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# A weak Schwarz inequality for semi-inner products on groupoids

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# A WEAK SCHWARZ INEQUALITY FOR SEMI-INNER PRODUCTS ON GROUPOIDS

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Abstract. By introducing appropriate notions of semi-inner products and their induced generalized seminorms on groupoids, we shall prove a weak form of the famous Schwarz inequality.

In case of groups, this will be sufficient to prove the subadditivity of the induced generalized seminorms. Thus, some of the results of the theory of inner product spaces can be extended to inner product groups.

However, in the near future, we shall only be interested in the corresponding extensions of some fundamental theorems of Gy. Maksa, P. Volkmann, A. Gilányi, J. Rätz and W. Fechner on additive and quadratic functions.

## 1. Introduction

By introducing appropriate notions of semi-inner products and their induced generalized seminorms on groupoids, we shall prove a weak form of the famous Schwarz inequality.

More concretely, if $X$ is an additively written groupoid and $P$ is a function of $X^2$ to $\mathbb{C}$ such that

$$P(x, x) \geq 0, \qquad P(y, x) = \overline{P(x, y)}, \qquad P(x+y, z) = P(x, z) + P(y, z)$$

for all $x, y, z \in X$, then by using the notation

$$p(x) = \sqrt{P(x, x)}$$

with $x \in X$, we shall prove that

$$-P_1(x, y) \leq p(x)\, p(y)$$

for all $x, y \in X$, where $P_1$ denotes the real part, i. e., the first coordinate function of $P$.

If in particular $X$ is a group, then this weak Schwarz inequality already implies that $P_1(x, y) \leq p(x)\, p(y)$ also holds for all $x, y \in X$. Therefore, in this important particular case, the generalized seminorm $p$ can be proved to be a seminorm on $X$ in the sense it is an even, $\mathbb{N}$–homogeneous, subadditive function of $X$ to $\mathbb{R}$.

Thus, some of the results of the theory of inner product spaces can be naturally extended to inner product groups. However, in the near future, we shall only be interested in the corresponding extensions of some fundamental theorems of

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Maksa and Volkmann [14], Gilányi [8], Rätz [15] and Fechner [6] on additive and quadratic functions.

## 2. Additive functions of groupoids

If $X$ is a set, then a function $+$ of $X^2$ to $X$ is called an operation on $X$, and the ordered pair $X(+) = (X, +)$ is called a groupoid.

In the sequel, as is customary, we shall simply write $X$ in place of $X(+)$. And, for any $x, y \in X$, we shall write $x + y$ in place of the value $+(x, y)$.

Moreover, for any $x \in X$ and $n \in \mathbb{N}$, with $n > 1$, we define

$$1\,x = x \qquad \text{and} \qquad n\,x = (n-1)\,x + x\,.$$

If in particular, $X$ is group, then for any $x \in X$ and $n \in \mathbb{N}$ we may also naturally define

$$0\,x = 0 \qquad \text{and} \qquad (-n)\,x = n\,(-x)\,.$$

A function $f$ of one groupoid $X$ to another $Y$ is called additive if

$$f\,(x+y) = f\,(x) + f\,(y)$$

for all $x, y \in X$.

Moreover, the function $f$ may be naturally called $\mathbb{N}$–homogeneous if it is $n$–homogeneous for all $n \in \mathbb{N}$ in the sense that $f\,(n\,x) = n\,f(x)$ for all $x \in X$.

Additive functions were first studied only on $\mathbb{R}$ or $\mathbb{R}^n$ (see Kuczma [12]). However, later they have also been intensively investigated on arbitrary groups (see Stetkaer [21]).

Some of the results obtained in groups can be naturally extended to monoids and semigroups. In [17] and [10], additive functions and relations were considered on groupoids too.

For instance, by induction, we can easily prove the following

**Theorem 2.1.** *If $f$ is an additive function of a groupoid $X$ to another $Y$, then $f$ is $\mathbb{N}$–homogeneous.*

*Proof.* To check this, note that if $f\,(n\,x) = n\,f(x)$ holds for some $x \in X$ and $n \in \mathbb{N}$, then we also have

$$f\,\big((n+1)\,x\big) = f\,(n\,x + x) = f\,(n\,x) + f\,(x) = n\,f\,(x) + f\,(x) = (n+1)\,f(x)\,.$$

**Remark 2.2.** If $f$ is an additive function of a groupoid $X$, with zero, to a group $Y$, then $f$ is 0–homogeneous too.

