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MILD CONTINUITY PROPERTIES OF RELATIONS AND RELATORS IN RELATOR SPACES

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Abstract. In this paper, we establish several useful consequences of the following, and some other closely related, basic definitions introduced in some former papers by the first author.

A family $\mathcal{R}$ of relations on one set X to another Y is called a relator on X to Y. Moreover, the ordered pair $(X, Y)(\mathcal{R}) = \big((X, Y), \mathcal{R}\big)$ is called a relator space.

A function $\Box$ of the class of all relator spaces to the class of all relators is called a direct unary operation for relators if, for any relator $\mathcal{R}$ on X to Y, the value $\mathcal{R}^{\Box} = \mathcal{R}^{\Box_{XY}} = \Box\big((X, Y)(\mathcal{R})\big)$ is also relator on X to Y.

If $(X, Y)(\mathcal{R})$ and $(Z, W)(\mathcal{S})$ are relator spaces and $\Box$ is a direct unary operation for relators, then a pair $(\mathcal{F}, \mathcal{G})$ of relators $\mathcal{F}$ on X to Z and $\mathcal{G}$ on Y to W is called mildly $\Box$–continuous if, under the elementwise inversion and compositions of relators, we have $\big((\mathcal{G}^{\Box})^{-1} \circ \mathcal{S}^{\Box} \circ \mathcal{F}^{\Box}\big)^{\Box} \subseteq \mathcal{R}^{\Box\Box}$.

Contents

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1. Introduction

In this paper, we continue the investigations initiated by the first author in [32, 34, 42, 29, 43, 51, 52] on the basic continuity properties of a single relation, and also of a pair of relations, on one relator (generalized uniform) space to another.

Meantime, we have observed that, much more generally, the corresponding, and some other, continuity properties of pairs of relators (families of relations) can also be naturally investigated [56, 59].

Here, a family $\mathcal{R}$ of relations on one set X to another X is called a *relator* on X to Y, and the ordered pair $(X, Y)(\mathcal{R}) = \big((X, Y), \mathcal{R}\big)$ is called a *relator space*. Thus, relator spaces are common generalizations of *ordered sets* [4], *formal contexts* [11], and *uniform spaces* [10].

Note that even on the real line $\mathbb{R}$ we already have two natural relators. Namely, $\mathcal{R} = \{\leq\}$ where $\leq$ is the usual ordering on $\mathbb{R}$, and $\mathcal{S} = \{B_r : \ r > 0\}$ where $B_r = \{x \in \mathbb{R}^2 : \ |x_1 - x_2| < r\}$. Thus, *birelator spaces* should also be studied.

In a recent paper [59], to motivate the definitions of the corresponding continuity properties of pairs of relators on one relator space to another, the first author offered the following convincing arguments.

Example 1.1. Suppose that $X = X(\leq_X)$ and $Y = Y(\leq_Y)$ are *generalized ordered sets* in the sense that $\leq_X$ and $\leq_Y$ are arbitrary relations on the sets X and Y, respectively.

Then, a function f of X to Y may be naturally called *increasing*, with respect to the inequalities $\leq_X$ and $\leq_Y$, if for every $u, v \in X$

$$u \leq_X v \quad \Longrightarrow \quad f(u) \leq_Y f(v).$$

Now, by using the more convenient notations $R = \leq_X$ and $S = \leq_Y$, the above implication can be reformulated in the form that

$$u\, R\, v \quad \Longrightarrow \quad f(u)\, S\, f(v),$$

or equivalently

$$(u, v) \in R \quad \Longrightarrow \quad \big(f(u), f(v)\big) \in S.$$

Example 1.2. Suppose that $X = X(d_X)$ and $Y = Y(d_Y)$ are *generalized metric spaces* in the sense that d_X and d_Y are arbitrary functions of X^2 and Y^2 to $[0, +\infty]$, respectively.

Then, a function f of X to Y may be naturally called *uniformly continuous*, with respect to the distance functions d_X and d_Y, if for each $s > 0$ there exists $r > 0$ such that for every $u, v \in X$

$$d_X(u, v) < r \quad \Longrightarrow \quad d_Y\big(f(u)\ f(v)\big) < s.$$

Now, by using the surroundings

$$R = B_r^{d_X} = \big\{ x \in X^2 : \quad d_X(x_1, x_2) < r \big\}$$

and

$$S = B_r^{d_Y} = \{ \, y \in Y^2 : \quad d_X(y_1, \, y_2) < s \, \} \, ,$$

the above implication can be reformulated in the form that

$$(u, \, v) \in R \qquad \Longrightarrow \qquad \big(f(u), \; f(v) \big) \in S \, .$$

The above examples clearly reveal that the seemingly quite different algebraic and topological notions such as "increasingness" and "uniform continuity" are essentially equivalent.

