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A MOST GENERAL SCHWARZ INEQUALITY FOR GENERALIZED SEMI-INNER PRODUCTS ON GROUPOIDS

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Abstract. By introducing an appropriate notion of generalized semi-inner products on groupoids, we shall prove a very general form of the famous Schwarz inequality.

In case of groups, this will be sufficient to prove the subadditivity of the induced generalized seminorms. Thus, some results on inner product spaces can be extended to inner product groups.

1. Introduction

Semi-inner products on groups were first introduced by the second author in [16] to prove a natural generalization of a basic theorem of Maksa and Volkmann [13] on additive functions without any particular tricks preferred by functional equationalists.

In [16], the second author claimed that even a weaker form of Schwarz inequality cannot be proved for semi-inner products on groups. Moreover, he asked several mathematicians in Debrecen and Cluj-Napoca, at a conference, to justify his statement by providing an appropriate example.

However, the first author in [3] could disprove this claim by proving a weak form of Schwarz inequality which is slightly more than that is sufficient to prove the subadditivity the induced generalized seminorms. Thus, some results on inner product spaces can be extended to inner product groups.

This weak Schwarz inequality has been greatly utilized and further generalized by the second author. It has turned out that a consequence of it can be proved even for semi-inner products on groupoids. The corresponding results have been presented in our technical reports [6] and [7].

Later, the second author has noticed that even the weak Schwarz inequality proved in [7] can still be further generalized by using the smallest denominator

$$n(r) = \min \left\{ n \in \mathbb{N} : \quad n\, r \in \mathbb{Z} \right\}$$

and the corresponding numerator $m(r) = n(r)\, r$ of a rational number r studied and applied in our former papers [4] and [5].

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For this, in the present paper, we shall suppose that X is a groupoid and P is a function of X^2 to $\mathbb{C}$ such that, under the notations

$$\Delta_P(x) = P(x, x) \qquad \text{and} \qquad P_1(x, y) = 2^{-1}\Big(P(x, y) + \overline{P(x, y)}\Big),$$

for any $n \in \mathbb{N}$ and $x, y \in X$ we have

(a) $\Delta_P(x) = \Delta_{P_1}(x)$,

(b) $P_1(nx, y) = n P_1(x, y) = P_1(x, ny)$,

(c) $\Delta_P(x+y) = \Delta_P(x) + \Delta_P(y) + 2 P_1(x, y)$.

By using this generalized semi-inner product P and the notations

$$\alpha_P(x, y) = \inf_{r \in Q_+} \big(r\, \Delta_P(x) + r^{-1} \Delta_P(y)\big)$$

and

$$\beta_P(x, y) = \inf_{r \in \mathbb{Q}_+} \frac{1}{m(r)\, n(r)}\, \Delta_P\big(m(r)\, x + n(r)\, y\big),$$

we shall show that

$$\beta_P(x, y) \leq \alpha_P(x, y) + 2 P_1(x, y)$$

for all $x, y \in X$.

Hence, by assuming that Δ_P is nonnegative and proving that

$$\alpha_P(x, y) = 2\sqrt{\Delta_P(x)\, \Delta_P(y)},$$

we can already derive the corresponding weak form

$$-P_1(x, y) \leq p(x)\, p(y)$$

of Schwarz inequality with $p(x) = \sqrt{\Delta_P(x)}$.

2. A generalized semi-inner product

Notation 2.1. Let X be a groupoid, and for any $n \in \mathbb{N}$ and $x \in X$ define $nx = x$ if $n = 1$ and $nx = (n-1)x + x$ if $n > 1$. Moreover, suppose that P is a function of X^2 to $\mathbb{C}$.

For any $x, y \in X$, define

$$\Delta_P(x) = P(x, x) \qquad \text{and} \qquad P_1(x, y) = 2^{-1}\Big(P(x, y) + \overline{P(x, y)}\Big).$$

Moreover, suppose that for any $n \in \mathbb{N}$ and $x, y \in X$ we have

(a) $\Delta_P(x) = \Delta_{P_1}(x)$,

(b) $P_1(nx, y) = n P_1(x, y) = P_1(x, ny)$,

(c) $\Delta_P(x+y) = \Delta_P(x) + \Delta_P(y) + 2 P_1(x, y)$.

