3} \subseteq \left(\left(\mathcal{G}^{\Box_4}\right)^{-1} \circ \mathcal{R}^{\Box_5}\right)^{\Box_6},$$

(2) $\mathcal{F}$ is *increasingly mildly right* $\Box$–$\mathcal{G}$–*normal* with respect to the relators $\mathcal{R}$ and $\mathcal{S}$ if

$$\left(\mathcal{G}^{\Box_1} \circ \mathcal{S}^{\Box_2} \circ \mathcal{F}^{\Box_3}\right)^{\Box_4} \subseteq \mathcal{R}^{\Box_5 \Box_6},$$

(3) $\mathcal{F}$ is *increasingly vaguely right* $\Box$–$\mathcal{G}$–*normal* with respect to the relators $\mathcal{R}$ and $\mathcal{S}$ if

$$\mathcal{S}^{\Box_1 \Box_2} \subseteq \left(\left(\mathcal{G}^{\Box_3}\right)^{-1} \circ \mathcal{R}^{\Box_4} \circ \left(\mathcal{F}^{\Box_5}\right)^{-1}\right)^{\Box_6},$$

(4) $\mathcal{F}$ is *increasingly lower right* $\Box$–$\mathcal{G}$–*normal* with respect to the relators $\mathcal{R}$ and $\mathcal{S}$ if

$$\left(\mathcal{G}^{\Box_1} \circ \mathcal{S}^{\Box_2}\right)^{\Box_3} \subseteq \left(\mathcal{R}^{\Box_4} \circ \left(\mathcal{F}^{\Box_5}\right)^{-1}\right)^{\Box_6}.$$

Thus, for instance, we can easily establish the following

Theorem 5.8. *If in particular the operation $\Box_4$ is inversion compatible, then the following assertions are equivalent:*

(1) $\left(\mathcal{F}, \mathcal{G}^{-1}\right)$ *is upper $\Box$–continuous;*

(2) $\mathcal{F}$ *is increasingly upper right $\Box$–$\mathcal{G}$–normal.*

By [46, 54], we may also naturally introduce the following

Definition 5.9. Let $X(R)$ and $Y(S)$ be simple relator spaces. Moreover, let f be a function of X to Y, and φ be a function of X to itself.

Then, we say that f is *increasingly right φ–regular* if for any $u, v \in X$

$$f(u)\, S\, f(v) \quad \Longrightarrow \quad u\, R\, \varphi(v).$$

Remark 5.10. In this case, we may also say that the functions f and φ form an *increasing right Pataki connection* between $X(R)$ and $Y(S)$.

To clarify the relationship between Definitions 5.1 and 5.9, we can easily prove the following two lemmas.

Lemma 5.11. *If f is increasingly right g–normal for some function g of Y to X and $\varphi = g \circ f$, then f is increasingly right φ–regular.*

Lemma 5.12. *If f is increasingly right φ–regular, $Y = f[X]$ and g is a function of Y to X such that $\varphi = g \circ f$, then f is increasingly right g–normal.*

Proof. Suppose that $x \in X$ and $y \in Y$. Then, since f is onto Y, there exists $v \in X$ such that $y = f(v)$.

Now, we can easily see that

$$f(x) S y \iff f(x) S f(v) \iff x R \varphi(v)$$
$$\iff x R (g \circ f)(v) \iff x R g\big(f(v)\big) \iff x R g(y).$$

Therefore, the required assertion is true.

Now, analogously to Theorem 5.3, we can only prove the following

Theorem 5.13. *Under the notations of Definition 5.9, the following assertions are equivalent :*

(1) f *is right* φ*–regular ;* *(2)* $f^{-1} \circ S \circ f \subset \varphi^{-1} \circ R$,

Proof. To check this, note that, for any $u, v \in X$, the following asertions are equivalent :

$$f(u) S f(v) \implies u R \varphi(v),$$
$$f(v) \in S\big(f(u)\big) \implies \varphi(v) \in R(u),$$
$$v \in f^{-1}\big[S\big(f(u)\big)\big] \implies v \in \varphi^{-1}[R(u)],$$
$$f^{-1}\big[S\big(f(u)\big)\big] \subseteq \varphi^{-1}[R(u)]$$
$$\big(f^{-1} \circ S \circ f\big)(u) \subseteq \big(\varphi^{-1} \circ R\big)(u).$$

Remark 5.14. From this theorem, by using the operation $*$, we can easily derive an analogue of Theorem 4.6.

However, again it is more important to note that, by using Theorem 5.13 and the operation $\circledast = c * c$, we can also prove the following

Theorem 5.15. *Under the notations*

$$\mathcal{F} = \{f\}, \qquad \Phi = \{\varphi\}, \qquad \mathcal{R} = \{R\} \qquad \textit{and} \qquad \mathcal{S} = \{S\},$$

the following assertions are equivalent :

(1) f *is increasingly right* φ*–regular ;*

(2) $\Big(\big(\mathcal{F}^{\circledast}\big)^{-1} \circ \mathcal{S}^{\circledast} \circ \mathcal{F}^{\circledast}\Big)^{\circledast} \subseteq \Big(\big(\Phi^{\circledast}\big)^{-1} \circ \mathcal{R}^{\circledast}\Big)^{\circledast}$.

Now, analogously to Definition 5.7, we can only introduce the following

Definition 5.16. Let $(X, Y)(\mathcal{R})$ and $Z(\mathcal{S})$ be relator spaces, Moreover, let $\mathcal{F}$ be a relator on X to Z, and Φ be a relator on X to Y.

Then, for a family $\Box = (\Box_i)_{i=1}^{7}$ of direct unary operations for relators, we say that $\mathcal{F}$ is *increasingly right* $\Box$*–*Φ*–regular* with respect to the relators $\mathcal{R}$ and $\mathcal{S}$ if

$$\Big(\big(\mathcal{F}^{\Box_1}\big)^{-1} \circ \mathcal{S}^{\Box_2} \circ \mathcal{F}^{\Box_3}\Big)^{\Box_4} \subseteq \Big(\big(\Phi^{\Box_5}\big)^{-1} \circ \mathcal{R}^{\Box_6}\Big)^{\Box_7}.$$

Thus, for instance, we can easily establish the following

Theorem 5.17. *If in particular* $\Box_5 = \Box_6 = \Box_7$ *is an inversion and composition compatible closure operation for relators and* $\Diamond = \big(\Box_i\big)_{i=1}^{6}$, *then the following assertions are equivalent :*

(1) $\mathcal{F}$ is mildly $\lozenge$–continuous with respect to the relators $\Phi \circ \mathcal{R}$ and $\mathcal{S}$;

(2) $\mathcal{F}$ is increasingly right $\square$–Φ–regular with respect to the relators $\mathcal{R}$ and $\mathcal{S}$.

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