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GENERALIZATIONS OF AN ASYMPTOTIC STABILITY THEOREM OF BAHYRYCZ, PÁLES AND PISZCZEK ON CAUCHY DIFFERENCES TO GENERALIZED COCYCLES

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Abstract. We prove some straightforward analogues and generalizations of a recent asymptotic stability theorem of A. Bahyrycz, Zs. Páles and M. Piszczek on Cauchy differences to semi-cocycles and pseudo-cocycles introduced in a former paper by the present author.

1. Introduction

In [5], Bahyrycz, Páles and Piszczek have proved a metric form of the following theorem on restricted and asymptotic stabilities with mentioning only a few former results on these stabilities.

The most closely related ones are [37, Theorem 1] of Losonczi with the same constant 5, and the results of Jung [32] and Chung [12, 13] with some other natural constants in the concluded estimates.

Theorem 1.1. *If f is a function of an unbounded commutative preseminormed group X to a commutative preseminormed one Y and*

$$\varepsilon = \limsup_{\|x\| \wedge \|y\| \to \infty} \| f(x+y) - f(x) - f(y) \| ,$$

then

$$\| f(x+y) - f(x) - f(y) \| \leq 5\,\varepsilon$$

for all $x, y \in X$.

Remark 1.2. Moreover, by taking $\varepsilon > 0$ and $x_0 \in X \setminus \{0\}$, and defining

$$f(x_0) = 3\,\varepsilon \qquad \text{and} \qquad f(x) = \varepsilon \quad \text{for} \quad x \in X \setminus \{x_0\},$$

they have also proved that 5 is the smallest possible constant in their theorem.

From Theorem 1.1, one can immediately derive

Corollary 1.3. *If f is a function of an unbounded commutative preseminormed group X to a commutative prenormed one Y such that*

$$\limsup_{\|x\| \wedge \|y\| \to \infty} \| f(x+y) - f(x) - f(y) \| = 0 ,$$

then f is additive. (That is, $f(x+y) = f(x) + f(y)$ for all $x, y \in X$.)

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However, it is now more important to note that Bahyrycz, Páles and Piszczek, in the proof of their [5, Theorem 1], have used, but not explicitly stated, the equality

$$(1)\quad f(x+y)-f(x)-f(y)=f(x-u)+f(u)-f(x) \\ +f(y-v)+f(v)-f(y)+f(x+y-u-v)-f(x-u)-f(y-v) \\ +f(u+v)-f(u)-f(v)+f(x+y)-f(x+y-u-v)-f(u+v)\,.$$

In a former paper [57], by using the *Cauchy difference*

$$(2)\qquad F(x,\,y)=f(x+y)-f(x)-f(y)\,,$$

we have noticed that, instead of equation (1), it is more convenient to consider the equation

$$(3)\quad F(x,\,y)=F(u,\,v)-F(x-u,\,u)-F(y-v,\,v) \\ +F(x-u,\,y-v)+F(x+y-u-v,\,u+v)\,.$$

Namely, thus Theorem 1.1 can be easily extended to the solutions of (3). Moreover, we can prove that every *symmetric cocycle* F on X to Y is a solution of this equation.

That is, if F is a function of X^2 to Y such that $F(x,\,y)=F(y,\,x)$ and

$$(4)\qquad F(x,\,y)+F(x+y,\,z)=F(x,\,y+z)+F(y,\,z)$$

for all $x,\,y,\,z\in X$, then (3) also holds for all $x,\,y,\,u,\,v\in X$.

It is well-known that every Cauchy–difference is a symmetric cocycle. Moreover, Davison and Ebanks [17, Lemma 2] have proved that if F is a symmetric cocycle on X to Y, then

$$(5)\quad F(x+y,\,u+v)=F(x+u,\,y+v) \\ +F(x,\,u)+F(y,\,v)-F(x,\,y)-F(u,\,v)$$

also holds for all $x,\,y,\,u,\,v\in X$.