Remark 2.2. Properties (a), (b) and (c) will be called the the *reality* of Δ_P, $\mathbb{N}$-*bihomogeneity* of P_1 and *polarization identity* for P, respectively.

The functions Δ_P and P_1 will be called the *diagonalization* and *real part* (*first coordinate function* of P, respectively. Moreover, the function P itself will be called a *generalized semi-inner product* on X.

If in particular the groupoid X has a zero element 0, then the generalized semi-inner product P will be called a *generalized inner product* if $\Delta_P(x) = 0$ implies $x = 0$ for all $x \in X$.

Some further useful additional properties of X and P will be assumed gradually as the forthcoming definitions and theorems need. For instance, we may naturally require X to be a group and P_1 to be $\mathbb{Z}$–bihomogeneous.

The following theorem clarifies our present generalization of the ordinary semi-inner products [15] which may be modified according to the ideas of Lumer [12], [11], Nath [14], Bognár [2], Antoine and Grossmann [1] and Drygas [9].

Theorem 2.3. *If Q is a conjugate-symmetric function of X^2 to $\mathbb{C}$ such that Q is additive in its first (second) variable, then Q is a generalized semi-inner product on X with several useful additional properties.*

Proof. By the conjugate-symmetry of Q, for any $x, y \in X$, we have

$$Q(y, x) = \overline{Q(x, y)}\,.$$

Hence, in particular, it clear that

$$Q(x, x) = \overline{Q(x, x)}\,, \qquad \text{and thus} \qquad Q(x, x) = Q_1(x, x)\,.$$

Therefore, property (a) holds for Q.

Moreover, we can also at once see that

$$Q_1(x, y) = 2^{-1}\big(Q(x, y) + \overline{Q(x, y)}\big) = 2^{-1}\big(Q(x, y) + Q(y, x)\big)$$

Thus, Q_1 is just the symmetric part of Q. Concerning the imaginary part (second coordinate function) Q_2 of Q, we can quite similarly see that

$$Q_2(x, y) = (i\,2)^{-1}\big(Q(x, y) - \overline{Q(x, y)}\big) = i^{-1}\,2^{-1}\big(Q(x, y) - Q(y, x)\big)\,.$$

for all $x, y \in X$. Therefore, $i\,Q_2$ is just the skew-symmetric part of Q.

Furthermore, by using the conjugate-symmetry and the additivity of Q in its first variable, we can easily see that

$$\begin{aligned} Q(x, y+z) &= \overline{Q(y+z, x)} = \overline{Q(y, x) + Q(z, x)} = \\ &= \overline{Q(y, x)} + \overline{Q(z, x)} = Q(x, y) + Q(x, z) \end{aligned}$$

for all $x, y, z \in X$. Therefore, Q is additive in its second variable too. That is, Q is actually biadditive.

Thus, for any $x, y \in X$, the first and second partial functions of Q, defined by

$$\varphi_Q(u) = Q(u, y) \qquad \text{and} \qquad \psi_Q(v) = Q(x, v)$$

for all $u, v \in X$, are additive functions of X to $\mathbb{C}$.

Hence, since an additive function of one groupoid to another can easily be seen to be $\mathbb{N}$–homogeneous, it is clear that φ_Q and ψ_Q are $\mathbb{N}$–homogeneous. Therefore, Q, and thus its coordinate functions Q_1 and Q_2 are, as well, $\mathbb{N}$–bihomogeneous. Thus, in particular property (b) also holds.

Moreover, by using the biadditivity of Q and our former equality on Q_1, we can also see that

$$\begin{aligned} &\Delta_Q(x+y) = Q(x+y,\ x+y) = Q(x,\ x+y) + Q(y,\ x+y) \\ &= Q(x, x) + Q(x, y) + Q(y, x) + Q(y, y) = \Delta_Q(x) + 2\,Q_1(x, y) + \Delta_Q(y) \end{aligned}$$

for all $x, y \in X$. Thus, property (c) also holds for Q.

Remark 2.4. Note that if in particular X is a group, then instead of the $\mathbb{N}$–bihomogeneity of Q we can also state the $\mathbb{Z}$–bihomogeneity of Q.