Namely, in this case, we have

$$f\,(0) + f\,(0) = f\,(\,0 + 0\,) = f\,(0)\,,$$

and thus $f\,(0) = 0$. Therefore,

$$f\,(0\,x) = f\,(0) = 0 = 0\,f\,(x)$$

also holds for all $x \in X$.

Now, by using the above observations and the corresponding definitions, we can also easily prove the following

**Theorem 2.3.** *If $f$ is an additive function of a group $X$ to another $Y$, then $f$ is $\mathbb{Z}$–homogeneous.*

*Proof.* If $x \in X$, then by using Remark 2.2 we can see that

$$f(-x) + f(x) = f(-x + x) = f(0) = 0\,,$$

and thus $f(-x) = -f(x)$. Now, if $n \in \mathbb{N}$, then by using Theorem 1.1 we can also see that

$$f\big((-n)\,x\big) = f\big(n(-x)\big) = n\,f(-x) = n\big(-f(x)\big) = (-n)\,f(x)\,.$$

Therefore, $f$ is also $-N$–homogeneous. Thus, by Theorem 2.1, the required assertion is also true.

In addition to the above theorems, sometimes we shall also need the following

**Theorem 2.4.** *If $f$ is an additive function of an arbitrary groupoid $X$ to a commutative one $Y$, then for any $x, y \in X$ we have*

$$f(y + x) = f(x + y)\,.$$

*Proof.* By the above assumptions, we evidently have

$$f(y + x) = f(y) + f(x) = f(x) + f(y) = f(x + y)\,.$$

**Remark 2.5.** In this case, in contrast to the termilogy of Stetkaer [21, p. 315], we would rather say that $f$ is commutative.

## 3. Semi-inner products on groupoids

**Notation 3.1.** Suppose that $X$ is a groupoid and $P$ is a function of $X^2$ to $\mathbb{C}$ such that, for any $x, y, z \in X$, we have

(a) $P(x, x) \geq 0$,

(b) $P(y, x) = \overline{P(x, y)}$,

(c) $P(x + y, z) = P(x, z) + P(y, z)$.

**Remark 3.2.** In this case, the function $P$ will be called a *semi-inner product* on $X$.

Moreover, if in particular $X$ has a zero, then the semi-inner product $P$ will be called an inner product if

(d) $P(x, x) = 0$ implies $x = 0$ for all $x \in X$.

**Remark 3.3.** Thus, our present definition is in accordance with that of [16], but differs from that used by Lumer [11] and Giles [9]. (See also Dragomir [4, p. 19] for some further developments.)

The definition and results of the above mentioned authors allowed to carry over some arguments in inner product spaces to those in normed spaces. While, our ones will only allow of a similar transition from inner product spaces to inner product groups.

**Example 3.4.** If $a$ is an additive function of $X$ to an inner product space $H$ and

$$Q(x, y) = \langle a(x), a(y) \rangle$$

for all $x, y \in X$, then $Q$ is a semi-inner product on $X$. Moreover, if if in particular $X$ is a group, then $Q$ is an inner product if and only if $a$ is injective.

Despite this, $Q$ may be a rather curious function even if $X = \mathbb{R}^n$ $H = \mathbb{R}$. Namely, by Kuczma [12, p. 292], there exist discontinuous, injective additive functions of $\mathbb{R}^n$ to $\mathbb{R}$. In the case $n = 1$, by Makai [13], Kuczma [12, p. 293] and Baron [1], we can say even more.

The most basic properties of the semi-inner product $P$ can be listed in the next

**Theorem 3.5.** *For any $x, y, z \in X$ and $n \in \mathbb{N}$, we have*

*(1)* $P(y + x, z) = P(x + y, z)$,

*(2)* $P(x, z + y) = P(x, y + z)$,

*(3)* $P(x, y + z) = P(x, y) + P(x, z)$,

*(4)* $P(nx, y) = n\,P(x, y) = P(x, ny)$.