At first seeing, I considered equations (3) and (5) to be very similar, but still quite independent. However, Gyula Maksa, my close colleague, has noticed that they are actually equivalent.

Namely, (5) can be immediately derived from (3) by replacing x by $x+u$ and y by $y+v$. And conversely, (3) can be immediately derived from (5) by replacing x by $x-u$ and y by $y-v$. Thus, equation (1) is a consequence of (5) too.

Inspired by the above observations, in our former paper [57], we have also considered the more difficult equations

$$(6)\quad F(x,\,y)+F(u,\,y+v)+F(x+y,\,u+v) \\ =F(x,\,u)+F(y,\,u+v)+F(x+u,\,y+v)\,,$$

and

$$(7)\quad F(x,\,y)+F(x-u,\,u)+F(y-v,\,u)+F(y-v,\,v) \\ =F(u,\,v)+F(u,\,y-v)+F(x-u,\,y-v)+F(x+y-u-v,\,u+v)\,.$$

Note that if in particular F is symmetric, then equation (7) is equivalent to (3), which is in turn equivalent to (5). Moreover, it can be easily shown that if F is additive in its second variable, then equations (6) and (7) are also equivalent.

In our former paper [57], by using some more difficult computations, we have also proved that equations (6) and (7) are also natural generalizations of (4) too. Therefore, their solutions may be naturally called *semi-cocycles* and *pseudo-cocycles*, respectively.

In the light of the above observations, it seems to be a reasonable research program to extend some of the basic theorems on cocycles to these generalized cocycles. And, to establish some deeper relationships among the various generalizations of cocycles.

However, now we shall only prove some straightforward analogues and generalizations of Theorem 1.1 to semi-cocycles and pseudo–cocycles. For instance, we shall prove the following natural analogue of Theorem 1.1.

Theorem 1.4. *If F is a semi-cocycle on an unbounded commutative preseminormed group X to a commutative preseminormed one Y and*

$$\varepsilon = \overline{\lim_{\|z\| \to \infty}} \, \| F(z) \| ,$$

then $\| F(z) \| \leq 5\,\varepsilon$ for all $z \in X^2$.

Remark 1.5. If F is a symmetric pseudo-cocycle, then the same conclusion also holds with

$$[\![z]\!] = \| z_1 \| \wedge \| z_2 \| = \min\big\{ \| z_1 \| , \ \| z_2 , \| \big\}$$

in place of $\| z \| = \| z_1 \| + \| z_2 \|$ or

$$\| z \| = \| z_1 \| \vee \| z_2 \| = \max\big\{ \| z_1 \| , \ \| z_2 \| \big\} .$$

However, if the pseudo-cocycle F fails to be symmetric, then we have to write 7 in place of 5 in the above estimation. Thus, by letting $\varepsilon = 0$, we can also at once state

Corollary 1.6. *If F is a semi-cocyle (pseudo-cocycle) on an unbounded commutative preseminormed group X to a commutative prenormed one Y such that*

$$\overline{\lim_{\|z\| \to \infty}} \, \| F(z) \| = 0 \qquad\qquad \Big(\overline{\lim_{[\![z]\!] \to \infty}} \, \| F(z) \| = 0 \Big) ,$$

then F is identically zero. (That is, $F(z) = 0$ for all $z \in X^2$.)

Remark 1.7. Here, motivated by the corresponding definitions of [55, 22, 56] and [41, 27, 11] and the proofs of our forthcoming theorems, an even subadditive function $\| \ \|$ of a group X to $\mathbb{R}$ is called a *preseminorm* on X.

Thus, under the notation $\| x \| = \| \ \| (x)$ with $x \in X$, we have

$$\| 0 \| = \| \| 0 + 0 \| \leq 2 \| 0 \| \quad \text{ and } \quad \| 0 \| = \| x + (-x) \| \leq \| x \| + \| -x \| = 2 \| x \|$$

for all $x \in X$. Therefore, $0 \leq \| 0 \|$, and thus also $0 \leq \| x \|$ for all $x \in X$.