Therefore, if in particular X is a group, then instead of property (b) it is convenient to assume the $\mathbb{Z}$–bihomogenity of P_1.

Moreover, if P is not conjugate symmetric, then in Notation 2.1 it may be convenient to write the symmetric part of P instead of P_1.

The appropriateness of assumptions (a)–(c) is also apparent from the following

Theorem 2.5. *For any* $n, m \in \mathbb{N}$ *and* $x, y \in X$,

(1) $\Delta_P(n\,x) = n^2\,\Delta_P(x)$,

(2) $\Delta_P\big(n\,(x+y)\big) = \Delta_P(n\,x + n\,y)$,

(3) $\Delta_P(m\,x + n\,y) = m^2\Delta_P(x) + n^2\Delta_P(y) + 2\,m\,n\,P_1(x, y)$.

Proof. By (a) and (b), we have

$$\begin{aligned}\Delta_P(n\,x) = \Delta_{P_1}(n\,x) = P_1(n\,x,\, n\,x) = n\,P_1(x,\, n\,x)\\ = n^2\,P_1(x,\, x) = n^2\,\Delta_{P_1}(x) = n^2\,\Delta_P(x),\end{aligned}$$

and thus (1) is true.

To prove (2) and (3), note that by (1) and (c) we have

$$\Delta_P\big(n\,(x+y)\big) = n^2\,\Delta_P(x+y) = n^2\,\Delta_P(x) + n^2\,\Delta_P(y) + 2\,n^2\,P_1(x, y).$$

Moreover, by (c), (1) and (b), we also have

$$\begin{aligned}\Delta_P(m\,x + n\,y) = \Delta_P(m\,x) + \Delta_P(n\,x) + 2\,P_1(m\,x,\, n\,y)\\ = m^2\,\Delta_P(x) + n^2\,\Delta_P(y) + 2\,m\,n\,P_1(x, y),\end{aligned}$$

and thus in particular

$$\Delta_P(n\,x + n\,y) = n^2\,\Delta_P(x) + n^2\,\Delta_P(y) + 2\,n^2\,P_1(x, y).$$

Remark 2.6. If in particular P_1 is symmetric, then because of (3) we can also state that

$$\Delta_P(n\,y + m\,x) = \Delta_P(m\,x + n\,y).$$

Remark 2.7. While, if in particular X is a group and P_1 is $\mathbb{Z}$–bihomogeneous, then the above equalities can also be stated for all $n, m \in \mathbb{Z}$.

3. A generalized Schwarz inequality

Definition 3.1. For any $x, y \in X$, we define

$$\alpha_P(x, y) = \inf_{r \in \mathbb{Q}_+} \big(r\,\Delta_P(x) + r^{-1}\,\Delta_P(y)\big)$$

and

$$\beta_P(x, y) = \inf_{r \in \mathbb{Q}_+} \frac{1}{m(r)\,n(r)}\,\Delta_P\big(m(r)\,x + n(r)\,y\big),$$

where

$$m(r) = r\, n(r) \qquad \text{with} \qquad n(r) = \min\left\{ n \in \mathbb{N} : \quad n\, r \in \mathbb{Z} \right\}.$$

Remark 3.2. The natural numbers $n(r)$ and $m(r)$ are called the *smallest denominator* and the *associated numerator* of $r \in \mathbb{Q}_+$, respectively.

Several remarkable properties and important applications of their obvious extensions for $r \in \mathbb{Q}$ have been established in our former papers [4] and [5].

For instance, it has been proved that if $r = k/l$, for some $k \in \mathbb{Z}$ and $l \in \mathbb{N}$, then

$$m\big(k/l\big) = k/(k;\, l) \qquad \text{and} \qquad n\big(k/l\big) = l/(k;\, l),$$

where $(k;\, l)$ is the greatest common divisor of k and l. Thus, in particular $m(r)$ and $n(r)$ are relatively prime in the sense that $\big(m(r);\, n(r)\big) = 1$.

Remark 3.3. We shall soon see that the extended real number $\alpha_P(x,\, y)$ can be quite easily determined.