Moreover, they naturally lead us to the following simple unifying

Definition 1.3. Assume that $X = X(R)$ and $Y = Y(S)$ are *relational spaces* in the sense that R and S are arbitrary relations on X and Y, respectively.

Then, a function f of X to Y will be called *increasing* or *continuous*, with respect to the relations R and S, if

$$(u, \, v) \in R \qquad \Longrightarrow \qquad \big(f(u), \; f(v) \big) \in S$$

for all $u, \, v \in X$. That is, the function f is, in a certain sense, relation-preservig.

Now, by using this definition, we can easily prove the following theorem of [59] presented partly also in [57].

Theorem 1.4. *For any function f of one relational space $X(R)$ to another $Y(S)$, the following assertions are equivalent:*

(1) f *is increasing (continuous),*

(2) $f \circ R \subseteq S \circ f$, *(3)* $R \subseteq f^{-1} \circ S \circ f$,

(4) $f \circ R \circ f^{-1} \subseteq S$, *(5)* $R \circ f^{-1} \subseteq f^{-1} \circ S$.

Proof. By the corresponding definitions, it is clear that the following assertions are equivalent:

(a) $f \circ R \subseteq S \circ f$,

(b) $\forall \; u \in X$: $\qquad (f \circ R)(u) \subseteq (S \circ f)(u)$,

(c) $\forall \; u \in X$: $\qquad f\,[R(u)] \subseteq S\big(f(u)\big)$,

(d) $\forall \; u \in X$: $\qquad \forall \; v \in R(u) : \; f(v) \in S\big(f(u)\big)$,

(e) $\forall \; u, \, v \in X$: $\qquad \big((u, v) \in R \implies \big(f(u), \, f(v) \big) \in S \big)$.

Therefore, assertions (2) and (1) are equivalent.

The proofs of the remaining equivalences depend on the increasingness and associativity of composition, and the inclusions

$$\Delta_X \subseteq f^{-1} \circ f \qquad \text{and} \qquad f \circ f^{-1} \subseteq \Delta_Y \, ,$$

where Δ_X and Δ_Y are the identity functions of X and Y, respectively.

Remark 1.5. The latter inclusions indicate that assertions (2)–(5) need not be equivalent for an arbitrary relation f on $X(R)$ to $Y(S)$. Therefore, they can be used to define different increasingness or continuity properties of relations.

Remark 1.6. In [54], having in mind set-valued functions, a relation F on a generalized ordered set $X(\leq)$ to a set Y has been called *increasing* if $u \leq v$ implies $F(u) \subseteq F(v)$ for all $u, \, v \in X$.

Thus, it can be easily shown that the relation F is increasing if and only if its inverse F^{-1} is *ascending-valued* in the sense that $F^{-1}(y)$ is an ascending subset of $X(\leq)$ for all $y \in Y$.

By using the more convenient notation $R = \leq$, the latter statement can be reformulated in the form that $R[F^{-1}(y)] \subseteq F^{-1}(y)$ for all $y \in Y$. That is, $R \circ F^{-1} \subseteq F^{-1}$.

The latter inclusion can be reformulated in the form that $R \circ F^{-1} \subseteq F^{-1} \circ \Delta_Y$. This shows that the $R = \Delta_X$ and $S = \Delta_Y$ particular cases of Theorem 1.4 may also be of some interest.

In [59], it has been proved that, under the notations of Definition 1.3, we have $F \circ R \circ F^{-1} \subseteq S$ if and only if, for every $u, v \in X$, we have $F(u)\, S\, F(v)$ in the sense that $y\, S\, z$ for all $y \in F(u)$ and $z \in F(v)$.

However, it is now more important to note that, by using the corresponding particular cases of the plausible operations, defined by $\mathcal{R}^{-1} = \{ R^{-1} : \ R \in \mathcal{R} \}$,

$$\mathcal{R}^* = \big\{ S \subseteq X \times Y : \quad \exists\ R \in \mathcal{R} : \quad R \subseteq S \big\},$$

and $\mathcal{S} \circ \mathcal{R} = \big\{ S \circ R : \ R \in \mathcal{R},\ S \in \mathcal{S} \big\}$ for any relators $\mathcal{R}$ on X to Y and $\mathcal{S}$ on Y to Z, Theorem 1.4 can be reformulated in the following more instructive form.