Moreover, by using the corresponding definitions, we can also easily see that

$$\| n\, x \| \leq n \| x \| \qquad \text{and} \qquad \| (-n)\, x \| = \| n\,(-x) \| \leq n \, \| -x \| = n \| x \|$$

for all $n \in \mathbb{N}$ and $x \in X$.

Therefore, a preseminorm $\| \ \|$ on X may be naturally called a *seminorm* if $\| k\, x \| = | k | \, \| x \|$ for all $x \in X$ and $k \in \mathbb{Z} \setminus \{0\}$. Moreover, a seminorm (preseminorm) $\| \ \|$ on X may be naturally called a *norm (prenorm)* if $\| x \| = 0$ implies $x = 0$.

To feel the importance of seminorms, note that if in particular X is a non-trivial seminormed group in the sense that $\| x \| \neq 0$ for some $x \in X$, then because of the equality $\| n x \| = n \| x \|$ with $n \in \mathbb{N}$, the seminormed group X is unbounded.

2. Analogues of Theorem 1.1 for generalized cocycles

Notation 2.1. *In the sequel, we shall assume that F is a function of an unbounded, commutative preseminormed group X to a commutative preseminormed group Y.*

Remark 2.2. Note that now, by defining

$$(x, y) + (u, v) = (x + u, \ y + v)$$

and $\| (x, y) \| = \| x \| + \| y \|$ or

$$\| (x, y) \| = \| x \| \vee \| y \| = \max\{ \| x \|, \ \| y \| \}$$

for all $x, y, u, v \in X$, the set X^2 can also be turned into a commutative preseminormed group.

Now, by using a more simple argument than that used by Bahyrycz, Páles and Piszczek in [5], we can prove the following natural analogue of Theorem 1.1.

Theorem 2.3. *If F is a semi-cocycle and*

$$\varepsilon = \overline{\lim_{\| z \| \to +\infty}} \| F(z) \|,$$

then

$$\| F(z) \| \leq 5 \varepsilon$$

for all $z \in X^2$.

Proof. By the corresponding definitions, for any $\eta > \varepsilon$, we have

$$\inf_{r > 0} \sup_{\|z\| > r} \| F(z) \| < \eta .$$

Therefore, there exists $r > 0$ such that $\sup_{\|z\| > r} \| F(z) \| < \eta$, and thus

$$\| F(z) \| < \eta$$

for all $z \in X^2$ with $\| z \| > r$.

Hence, since $\| z \| = \| (z_1, z_2) \| \geq \| z_i \|$ for $i = 1, 2$, it is clear that in particular we have

$$\| F(s, t) \| < \eta$$

for all $s, t \in X$ with either $\| s \| > r$ or $\| t \| > r$.

Now, by taking $x, y \in X$ and using equation (6), we can see that

$$\begin{aligned} \| F(x, y) \| &= \| F(x, u) + F(y, u + v) - F(u, y + v) \\ &\quad + F(x + u, y + v) - F(x + y, u + v) \| \\ &\leq \| F(x, u) \| + \| F(y, u + v) \| + \| F(u, y + v) \| \\ &\quad + \| F(x + u, y + v) \| + \|F(x + y, u + v) \| < 5 \eta \end{aligned}$$

whenever for instance $u, v \in X$ such that

$$\| u \| > r , \qquad \| u + v \| > r , \qquad \| x + u \| > r .$$

Therefore, if such u and v exist, then

$$\| F(x, y) \| < 5\eta, \qquad \text{and thus} \qquad \| F(x, y) \| \leq 5\varepsilon$$

Now, to complete the proof, it remains to show only that the required u and v exist. For this, we can note that, because of the assumed unboundedness of X, there exist $u, v \in X$ such that

$$\| u \| > r + \| x \| \qquad \text{and} \qquad \| v \| > r + \| u \| .$$

Thus, we evidently have $\| u \| > r$. Moreover, by using the inequality $\| s + t \| \geq \| t \| - \| s \|$, we can also see that

$$\| x + u \| \geq \| u \| - \| x \| > r + \| x \| - \| x \| = r$$

and

$$\| u + v \| \geq \| v \| - \| u \| > r + \| u \| - \| u \| = r .$$

From the above theorem, we can immediately derive

Corollary 2.4. *If* Y *is prenormed,* F *is a semi-cocycle and*

$$\overline{\lim_{\| z \| \to +\infty}} \| F(z) \| = 0 ,$$

then $F(z) = 0$ *for all* $z \in X^2$.