*Proof.* By using (b) and (c), and the additivity of complex conjugation, we can see that (3) is true.

Thus, $P$ is actually a biadditve function of $X^2$ to $\mathbb{C}$. Hence, by Theorem 2.1, it is clear that (4) is also true.

Moreover, by using (c) and (3) and the commutativity of the addition in $\mathbb{C}$, we can see that (1) and (2) are also true.

**Remark 3.6.** Note that if in particular $X$ has a zero, then by Remark 2.2 we have $P(x, 0) = 0$ and $P(0, y) = 0$, and thus also

$$P(0x, y) = 0\,P(x, y) = P(x, 0y)$$

for all $x, y \in X$.

Moreover, if more specially is a group, then by Theorem 2.3 we have

$$P(kx, y) = k\,P(x, y) = P(x, ky)$$

for all $k \in \mathbb{Z}$ and $x, y \in X$.

**Remark 3.7.** Note that the first and second coordinate functions $P_1$ and $P_2$ of $P$ also have the same commutativity and bilinearity properties as $P$.

Furthermore, by properties (a) and (b), for any $x, y \in X$ we have

(1) $P_1(x, x) = P(x, x)$ and $P_2(x, x) = 0$,

(2) $P_1(y, x) = P_1(x, y)$ and $P_2(y, x) = -P_2(x, y)$.

Thus, in particular $P_1$ is also a semi-inner product on $X$. However, because of its skew-symmetry, $P_2$ cannot be a semi-inner product on $X$ whenever $P_2 \neq 0$.

More exactly, one can easily prove the following

**Theorem 3.8.** *A function $Q$ of $X^2$ to $\mathbb{C}$ is a semi-inner product if and only if for any $x, y \in X$ we have*

*(1)* $Q_1(x, x) \geq 0$ *and* $Q_2(x, x) = 0$,

*(2)* $Q_1(y, x) = Q_1(x, y)$ *and* $Q_2(y, x) = -Q_2(x, y)$,

*(3)* $Q_i(x+y, z) = Q_i(x, z) + Q_i(y, z)$ *for* $i = 1$ *and* $i = 2$.

**Remark 3.9.** Note that the second part of (2) implies that of (1). Moreover, the second parts of (2) and (3) imply that $P_2$ is additive in its second variable too.

Therefore, by the above theorem, we can also state that a function $Q$ of $X^2$ to $\mathbb{C}$ is a semi-inner product if and only if $P_1$ is a semi-inner product and $P_2$ is a skew-symmetric and biadditive.

## 4. The induced generalized norm

**Definition 4.1.** For any $x \in X$, we define

$$p(x) = \sqrt{P(x, x)}.$$

**Example 4.2.** If in particular $Q$ is as in Example 3.4, then

$$q(x) = \sqrt{Q(x, x)} = \| a(x) \|$$

for all $x \in X$.

The most immediate properties of the function $p$ can be listed in the following

**Theorem 4.3.** *For any* $x, y \in X$ *and* $n \in \mathbb{N}$, *we have*

*(1)* $p(x) \geq 0$,

*(2)* $p(nx) = n\, p(x)$,

*(3)* $p(x+y) = p(y+x)$,

*(4)* $p\big(n(x+y)\big) = p(nx+ny)$,

*(5)* $p(x+y)^2 = P_1(x+y, x) + P_1(x+y, y)$,

*(6)* $p(x+y)^2 = p(x)^2 + p(y)^2 + 2\, P_1(x, y)$.

*Proof.* To prove (5) and (6), note that by the Definition 4.1 and Remark 3.7 we have

$$p(x) = \sqrt{P_1(x, x)}$$

and

$$p(x+y)^2 = P_1(x+y,\ x+y) = P_1(x+y, x) + P_1(x+y, y)$$
$$= P_1(x, x) + P_1(y, x) + P_1(x, y) + P_1(y, y) = p(x)^2 + 2\, P_1(x, y) + p(y)^2.$$

Hence, by the symmetry of $P_1$ and the commutativity of the addition in $\mathbb{R}$, it is clear that (3) is also true.