Theorem 1.7. *If f is a function of one relational space $X(R)$ to another $Y(S)$, then under the notations*

$$\mathcal{F} = \{ f \}, \qquad \mathcal{R} = \{ R \} \qquad \text{and} \qquad \mathcal{S} = \{ S \}$$

the following assertions are equivalent:

(1) f *is increasing (continuous),*

(2) $\big(\mathcal{S}^* \circ \mathcal{F}^* \big)^* \subseteq \big(\mathcal{F}^* \circ \mathcal{R}^* \big)^*$, *(3)* $\Big(\big(\mathcal{F}^* \big)^{-1} \circ \mathcal{S}^* \circ \mathcal{F}^* \Big)^* \subseteq \mathcal{R}^{**}$,

(4) $\mathcal{S}^{**} \subseteq \Big(\mathcal{F}^* \circ \mathcal{R}^* \circ \big(\mathcal{F}^* \big)^{-1} \Big)^*$, *(5)* $\Big(\big(\mathcal{F}^* \big)^{-1} \circ \mathcal{S}^* \Big)^* \subseteq \Big(\mathcal{R}^* \circ \big(\mathcal{F}^* \big)^{-1} \Big)^*$.

Hint. The proof of the equivalences of the assertions (2)–(5) of this theorem to those of Theorem 1.4 depend on the fact that $*$ is an *inversion and composition compatible closure operation for relators* in the sense that:

(a) $\big(\mathcal{R}^* \big)^{-1} = \big(\mathcal{R}^{-1} \big)^*$ for any relator $\mathcal{R}$ on X to Y,

(b) $\mathcal{R}^* \subseteq \mathcal{S}^*$ is equivalent to $\mathcal{R} \subseteq S^*$ for any relators $\mathcal{R}$ and $\mathcal{S}$ on X to Y,

(c) $\big(\mathcal{S} \circ \mathcal{R} \big)^* = \big(\mathcal{S} \circ \mathcal{R}^* \big)^* = \big(\mathcal{S}^* \circ \mathcal{R} \big)^*$ for any relators $\mathcal{R}$ on X to Y and $\mathcal{S}$ on Y to Z.

Now, the Pexiderizations of the inclusions in Theorem 1.7, and an abstraction of the operation $*$, naturally lead us to the following straightforward extension of [43, Definition 4.1].

Definition 1.8. Suppose that $(X, Y)(\mathcal{R})$ and $(Z, W)(\mathcal{S})$ are relator spaces, $\mathcal{F}$ is a relator on X to Z, and $\mathcal{G}$ is a relator on Y to W.

Moreover, assume that $\square$ is a *direct unary operation for relators* in the sense that it is function of the class of all relator spaces to the class of all relators such that, for any relator $\mathcal{R}$ on X to Y, the value $\mathcal{R}^{\square} = \mathcal{R}^{\square_{XY}} = \square\big((X, Y)(\mathcal{R}) \big)$ is also a relator on X to Y.

Then, we say that the ordered pair

(1) $(\mathcal{F}, \mathcal{G})$ is *upper* $\square$*–continuous* with respect to the relators $\mathcal{R}$ and $\mathcal{S}$ if

$$\left(\mathcal{S}^{\square} \circ \mathcal{F}^{\square}\right)^{\square} \subseteq \left(\mathcal{G}^{\square} \circ \mathcal{R}^{\square}\right)^{\square},$$

(2) $(\mathcal{F}, \mathcal{G})$ is *mildly* $\square$*–continuous* with respect to the relators $\mathcal{R}$ and $\mathcal{S}$ if

$$\left(\left(\mathcal{G}^{\square}\right)^{-1} \circ \mathcal{S}^{\square} \circ \mathcal{F}^{\square}\right)^{\square} \subseteq \mathcal{R}^{\square\,\square},$$

(3) $(\mathcal{F}, \mathcal{G})$ is *vaguely* $\square$*–continuous* with respect to the relators $\mathcal{R}$ and $\mathcal{S}$ if

$$\mathcal{S}^{\square\,\square} \subseteq \left(\mathcal{G}^{\square} \circ \mathcal{R}^{\square} \circ \left(\mathcal{F}^{\square}\right)^{-1}\right)^{\square},$$

(4) $(\mathcal{F}, \mathcal{G})$ is *lower* $\square$*–continuous* with respect to the relators $\mathcal{R}$ and $\mathcal{S}$ if

$$\left(\left(\mathcal{G}^{\square}\right)^{-1} \circ \mathcal{S}^{\square}\right)^{\square} \subseteq \left(\mathcal{R}^{\square} \circ \left(\mathcal{F}^{\square}\right)^{-1}\right)^{\square}.$$

Remark 1.9. To keep in mind the above assumptions, for any $R \in \mathcal{R}$, $S \in \mathcal{S}$, $F \in \mathcal{F}$ and $G \in \mathcal{G}$, we can use the diagram:

$$\begin{array}{ccc} X & \xrightarrow{\;F\;} & Z \\ {\scriptstyle R}\big\downarrow & & \big\downarrow{\scriptstyle S} \\ Y & \xrightarrow{\;G\;} & W \end{array}$$

Moreover, to clarify the notion of a direct unary operation for relators, we can note that $*$ is a direct, but -1 is a non-direct unary operation for relators. Of course, if we restrict ourself to relator spaces of the simpler type $X(\mathcal{R}) = (X, X)(\mathcal{R})$, then the latter inconvenience does not occur.