However, concerning $\beta_P(x,\, y)$, we can only note that if $\Delta_P(m\,x + n\,y) \geq 0$ for all $m,\, n \in \mathbb{N}$, with $(m;\, n) = 1$, then $\beta_P(x,\, y) \geq 0$.

Fortunately, this simple observation will already allow us to establish some applicable particular cases of the following generalized Schwarz inequality.

Theorem 3.4. *For any $x,\, y \in X$, we have*

$$\beta_P(x,\, y) \leq \alpha_P(x,\, y) + 2\, P_1(x,\, y).$$

Proof. By (3) in Theorem 2.5, for any $r \in \mathbb{Q}_+$, we have

$$\Delta_P(m(r)\, x + n(r)\, y) = m(r)^2 \Delta_P(x) + n(r)^2 \Delta_P(y) + 2\, m(r)\, n(r)\, P_1(x,\, y),$$

and thus

$$\frac{1}{m(r)\, n(r)}\, \Delta_P\big(m(r)\, x + n(r)\, y\big) = \frac{m(r)}{n(r)}\, \Delta_P(x) + \frac{n(r)}{m(r)}\, \Delta_P(y) + 2\, P_1(x,\, y)$$
$$= r\, \Delta_P(x) + r^{-1}\, \Delta_P(y) + 2\, P_1(x,\, y).$$

Hence, by the lower bound property of infimum, we can see that

$$\beta_P(x,\, y) \leq r\, \Delta_P(x) + r^{-1}\, \Delta_P(y) + 2\, P_1(x,\, y),$$

and thus

$$\beta_P(x,\, y) - 2\, P_1(x,\, y) \leq r\, \Delta_P(x) + r^{-1}\, \Delta_P(y)$$

even if $\beta_P(x,\, y) = -\infty$.

Hence, by the maximality property of infimum, we can see that

$$\beta_P(x,\, y) - 2\, P_1(x,\, y) \leq \inf_{r \in \mathbb{Q}_+} \big(r\, \Delta_P(x) + r^{-1}\, \Delta_P(y)\big) = \alpha_P(x,\, y),$$

and thus the required inequality is also true. (Namely, if $\beta_P(x,\, y) = -\infty$, then it trivially holds.)

From this theorem, we can immediately derive

Corollary 3.5. *If $x,\, y \in X$ such that $\beta_P(x,\, y)$ is finite, then $\alpha_P(x,\, y)$ is also finite and*

$$-P_1(x,\, y) \leq 2^{-1}\big(\alpha_P(x,\, y) - \beta_P(x,\, y)\big).$$

Hence, by Remark 3.3, it is clear that in particular we also have

Corollary 3.6. *If* $x, y \in X$ *such that* $\Delta_P(mx+ny) \geq 0$ *for all* $m, n \in \mathbb{N}$ *with* $(m; n) = 1$, *then*

$$-P_1(x, y) \leq 2^{-1} \alpha_P(x, y).$$

Now, by using this corollary, we can also easily prove the following

Theorem 3.7. *If* X *is a group,* Δ_P *is nonnegative and* P_1 *is biodd, then for any* $x, y \in X$ *we have*

$$|P_1(x, y)| \leq 2^{-1} \alpha_P(x, y).$$

Proof. By property (a) and the biodness of P_1, we have

$$\Delta_P(-x) = P_1(-x, -x) = P_1(x, x) = \Delta_P(x)$$

for all $x \in X$. Therefore, Δ_P is even, and thus α_P is bieven.

Moreover, by the nonnegativity of Δ_P and Corollary 3.6, we have

$$-P_1(x, y) \leq 2^{-1} \alpha_P(x, y)$$

for all $x, y \in X$.

Hence, by using the oddness of P_1 and the evenness of α_P in their fist arguments, we can already see that

$$P_1(x, y) = -P_1(-x, y) \leq 2^{-1} \alpha_P(-x, y) = 2^{-1} \alpha_P(x, y)$$

also holds for all $x, y \in X$. Therefore, by the definition of the absolute value, the required inequality is also true.

Remark 3.8. Now, to obtain some applicable particular cases of Theorem 3.7, it is enough to compute $\alpha_P(x, y)$ for all $x, y \in X$.