From equation (7) and the proof of Theorem 2.3, it is clear that we also have

Theorem 2.5. *If* F *is a pseudo-cocycle and*

$$\varepsilon = \overline{\lim_{\| z \| \to +\infty}} \| F(z) \| ,$$

then

$$\| F(z) \| \leq 7\varepsilon$$

for all $z \in X^2$.

Hence, we can immediately derive

Corollary 2.6. *If* Y *is prenormed,* F *is a pseudo-cocycle and*

$$\overline{\lim_{\| z \| \to +\infty}} \| F(z) \| = 0 ,$$

then $F(z) = 0$ *for all* $z \in X^2$.

Remark 2.7. However, because of Remark 2.2, from Theorems 2.3 and 2.5 we cannot get proper generalizations of Theorem 1.1. Therefore, in the next section we shall prove some modification and improvement of Theorem 2.5.

3. Proper and partial generalizations of Theorem 1.1 for pseudo-cocycles

Remark 3.1. Because of Remark 2.7, in the sequel we shall use the quantity

$$[\![(x, y)]\!] = \| x \| \wedge \| y \| = \min\{ \| x \|, \| y \| \},$$

for all $(x, y) \in X^2$ instead of the usual preseminorms mentioned in Remark 2.2.

Thus, the function $[\![\ ]\!]$ is not a preseminorm on X^2. However, despite this, it can be well used to measure the magnitude of the points of X^2.

Moreover, it can as well be used to prove the following proper and partial generalizations of Theorem 1.1 to pseudo-cocycles. The proof of the first one is quite similar to the second one. Therefore, it will be omitted.

Theorem 3.2. *If F is a symmetric pseudo-cocycle and*

$$\varepsilon = \overline{\lim_{[\![z]\!] \to +\infty}} \| F(z) \|,$$

then

$$\| F(z) \| \leq 5\varepsilon$$

for all $z \in X^2$.

Hence, we can immediately derive

Corollary 3.3. *If Y is prenormed, F is a symmetric pseudo-cocycle and*

$$\overline{\lim_{[\![z]\!] \to +\infty}} \| F(z) \| = 0,$$

then $F(z) = 0$ for all $z \in X^2$.

The proof of the following theorem is again quite similar, but a little more readable, than the one given by Bahyrycz, Páles and Piszczek in [5].

Theorem 3.4. *If F is a pseudo-cocycle and*

$$\varepsilon = \overline{\lim_{[\![z]\!] \to +\infty}} \| F(z) \|,$$

then

$$\| F(z) \| \leq 7\varepsilon$$

for all $z \in X^2$.

Proof. By the corresponding definitions, for any $\eta > \varepsilon$, we have

$$\inf_{r>0} \sup_{[\![z]\!] > r} \| F(z) \| < \eta .$$

Therefore, there exists $r > 0$ such that $\sup_{[\![z]\!] > r} \| F(z) \| < \eta$, and thus

$$\| F(z) \| < \eta$$

for all $z \in X^2$ with $[\![z]\!] > r$.

Hence, since $[\![z]\!] = [\![(z_1, z_2)]\!] = \min\{ \| z_1 \|, \| z_2 \| \}$, it is clear that in particular we have

$$\| F(s, t) \| < \eta$$

for all $s, t \in X$ with $\| s \| > r$ and $\| t \| > r$.