Now, since there is a great number of important direct unary operations for relators, the investigation of the above continuity properties and their relationships to each other offer an exhausting work for hundreds of mathematicians.

In this paper, to let the reader feel the main directions in the above mentioned investigations, we shall only point out some basic facts concerning the various mild continuities of relators, relations, and functions.

For instance, we shall prove the following two basic theorems.

Theorem 1.10. *If in particular* $\square$ *is an inversion and composition compatible closure operation for relators, then the following assertions are equivalent:*

(1) $(\mathcal{F}, \mathcal{G})$ *is mildly* $\square$*–continuous with respect to the relators* $\mathcal{R}$ *and* $\mathcal{S}$*,*

(2) $(\mathcal{F}, \mathcal{G})$ *is properly mildly continuous with respect to the relators* $\mathcal{R}^{\square}$ *and* $\mathcal{S}$ *in the sense that* $\mathcal{G}^{-1} \circ \mathcal{S} \circ \mathcal{F} \subseteq \mathcal{R}^{\square}$*,*

(3) $(\mathcal{F}, \mathcal{G})$ *is elementwise mildly* $\square$*–continuous with respect to the relators* $\mathcal{R}$ *and* $\mathcal{S}$ *in the sense that, for any* $F \in \mathcal{F}$ *and* $G \in \mathcal{G}$*, the pair* (F, G)*, i.e., the pair* $(\{F\}, \{G\})$*, is mildly* $\square$*–continuous with respect to the relators* $\mathcal{R}$ *and* $\mathcal{S}$*.*

Theorem 1.11. *For any* $F \in \mathcal{F}$ *and* $G \in \mathcal{G}$*, the following assertions are equivalent:*

(1) $G^{-1} \circ \mathcal{S} \circ F \subseteq \mathcal{R}^{*}$*,* *(2)* (F, G) *is mildly* $*$*–continuous,*

(3) for each $S \in \mathcal{S}$ *there exists* $R \in \mathcal{R}$ *such that* $R \subseteq G^{-1} \circ S \circ F$ *,*

(4) for each $S \in \mathcal{S}$ *there exists* $R \in \mathcal{R}$ *such that for every* $x \in X$ *and* $y \in R(x)$ *we have* $G(y) \cap S\,[\,F(x)\,] \neq \emptyset$ *,*

(5) for each $S \in \mathcal{S}$ *there exists* $R \in \mathcal{R}$ *such that for every* $x \in X$ *and* $y \in R(x)$ *there exist* $z \in F(x)$ *and* $w \in G(y)$ *such that* $w \in S(z)$ *.*

In this respect, it is also worth noticing that, by using the the notations

$$\mathcal{R}^{\#} = \big\{\, S \subseteq X \times Y : \quad \forall\ A \subseteq X : \ \ \exists\ \ R \in \mathcal{R} : \quad R\,[\,A\,] \subseteq S\,[\,A\,]\,\big\}$$

for any relator $\mathcal{R}$ on X to Y, and

$$\operatorname{Int}_{\mathcal{R}}(B) = \big\{\, A \subseteq X : \quad \exists\ R \in \mathcal{R} : \quad R\,[\,A\,] \subseteq B\,\big\}$$

and

$$\operatorname{Cl}_{\mathcal{R}}(B) = \big\{\, A \subseteq X : \quad \forall\ R \in \mathcal{R} : \quad R\,[\,A\,] \cap B \neq \emptyset\,\big\}$$

for any $B \subseteq Y$, we can also prove the following

Theorem 1.12. *For any* $F \in \mathcal{F}$ *and* $G \in \mathcal{G}$ *, the following assertions are equivalent :*

(1) $G^{-1} \circ \mathcal{S} \circ F \subseteq \mathcal{R}^{\#}$*,*

(2) $(F,\, G)$ *is mildly #–continuous ,*

(3) $A \in \operatorname{Cl}_{\mathcal{R}}(B)$ *implies* $F\,[\,A\,] \in \operatorname{Cl}_{\mathcal{S}}\,(\,G\,[\,B\,]\,)$ *,*

(4) $F\,[\,A\,] \in \operatorname{Int}_{\mathcal{S}}\big(D\big)$ *implies* $A \in \operatorname{Int}_{\mathcal{R}}\big(\,G^{-1}\,[\,D\,]\,\big)$ *.*

Remark 1.13. Because of Example 1.2 and Theorem 1.12, the pair $(F,\, G)$ may be naturally called *uniformly (proximally) mildly continuous* if it is mildly $*$–continuous (#–continuous).