4. The determination of $\alpha_P(x, y)$

To compute the values of α_P, it is necessary to prove first the following

Lemma 4.1. *For any* $x, y \in X$, *we have*

$$\alpha_P(x, y) = \inf_{\lambda>0} \left(\lambda\,\Delta_P(x) + \lambda^{-1}\,\Delta_P(y)\right)$$

Proof. Because of the lower bound property of infimum, we have

$$\alpha_P(x, y) \leq r\,\Delta_P(x) + r^{-1}\,\Delta_P(y)$$

for all $r \in \mathbb{Q}_+$. Hence, by using the sequential denseness of $\mathbb{Q}$ in $\mathbb{R}$ and the sequential continuity of the operations in $\mathbb{R}$, we can infer that

$$\alpha_P(x, y) \leq \lambda\,\Delta_P(x) + \lambda^{-1}\,\Delta_P(y)$$

also holds for all $\lambda \in \mathbb{R}_+$. Therefore, by the maximality property of infimum, we have

$$\alpha_P(x, y) \leq \inf_{\lambda\in\mathbb{R}_+} \left(\lambda\,\Delta_P(x) + \lambda^{-1}\,\Delta_P(y)\right).$$

Moreover, by using the inclusion $\mathbb{Q}_+ \subseteq \mathbb{R}_+$ and the definition of infimum, we can also easily see that the converse inequality also holds. Therefore, the required equality is also true.

Remark 4.2. By using similar arguments, concerning $\beta_P(x, y)$, we can only prove that

$$\inf_{m,n\in\mathbb{N}} (m n)^{-1} \Delta_P(m x + n y) \leq \beta_P(x, y)$$

and

$$\beta_P(x, y) = \inf\left\{ (m n)^{-1} \Delta_P(m x + n y) : \quad m, n \in \mathbb{N}, \quad (m; n) = 1 \right\}$$

for all $x, y \in X$.

Now, by using Lemma 4.1 and some basic facts from calculus, we can also easily prove the following

Theorem 4.3. *For any* $x, y \in X$, *we have*

(1) $\alpha_P(x, y) = -\infty$ *if either* $\Delta_P(y) < 0$ *or* $\Delta_P(x) < 0$,

(2) $\alpha_P(x, y) = 2\sqrt{\Delta_P(x)\,\Delta_P(y)}$ *if* $\Delta_P(x) \geq 0$ *and* $\Delta_P(y) \geq 0$.

Proof. Here, we shall prove somewhat more than that is stated. For this, define

$$a = \Delta_P(x), \qquad b = \Delta_P(y) \qquad \text{and} \qquad c = \alpha_P(x, y).$$

Moreover, define

$$f(\lambda) = a\lambda + b\lambda^{-1}$$

for all $\lambda > 0$.

Then, by Lemma 4.1, we have

$$c = \inf_{\lambda>0} f(\lambda).$$

Moreover, we can note that f is a differentiable function of $\mathbb{R}_+$ such that

$$f'(\lambda) = a - b\lambda^{-2}$$

for all $\lambda > 0$. Therefore,

$$f'(\lambda) < 0 \iff a\lambda^2 < b \qquad \text{and} \qquad f'(\lambda) > 0 \iff b < a\lambda^2.$$

Hence, if $a > 0$ and $b > 0$, then by defining $\lambda_0 = \sqrt{b/a}$ we can see that f is strictly decreasing on $]-\infty, \lambda_0]$ and f is strictly increasing on $[\lambda_0, +\infty[$. Therefore,

$$f(\lambda_0) = a\lambda_0 + b\lambda_0^{-1} = 2\sqrt{a\,b}$$

is a strict global minimum of f. Thus, in particular $c = 2\sqrt{a\,b}$.

Moreover, if $a = 0$ and $b > 0$, then we can see that f is strictly decreasing on $\mathbb{R}_+$ and $\lim_{\lambda\to+\infty} f(\lambda) = 0$. Thus, in particular $c = 0 = 2\sqrt{a\,b}$.

While, if $a > 0$ and $b = 0$, then f is strictly increasing on $\mathbb{R}_+$ and $\lim_{\lambda\to 0} f(\lambda) = 0$. Thus, in particular $c = 0 = 2\sqrt{a\,b}$.