Moreover, by using Theorems 4.3 and 3.5, we can see that

$$p\big(n(x+y)\big)^2 = n^2 p(x+y)^2 = n^2 p(x)^2 + n^2 p(y)^2 + 2n^2 P_1(x, y)$$

and

$$p(nx+ny)^2 = p(nx)^2 + p(nx)^2 + 2\, P_1(nx, ny)$$
$$= n^2 p(x)^2 + n^2 p(y)^2 + 2n^2 P_1(x, y).$$

Therefore, $p\big(n\,(x+y)\big)^2 = p\,(n\,x + n\,y)^2$, and thus by the nonnegativity of $p$ (4) also holds.

**Remark 4.4.** If in particular $X$ has a zero, the by Remark 3.6 we have $p\,(0) = 0$, and thus also

$$p\,(0\,x) = |\,0\,|\,p\,(x) \qquad \text{and} \qquad p\big(0\,(x+y)\big) = p\,(0\,x + 0\,y)$$

for all $x,\,y \in X$.

Moreover, if more specially $X$ is a group, then by Remark 3.6 we have

$$p\,(k\,x) = |\,k\,|\,p\,(x) \qquad \text{and} \qquad p\big(k\,(x+y)\big) = p\,(k\,x + k\,y)$$

for all $k \in \mathbb{Z}$ and $x,\,y \in X$.

## 5. A weak Schwarz inequality

To prove a Schwarz type inequality for $P$, it is convenient to start with

**Lemma 5.1.** *For any $n, m \in \mathbb{N}$ and $x,\,y \in X$, we have*

$$p\,(n\,x + m\,y)^2 = n^2\,p\,(x)^2 + m^2\,p\,(y)^2 + 2\,n\,m\,P_1(x,\,y)\,.$$

*Proof.* By Theorem 4.3 and Remark 3.7, we have

$$\begin{aligned} p\,(n\,x + m\,y)^2 &= p\,(n\,x)^2 + p\,(m\,y)^2 + 2\,P_1(n\,x,\ m\,y) \\ &= n^2\,p\,(x)^2 + m^2\,p\,(y)^2 + 2\,n\,m\,P_1(x,\,y)\,. \end{aligned}$$

Now, by using this simple lemma, we can give two different proofs for the following theorem. The first one is more novel than the second one.

**Theorem 5.2.** *For any $x,\,y \in X$, we have*

$$-P_1(x,\,y) \le p\,(x)\,p\,(y)\,.$$

*Proof 1.* From Lemma 4.1, we can see that

$$-2\,P_1(x,\,y) \le (n/m)\,p\,(x)^2 + (m/n)\,p\,(y)^2\,.$$

for all $n,\,m \in \mathbb{N}$.

Therefore, by the definition of rational numbers, we actually have

$$-2\,P_1(x,\,y) \le r\,p\,(x)^2 + r^{-1}\,p\,(y)^2$$

for all $r \in \mathbb{Q}$ with $r > 0$.

Hence, by using that each real number is a limit of a sequence of rational numbers and the operation in $\mathbb{R}$ are continuous, we can already infer that

$$-2\,P_1(x,\,y) \le \lambda\,p\,(x)^2 + \lambda^{-1}\,p\,(y)^2$$

for all $\lambda \in \mathbb{R}$ with $\lambda > 0$.

Now, by defining

$$f\,(\lambda) = \lambda\,p\,(x)^2 + \lambda^{-1}\,p\,(y)^2$$

for all $\lambda > 0$, we can state that

$$-2\,P_1(x,\,y) \le \inf_{\lambda > 0}\,f\,(\lambda)\,.$$

Moreover, if $p(x) \neq 0$ and $p(y) \neq 0$, then by taking

$$\lambda_0 = p(y)/p(x)$$

we can note that $\lambda_0 > 0$ such that

$$f(\lambda_0) = 2\, p(x)\, p(y).$$

Therefore,

$$\inf_{\lambda>0} f(\lambda) \leq 2\, p(x)\, p(y), \qquad \text{and thus} \qquad -2\, P_1(x, y) \leq 2\, p(x)\, p(y).$$

Hence, the required inequality follows.