Unfortunately, the uniform continuity of $(F,\, G)$ can only be characterized in terms of the *convergence (adherence)* of one preordered net of sets to another, which is already a rather difficult notion.

In the subsequent preparatory sections, we shall list some basic facts on relations and relators, and structures and unary operations for relators, which are possibly unfamiliar to the reader. The proofs will be frequently omitted.

2. A few basic facts on relations

A subset F of a product set $X \times Y$ is called a *relation* on X to Y. Thus, the *empty relation* $\emptyset$ is the smallest and the *universal relation* $X \times Y$ is the largest relation on X to Y. And $\mathcal{P}(X \times Y)$ is the family of all relations on X to Y.

If in particular $F \subseteq X^{\,2}$, with $X^{\,2} = X \times X$, then we may simply say that F is a relation on X. In particular, $\Delta_X = \{\,(x,\, x) : \ \ x \in X\,\}$ is called the *identity relation* and its complement $\Delta_X^c = X^{\,2} \setminus \Delta_X$ is called the *diversity relation* on X.

If F is a relation on X to Y, then by the above definitions we can at once see that F is also a relation on $X \cup Y$. However, for our subsequent purposes, the latter view of the relation F would be quite unnatural.

If F is a relation on X to Y, then for any $x \in X$ and $A \subseteq X$ the sets $F(x) = \{\, y \in Y : \ (x,\, y) \in F\,\}$ and $F\,[\,A\,] = \bigcup_{a \in A} F(a)$ are called the *images* of x and A under F. If $(x,\, y) \in F$, then we may also write $x\, F\, y$.

Moreover, the sets $D_F = \{ x \in X : \ F(x) \neq \emptyset \}$ and $R_F = F[X]$ are called the *domain* and *range* of F. If in particular $D_F = X$, then we say that F is a relation of X to Y, or that F is a *total relation* on X to Y.

In particular, a relation f on X to Y is called a *function* if for each $x \in D_f$ there exists $y \in Y$ such that $f(x) = \{y\}$. In this case, by identifying singletons with their elements, we may simply write $f(x) = y$ in place of $f(x) = \{y\}$.

Moreover, a function $\star$ of X to itself is called a *unary operation* on X. While, a function $*$ of X^2 to X is called a *binary operation* on X. And, for any $x, y \in X$, we usually write $x^\star$ and $x * y$ instead of $\star(x)$ and $*\big((x, y)\big)$.

If F is a relation on X to Y, then $F = \bigcup_{x \in X} \{x\} \times F(x)$. Therefore, the values $F(x)$, where $x \in X$, uniquely determine F. Thus, a relation F on X to Y can also be naturally defined by specifying $F(x)$ for all $x \in X$.

For instance, the *complement relation* F^c can be naturally defined such that $F^c(x) = F(x)^c = Y \setminus F(x)$ for all $x \in X$. Thus, we also have $F^c = X \times Y \setminus F$. Moreover, it noteworthy that $F^c[A]^c = \bigcap_{a \in A} F(a)$ for all $A \subseteq X$. (See [53].)

Quite similarly, the *inverse relation* F^{-1} can be naturally defined such that $F^{-1}(y) = \{x \in X : \ y \in F(x)\}$ for all $y \in Y$. Thus, we have $F^{-1}[B] = \{x \in X : F(x) \cap B \neq \emptyset\}$ for all $B \subseteq Y$, and hence in particular $D_F = F^{-1}[Y]$.

Moreover, if in addition G is a relation on Y to Z, then the *composition relation* $G \circ F$ can be naturally defined such that $(G \circ F)(x) = G[F(x)]$ for all $x \in X$. Thus, we also have $(G \circ F)[A] = G\big[F[A]\big]$ for all $A \subseteq X$.

While, if G is a relation on Z to W, then the *box product relation* $F \boxtimes G$ can be naturally defined such that $(F \boxtimes G)(x, z) = F(x) \times G(z)$ for all $x \in X$ and $z \in Z$. Thus, we have $(F \boxtimes G)[A] = G \circ A \circ F^{-1}$ for all $A \subseteq X \times Z$. (See [53].)