Now, by taking $x, y \in X$ and using equation (7), we can see that

$$\begin{aligned} \| F(x, y) \| = \| F(u, v) + F(u, y-v) - F(y-v, u) \\ - F(x-u, u) - F(y-v, v) + F(x-u, y-v) + F(x+y-u-v, u+v) \| \\ \leq \| F(u, v) \| + \| F(u, y-v) | + \| F(y-v, u) \| \\ \| F(x-u, u) \| + \| F(y-v, v) | + \| F(x-u, y-v) \| + \| F(x+y-u-v, u+v) \| < 7\eta \end{aligned}$$

whenever $u, v \in X$ such that

$$\| u \| > r, \qquad \| v \| > r, \qquad \| x-u \| > r, \qquad \| y-v \| > r,$$
$$\| u+v \| > r, \qquad \| x+y-u-v \| > r.$$

Therefore, if such u and v exist, then

$$\| F(x, y) \| < 7\eta, \qquad \text{and thus} \qquad \| F(x, y) \| \leq 7\varepsilon.$$

Now, to complete the proof, it remains only to show that the required u and v exist. For this, following the arguments given [5], we can note that because of the assumed unboundedness of X there exist $u, v \in X$ such that

$$\| u \| > r + \| x \| \qquad \text{and} \qquad \| v \| > r + \| x \| + \| y \| + \| u \|.$$

Thus, we evidently have $\| u \| > r$ and $\| v \| > r$. Moreover, by using the inequality $\| s+t \| \geq \| t \| - \| s \|$, we can also see that

$$\| x-u \| \geq \| u \| - \| x \| > r + \| x \| - \| x \| = r,$$
$$\| y-v \| \geq \| v \| - \| y \| > r + \| x \| + \| y \| + \| u \| - \| x \| = r + + \| y \| + \| u \| \geq r,$$

and

$$\underline{\| u+v \|} \geq \| v \| - \| u \| > r + \| x \| + \| y \| + \| u \| - \| u \| = \underline{r + \| x \| + \| y \|} \geq r,$$
$$\| x+y-u-v \| \geq \| u+v \| - \| x+y \| > r + \| x \| + \| y \| - \| x \| - \| y \| = r.$$

Namely, by $\| x \| + \| y \| \geq \| x+y \|$, we also have $-\| x+y \| \geq -\| x \| - \| y \|$.

From this theorem, we can immediately derive

Corollary 3.5. *If Y is prenormed, F is a pseudo-cocycle and*

$$\overline{\lim_{\| z \| \to +\infty}} \| F(z) \| = 0,$$

then $F(z) = 0$ for all $z \in X^2$.

Remark 3.6. Recall that a Cauchy-difference is a symmetric cocycle. Moreover, a cocycle is both a semi-cocycle and a pseudo-cocycle.

Therefore, in Theorem 3.2 and its corollary F may, in particular, be a Cauchy-difference or a symmetric cocycle. While, in Theorems 2.3, 2.5 and 3.4 and their corollaries, F may already be an arbitrary cocycle.

4. An application of Theorem 2.3

In this section, we shall show that [37, Theorem 1] of Losonczi can be derived from the Cauchy-difference particular case Theorem 2.3 which is certainly more natural, but much weaker than that of Theorem 3.2.

For this, it is convenient to prove first the following intermediate theorem which may be of some interest for itself.

Theorem 4.1. *If F is a symmetric semicocycle and $\varepsilon \geq 0$ such there exists $S \subseteq X^2$ such that either the domain or the range of S is a bounded subset of X and*

$$\| F(x, y) \| \leq \varepsilon$$

for all $(x, y) \in S^c$, then

$$\| F(x, y) \| \leq 5\varepsilon$$

for all $x, y \in X$.

Proof. Now, by using that

$$(x, y) \in \big(S^{-1}\big)^c \implies (x, y) \notin S^{-1} \implies (y, x) \notin S \implies (y, x) \in S^c \implies \| F(y, x) \| \leq \varepsilon \implies \| F(x, y) \| \leq \varepsilon$$

for all $x, y \in X$, we can see that

$$\| F(x, y) \| \leq \varepsilon$$

also holds for all $(x, y) \in S^c \cup \big(S^{-1}\big)^c$, and thus also for all $(x, y) \in \big(S \cap S^{-1}\big)^c$.