While, if either $p(x) = 0$ or $p(y) = 0$, then from the definition of $f$ we can see that

$$\inf_{\lambda>0} f(\lambda) = 0, \qquad \text{and thus} \qquad -2\, P_1(x, y) \leq 0.$$

Therefore, $-P_1(x, y) \leq 0$, and thus the required inequality trivially holds.

**Remark 5.3.** If $p(x) \neq 0$ and $p(y) \neq 0$, then by computing $f'(\lambda)$ for all $\lambda > 0$, we can prove that $f(\lambda_0) < f(\lambda)$ for all $\lambda > 0$ with $\lambda \neq \lambda_0$.

*Proof 2.* From Lemma 4.1, we can also see that

$$0 \leq p(x)^2 + (m/n)^2\, p(y)^2 + 2\,(m/n)\, P_1(x, y)$$

for all $n, m \in \mathbb{N}$.

Therefore, by using a similar argument as in Proof 1, we can state that

$$0 \leq p(x)^2 + \lambda^2\, p(y)^2 + 2\,\lambda\, P_1(x, y),$$

and thus

$$0 \leq p(x)^2 + \lambda\, P_1(x, y) + \lambda \left( \lambda\, p(y)^2 + P_1(x, y) \right)$$

for all $\lambda > 0$.

Hence, if $p(y) > 0$ and $P_1(x, y) < 0$, then by taking $\lambda = -P_1(x, y)/p(y)^2$ we can see that

$$0 \leq p(x)^2 - P_1(x, y)^2/\, p(y)^2, \qquad \text{and thus} \qquad P_1(x, y)^2 \leq \left( p(x)\, p(y) \right)^2.$$

Therefore, because of $|\, P_1(x, y) \,| = -P_1(x, y)$, the required inequality is also true.

While, if $p(y) = 0$ and $P_1(x, y) < 0$, then by taking $\lambda = -n\, P_1(x, y)$ for some $n \in \mathbb{N}$ we can see that

$$0 \leq p(x)^2 - 2\, n\, P_1(x, y)^2, \qquad \text{and thus} \qquad P_1(x, y)^2 \leq p(x)^2/\, 2\, n.$$

Hence, by taking the limit $n \to \infty$, we can infer that $P_1(x, y) = 0$. Therefore, the required inequality trivially holds.

Now, to complete the proof, it remains only to note that if $P_1(x, y) \geq 0$, then the required inequality is also trivially true.

From Theorem 5.2, we can easily infer the following

**Corollary 5.4.** *If in particular $X$ is a group, then for any $x, y \in X$, we have*

$$|\, P_1(x, y) \,| \leq p(x)\, p(y).$$

*Proof.* By Theorem 5.2 and Remarks 3.6 and 4.4, now we also have

$$P_1(x, y) = -P_1(-x, y) \leq p(-x)\, p(y) = p(x)\, p(y).$$

Therefore, the required inequality is also true.

**Remark 5.5.** Note that if $x, y \in X$ such that $|P(x, y)| \leq p(x)p(y)$ holds, then we also have $|P_i(x, y)| \leq p(x)p(y)$ and hence $P_i(x, y) \leq p(x)p(y)$ and $-P_i(x, y)| \leq p(x)p(y)$ for $i = 1, 2$.

The following example shows that if in particular $X$ is a group and $P$ is an $\mathbb{R}$–bihomogeneous semi-inner product on $X$, then even the weak Scwarz inequality $-P_2(x, y) \leq p(x)p(y)$ need not be true.

**Example 5.6.** For any $x, y \in \mathbb{R}^2$, define

$$a(x) = x \qquad \text{and} \qquad b(y) = (y_2, -y_1),$$

and moreover

$$Q_1(x, y) = x_1 y_1 \qquad \text{and} \qquad Q_2(x, y) = \langle a(x), b(y) \rangle.$$

Then, $Q = (Q_1, Q_2)$ is an $\mathbb{R}$–bihogeneous semi-inner product on $\mathbb{R}^2$ such that, under the notation

$$q(x) = \sqrt{Q(x, x)}$$

with $x \in \mathbb{R}^2$, even the inequality

$$-Q_2(x, y) \leq q(x)q(y)$$

fails to hold for all $x, y \in \mathbb{R}^2$.