Hence, by taking $A = \{(x, z)\}$, and $A = \Delta_Y$ if $Y = Z$, one can at once see that the box and composition products are actually equivalent tools. However, the box product can be immediately defined for an arbitrary family of relations too.

If F is a relation on X to Y, then a function f of D_F to Y is called a *selection* of F if $f \subseteq F$, i.e., $f(x) \in F(x)$ for all $x \in D_F$. Thus, by the Axiom of Choice, every relation has a selection. Moreover, it is the union of its selections.

For any relation F on X to Y, we may naturally define two *set-valued functions*, $F^{\triangleright}$ of X to $\mathcal{P}(Y)$ and $F^{\blacktriangleright}$ of $\mathcal{P}(X)$ to $\mathcal{P}(Y)$, such that $F^{\triangleright}(x) = F(x)$ for all $x \in X$ and $F^{\blacktriangleright}(A) = F[A]$ for all $A \subseteq X$.

Functions of X to $\mathcal{P}(Y)$ can be identified with relations on X to Y. While, functions of $\mathcal{P}(X)$ to $\mathcal{P}(Y)$ are more general objects than relations on X to Y. They were briefly called *corelations* on X to Y in [55].

Now, a relation R on X may be briefly defined to be *reflexive* if $\Delta_X \subseteq R$, and *transitive* if $R \circ R \subseteq R$. Moreover, R may be briefly defined to be *symmetric* if $R^{-1} \subseteq R$, and *antisymmetric* if $R \cap R^{-1} \subseteq \Delta_X$.

Thus, a reflexive and transitive (symmetric) relation may be called a *preorder (tolerance) relation*. And, a symmetric (antisymmetric) preorder relation may be called an *equivalence (partial order) relation*.

For instance, for $A \subseteq X$, the *Pervin relation* $R_A = A^2 \cup A^c \times X$ is a preorder relation on X. (See [18] and [48].) While, for a *pseudo-metric* d on X and $r > 0$, the *surrounding* $B_r^d = \big\{(x, y) \in X^2 : \ d(x, y) < r\big\}$ is a tolerance relation on X.

Moreover, we may recall that if $\mathcal{A}$ is a *partition* of X, i. e., a family of pairwise disjoint, nonvoid subsets of X such that $X = \bigcup \mathcal{A}$, then $S_{\mathcal{A}} = \bigcup_{A \in \mathcal{A}} A^2$ is an equivalence relation on X, which can, to some extent, be identified with $\mathcal{A}$.

According to algebra, for any relation R on X, we may naturally define $R^0 = \Delta_X$, and $R^n = R \circ R^{n-1}$ if $n \in \mathbb{N}$. Moreover, we may also naturally define $R^{\infty} = \bigcup_{n=0}^{\infty} R^n$. Thus, R^{∞} is the smallest preorder relation containing R [12].

Now, in contrast to $(F^c)^c = F$ and $(F^{-1})^{-1} = F$, we have $(R^{\infty})^{\infty} = R^{\infty}$. Moreover, analogously to $(F^c)^{-1} = (F^{-1})^c$, we also have $(R^{\infty})^{-1} = (R^{-1})^{\infty}$. Thus, in particular R^{-1} is also a preorder on X if R is a preorder on X.

3. A few basic facts on relators

A family $\mathcal{R}$ of relations on one set X to another Y is called a *relator on X to Y*. And, the ordered pair $(X, Y)(\mathcal{R}) = \big((X, Y), \mathcal{R}\big)$ is called a *relator space.* (For the origins, see [32, 38, 44, 46] and the references in [32].)

If in particular $\mathcal{R}$ is a relator on X to itself, then we may simply say that $\mathcal{R}$ is a *relator on X*. And, by identifying singletons with their elements, we may naturally write $X(\mathcal{R})$ in place of $(X, X)(\mathcal{R})$, since $(X, X) = \{\{X\}\}$.

Relator spaces of this simpler type are already substantial generalizations of the various *ordered sets* [4] and *uniform spaces* [10]. However, they are insufficient for some important purposes. (See, for instance, [11] and [43].)

A relator $\mathcal{R}$ on X to Y, or a relator space $(X, Y)(\mathcal{R})$, is called *simple* if there exists a relation R on X to Y such that $\mathcal{R} = \{R\}$. In this case, by identifying singletons with their elements, we may write $(X, Y)(R)$ in place of $(X, Y)\big(\{R\}\big)$.