On the other hand, if either $a < 0$ or $b < 0$, then by not establishing the monotonicity properties f, we only note that $c = -\infty$. Namely, we evidently have

$$\lim_{\lambda\to 0} f(\lambda) = -\infty \quad \text{if} \quad b < 0, \qquad \text{and} \qquad \lim_{\lambda\to+\infty} f(\lambda) = -\infty \quad \text{if} \quad a < 0.$$

Thus, summarizing the above observations, we can state that

$$c = 2\sqrt{a\,b} \quad \text{if} \quad a \geq 0 \text{ and } b \geq 0, \qquad \text{and} \qquad c = -\infty \quad \text{if either} \quad a < 0 \text{ or } b < 0.$$

Therefore, the required assertions are also true.

Remark 4.4. Now, as an immediate consequence of Theorems 3.4 and 4.3, we can also state that if either $\Delta_P(y) < 0$ or $\Delta_P(x) < 0$, then $\beta_P(x, y) = -\infty$ also holds.

5. The induced generalized seminorm

By Theorem 4.3, we may also naturally introduce the following

Definition 5.1. If in particular Δ_P is nonnegative, then for any $x \in X$ we define

$$p(x) = \sqrt{\Delta_P(x)}\,.$$

Remark 5.2. The function p will be called the *generalized seminorm* derived from P.

Thus, as an immediate consequence of Theorems 3.7 and 4.3, we can also state

Theorem 5.3. *If X is a group, Δ_P is nonnegative and P_1 biodd, then for any $x, y \in X$ we have*

$$|P_1(x, y)| \leq p(x)\,p(y)\,.$$

Moreover, by using Theorem 2.5, we can also easily establish the following

Theorem 5.4. *If Δ_P is nonnegative, then for any $n, m \in \mathbb{N}$ and $x, y \in X$ we have*

(1) $p(nx) = n\,p(x)$,

(2) $p\big(n(x+y)\big) = p(nx+ny)$,

(3) $p(mx+ny)^2 = m^2 p(x)^2 + n^2 p(y)^2 + 2mn\,P_1(x, y)$.

Remark 5.5. If in particular P_1 is symmetric, then because of (3) we can also state that

$$p(ny+mx) = p(mx+ny)\,.$$

Remark 5.6. While, if in particular X is a group and P_1 is $\mathbb{Z}$–bihomogeneous, then instead of (1) we can also state that

$$p(nx) = |n|\,p(x)$$

for all $n \in \mathbb{Z}$. Moreover, the other equalities remain valid also for all $n, m \in \mathbb{Z}$.

Now, by using Theorem 5.3 and 5.4, we can also easily prove the following

Theorem 5.7. *If X is a group, Δ_P is nonnegative and P_1 is biodd, then for any $x, y \in X$ we have*

(1) $p(x+y) \leq p(x) + p(y)$, *(2)* $|p(x) - p(y)| \leq p(x-y)$.

Proof. By using Theorems 5.4 and 5.3, we can see that

$$\begin{aligned} p(x+y)^2 &= p(x)^2 + p(y)^2 + 2\,P_1(x, y) \\ &\leq p(x)^2 + p(y)^2 + 2\,p(x)\,p(y) = \big(p(x)+p(y)\big)^2. \end{aligned}$$

Therefore, by the nonnegativity of p, inequality (1) is also true.

Now, by using (1), we can also easily see that

$$p(x) = p(x-y+y) \leq p(x-y) + p(y) \quad \text{and thus} \quad p(x) - p(y) \leq p(x-y)\,.$$

Hence, it is clear that

$$-\big(p\,(x) - p\,(y) \big) = p\,(y) - p\,(x) \leq p\,(\,y - x\,) = p\,\big(-(x-y) \big) = p\,(\,x - y\,)\,,$$

and thus (2) is also true. Namely, by the bioddness of P_1, the function Δ_P is even, and thus p is also even.

Remark 5.8. Note that in Theorems 3.7, and thus also in Theorems 5.3 and 5.7, instead of the bioddness of P_1, it is enough to assume only that Δ_P is even and P_1 is odd in its first variable.

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