It is clear that $Q_1$ is a symmetric, bilinear function of $\left(\mathbb{R}^2\right)^2$ to $\mathbb{R}$. Moreover, we can easily see that $a$ and $b$ are linear functions of $\mathbb{R}^2$ to itself. Therefore, $Q_2$ is also a bilinear function of $\left(\mathbb{R}^2\right)^2$ to $\mathbb{R}$. Hence, it is clear that $Q$ is a bilinear function of $\mathbb{R}^2$ to itself.

Moreover, since

$$Q_2(x, y) = \langle a(x), b(y) \rangle = \langle (x_1, x_2), (y_2, -y_1) \rangle = x_1 y_2 - x_2 y_1$$

for all $x, y \in \mathbb{R}^2$, we can note that

$$Q_2(x, x) = 0 \qquad \text{and} \qquad Q_2(y, x) = -Q_2(y, x)$$

for all $x, y \in \mathbb{R}^2$. Hence, it is clear that $Q$ is an $\mathbb{R}$–bihogeneous semi-inner product on $\mathbb{R}^2$.

On the other hand, for instance, by taking

$$u = (0, 1) \qquad \text{and} \qquad v = (1, 0),$$

we can see that

$$q(u)q(v) = |u_1||v_1| = 0, \qquad \text{but} \qquad -Q_2(u, v) = u_2 v_1 - u_1 v_2 = 1.$$

**Remark 5.7.** Note that, by making the plausible change

$$Q_1(x, y) = \langle x, y \rangle$$

for all $x, y \in \mathbb{R}^2$, we could get

$$\begin{aligned} |Q(x, y)|^2 = Q_1(x, y)^2 + Q_2(x, y)^2 &= (x_1 y_1 + x_2 y_2)^2 + (x_1 y_2 - x_2 y_1)^2 \\ &= (x_1^2 + x_2^2)(y_1^2 + y_2^2) = |x|^2 |y|^2 = q(x)^2 q(y)^2, \end{aligned}$$

and thus $|Q(x, y)| = q(x)q(y)$ for all $x, y \in \mathbb{R}^2$.

However, it is now more important to note that, by using Corollary 5.4, we can give two different proofs for the subadditivity of $p$. The first one is more novel than the second one.

**Theorem 5.8.** *If in particular $X$ is a group, then for any $x, y \in X$, we have*

*(1)* $p(x+y) \leq p(x)+p(y)$, *(2)* $|p(x)-p(y)| \leq p(x-y)$.

*Proof 1.* By using Theorem 4.3 and the inequality $P_1(x, y) \leq p(x)\,p(y)$, we can see that

$$p(x+y)^2 = P_1(x+y, x) + P_1(x+y, y) \leq p(x+y)\,p(x) + p(x+y)\,p(y).$$

Therefore, by the nonnegativity of $p$, inequality (1) is also true.

*Proof 2.* By using Theorem 4.3 and the inequality $P_1(x, y) \leq p(x)\,p(y)$, we can also see that

$$\begin{aligned} p(x+y)^2 &= p(x)^2 + p(y)^2 + 2\,P_1(x, y) \\ &\leq p(x)^2 + p(y)^2 + 2\,p(x)\,p(y) = \big(p(x)+p(y)\big)^2. \end{aligned}$$

Therefore, by the nonnegativity of $p$, inequality (1) is also true.

**Remark 5.9.** Theorems 4.3 and 5.8, together with Remark 4.4, show that if in particular $X$ is a group, then $p$ is already a seminorm on $X$ in the sense it is an even, $\mathbb{N}$–homogeneous, subadditive function of $X$ to $\mathbb{R}$.

Hence, it can be easily seen that, in this case, the function $d$, defined by

$$d(x, y) = p(-x+y)$$

for all $x, y \in X$, is a both left and right translation invariant semimetric on $X$.

In an improved and enlarged version of [2], we shall show that, analogously to seminorms and semimetrics derived from the usual semi-inner products on vector spaces, the generalized seminorms and semimetrics derived from semi-inner products on groupoids and groups also have several useful additional properties.

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