According to Száz [45], a simple relator space $X(R)$ may be called a *goset* (generalized ordered set). Moreover, by Ganter and Wille [11, p. 17], a simple relator space $\big(X, Y\big)(R)$ may be called a *formal context* or *context space.*

A relator $\mathcal{R}$ on X, or a relator space $X(\mathcal{R})$, may, for instance, be naturally called *reflexive* if each member of $\mathcal{R}$ is reflexive. Thus, we may also naturally speak of *preorder, tolerance*, and *equivalence relators.*

For instance, for any family $\mathcal{A}$ of subsets of X, the family $\mathcal{R}_{\mathcal{A}} = \{R_A : A \in \mathcal{A}\}$, where $R_A = A^2 \cup A^c \times X$, is a preorder relator on X. Such relators were first used by Davis [5] and Pervin [28].

While, for any family $\mathcal{D}$ of pseudo-metrics on X, the family $\mathcal{R}_{\mathcal{D}} = \{B_r^d : r > 0, \ d \in \mathcal{D}\}$, where $B_r^d = \{(x, y) : d(x, y) < r\}$, is a tolerance relator on X. Such relators were first considered by Weil [**?**].

Moreover, if $\mathfrak{S}$ is a family of partitions of X, then the family $\mathcal{R}_{\mathfrak{S}} = \{S_{\mathcal{A}} : \mathcal{A} \in \mathfrak{S}\}$, where $S_{\mathcal{A}} = \bigcup_{A \in \mathcal{A}} A^2$, is an equivalence relator on X. Such practically important relators were first investigated by Levine [17].

A function $\square$ of the class of all relator spaces to the class of all relators is called a *direct (inderect) unary operation for relators* if, for any relator $\mathcal{R}$ on X to Y, the value $\mathcal{R}^{\square} = \mathcal{R}^{\square_{XY}} = \square\big((X, Y)(\mathcal{R})\big)$ is a relator on X to Y (on Y to X).

More generally, a function $\mathfrak{F}$ of the class of all relator spaces to some other class is called a *structure for relators* if, for any relator $\mathcal{R}$ on X to Y, the value $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_{\mathcal{R}}^{XY} = \mathfrak{F}\big((X, Y)(\mathcal{R})\big)$ is in a power set depending only on X and Y.

In accordance with [55], for a structure $\mathfrak{F}$ for relators, we say that:

(1) $\mathfrak{F}$ is *quasi-increasing* if $\mathfrak{F}_{\{R\}} \subseteq \mathfrak{F}_{\mathcal{R}}$ for any relator $\mathcal{R}$ on X to Y and $R \in \mathcal{R}$,

(2) $\mathfrak{F}$ is *increasing* if $\mathfrak{F}_{\mathcal{R}} \subseteq \mathfrak{F}_{\mathcal{S}}$ for any two relators $\mathcal{R}$ and $\mathcal{S}$ on X to Y with $\mathcal{R} \subseteq \mathcal{S}$,

(3) $\mathfrak{F}$ is *union-preserving* if $\mathfrak{F}_{\bigcup_{i \in I} \mathcal{R}_i} = \bigcup_{i \in I} \mathfrak{F}_{\mathcal{R}_i}$ for any family $\big(\mathcal{R}_i\big)_{i \in I}$ of relators on X to Y.

Thus, "union-preserving" implies "increasing" implies "quasi-increasing". Moreover, it can be shown that $\mathfrak{F}$ is union-preserving if and only if $\mathfrak{F}_{\mathcal{R}} = \bigcup_{R \in \mathcal{R}} \mathfrak{F}_{\{R\}}$ for any relator $\mathcal{R}$ on X to Y. Here, we may again write R instead of $\{R\}$.

A unary operation $\square$ for relators is called *extensive, intensive, involutive*, and *idempotent* if for any relator $\mathcal{R}$ on X to Y we have $\mathcal{R} \subseteq \mathcal{R}^{\square}$, $\mathcal{R}^{\square} \subseteq \mathcal{R}$, $\mathcal{R}^{\square\square} = \mathcal{R}$, and $\mathcal{R}^{\square\square} = \mathcal{R}^{\square}$, respectively.

In particular, an increasing idempotent operation for relators is called a *modification operation.* While, an extensive (intensive) modification operation for relators is called a *closure (interior) operation.*

Moreover, an increasing extensive (intensive) operation is called a *preclosore (preinterior) operation.* And, an extensive (intensive) idempotent operation is called a *semiclosure (semiinterior)* operation.