Moreover, concerning the corresponding domains and ranges, we can see that

$$D_{S\cap S^{-1}} \subseteq D_S \cap D_{S^{-1}} = D_S \cap R_S \quad \text{and} \quad R_{S\cap S^{-1}} \subseteq R_S \cap R_{S^{-1}} = R_S \cap D_S,$$

and thus

$$S \cap S^{-1} \subseteq D_{S\cap S^{-1}} \times R_{S\cap S^{-1}} \subseteq \big(D_S \cap R_S\big)^2.$$

Now, since either D_S or R_S is bounded, we can also note that $D_S \cap R_S$ is a bounded subset of X. Therefore, there exists $r > 0$ such that $D_S \cap R_S \subset B_r(0)$. Hence, we can see that $S \cap S^{-1} \subseteq B_r(0)^2$, and thus

$$\big(B_r(0)^c \times X\big) \cup \big(X \times B_r(0)^c\big) = \big(B_r(0)^2\big)^c \subseteq \big(S \cap S^{-1}\big)^c.$$

Therefore, if $x, y \in X$ such that $\| (x, y) \| > r$ holds with

$$\| (x, y) \| = \| x \| \vee \| y \| = \max\{ \| x \|, \| y \| \},$$

i. e., either $\| x \| > r$ or $\| y \| > r$, then in particular we also have

$$(x, y) \in \big(S \cap S^{-1}\big)^c, \qquad \text{and thus} \qquad \| F(x, y) \| \leq \varepsilon.$$

Hence, it is clear that

$$\overline{\lim_{\| z \| \to +\infty}} \| F(z) \| = \inf_{r>0} \sup_{\| z \| > r} \| F(z) \| \leq \varepsilon.$$

Therefore, by Theorem 2.3, the required assertion is also true.

From this theorem, by using a straightforward generalization of Hyers's theorem [28], we can more easily infer the following generalization of [37, Theorem 1] of Losonczi, in which it is necessary to assume that the domain normed space is non-trivial in the sense that it is not $\{0\}$.

Corollary 4.2. *If f is a function of X to a Banach space Z and $\varepsilon \geq 0$ such that there exists $S \subseteq X^2$ such that either the domain or the range of S is a bounded subset of X and*

$$\| f(x+y) - f(x) - f(y) \| \leq \varepsilon$$

for all $(x, y) \in S^c$, then there exists a unique additive function g of X to Z such that

$$\| f(x) - g(x) \| \leq 5\,\varepsilon$$

for all $x, y \in X$.

Proof. Define

$$F(x, y) = f(x+y) - f(x) - f(y)$$

for all $x, y \in X$. Then, we can note that F is a symmetric cocycle such that

$$\| F(x, y) \| \leq \varepsilon$$

for all $(x, y) \in S^c$. Thus, by the corresponding particular case of Theorem 4.1, we have $\| F(x, y) \| \leq 5\,\varepsilon$, and thus

$$\| f(x+y) - f(x) - f(y) \| \leq 5\,\varepsilon$$

for all $x, y \in X$. Hence, by a straightforward generalization of Hyers's theorem [28], it is clear that the required assertion is also true.

Remark 4.3. Note that if f is an arbitrary and g is an additive function of X to Y such that

$$\| f(x) - g(x) \| \leq 5\,\varepsilon$$

for all $x \in X$, then we can only state that

$$\begin{aligned} \| f(x+y) - f(x) - f(y) \| &= \| f(x, y) - g(x+y) + g(x) - f(x) + g(y) - f(y) \| \\ &\leq \| f(x+y) - g(x+y) \| + \| g(x) - f(x) \| + \| g(y) - f(y) \| \leq 15\,\varepsilon \end{aligned}$$

for all $x, y \in X$.

Therefore, the corresponding particular case of Theorem 4.1 is sharper than Corollary 4.2. This clearly reveal that the corresponding theorems on restricted stability have to split into two parts. The same idea is also apparent from the proofs of those theorems.

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