For instance, the functions c and -1, defined by

$$\mathcal{R}^{c} = \big\{ R^{c} : \quad R \in \mathcal{R} \big\} \qquad \text{and} \qquad \mathcal{R}^{-1} = \big\{ R^{-1} : \quad R \in \mathcal{R} \big\}$$

for any relator $\mathcal{R}$ on X to Y, are increasing involution operations for relators such that $\big(\mathcal{R}^{c}\big)^{-1} = \big(\mathcal{R}^{-1}\big)^{c}$. Thus, the operation c is *inversion compatible.*

And, the functions ∞ and ∂, defined by

$$\mathcal{R}^{\infty} = \big\{ R^{\infty} : \quad R \in \mathcal{R} \big\} \qquad \text{and} \qquad \mathcal{R}^{\partial} = \big\{ S \subseteq X^{2} : \quad S^{\infty} \in \mathcal{R} \big\}$$

for any relator $\mathcal{R}$ on X, are modification operations for relators such that, for any two relators $\mathcal{R}$ and $\mathcal{S}$ on X, we have

$$\mathcal{R}^{\infty} \subseteq \mathcal{S} \quad \iff \quad \mathcal{R} \subseteq \mathcal{S}^{\partial}.$$

Therefore, the operations ∞ and ∂ form a Galois connection [4, p. 155]. Thus, in particular $\infty\,\partial$ is a closure operation for relators such that $\infty = \infty\,\partial\infty$.

To investigate inclusions between generalized topologies derived from relations and relators, the operations ∞ and ∂ were first introduced by Mala [20] and Pataki [26], respectively. Moreover, by using several more powerful structures derived from relators, Száz [39] and Pataki [26] defined a great abundance of important closure operations for relators. Some of them were already considered by Kenyon [15] and H. Nakano and K. Nakano [23].

Beside the above mentioned unary operations, we may also naturally introduce several important binary operations for relators. For instance, for any two relators $\mathcal{R}$ on X to Y and $\mathcal{S}$ on Y to Z, we may naturally define

$$\mathcal{S} \circ \mathcal{R} = \big\{ S \circ R : \quad R \in \mathcal{R}, \quad S \in \mathcal{S} \big\}.$$

Hence, by using that $\big(S \circ R\big)^{-1} = R^{-1} \circ S^{-1}$ for all $R \in \mathcal{R}$ and $S \in \mathcal{S}$, we can easily see that $\big(\mathcal{S} \circ \mathcal{R}\big)^{-1} = \mathcal{R}^{-1} \circ \mathcal{S}^{-1}$. Moreover, it can also be easily seen that the composition of relators is also associative.

4. Some important structures for relators

If $\mathcal{R}$ is a relator on X to Y, then for any $A \subseteq X$, $B \subseteq Y$ and $x \in X$ we write:

(1) $A \in \operatorname{Int}_{\mathcal{R}}(B)$ if $R[A] \subseteq B$ for some $R \in \mathcal{R}$,

(2) $A \in \operatorname{Cl}_{\mathcal{R}}(B)$ if $R[A] \cap B \neq \emptyset$ for all $R \in \mathcal{R}$,

(3) $x \in \operatorname{int}_{\mathcal{R}}(B)$ if $\{x\} \in \operatorname{Int}_{\mathcal{R}}(B)$, (4) $x \in \operatorname{cl}_{\mathcal{R}}(B)$ if $\{x\} \in \operatorname{Cl}_{\mathcal{R}}(B)$,

(5) $B \in \mathcal{E}_{\mathcal{R}}$ if $\operatorname{int}_{\mathcal{R}}(B) \neq \emptyset$, (6) $B \in \mathcal{D}_{\mathcal{R}}$ if $\operatorname{cl}_{\mathcal{R}}(B) = X$.

Moreover, if in particular $\mathcal{R}$ is a relator on X, then for any $A \subseteq X$ we also write:

(7) $A \in \tau_{\mathcal{R}}$ if $A \in \operatorname{Int}_{\mathcal{R}}(A)$, (8) $A \in \tau\!\!\!-_{\mathcal{R}}$ if $A^c \notin \operatorname{Cl}_{\mathcal{R}}(A)$,

(9) $A \in \mathcal{T}_{\mathcal{R}}$ if $A \subseteq \operatorname{int}_{\mathcal{R}}(A)$, (10) $A \in \mathcal{F}_{\mathcal{R}}$ if $\operatorname{cl}_{\mathcal{R}}(A) \subseteq A$.

The relations $\operatorname{Int}_{\mathcal{R}}$ and $\operatorname{int}_{\mathcal{R}}$ are called *the proximal and topological interiors* induced by $\mathcal{R}$, respectively. While, the members of the families, $\tau_{\mathcal{R}}$, $\mathcal{T}_{\mathcal{R}}$ and $\mathcal{E}_{\mathcal{R}}$ are called the *proximally open, topologically open, and fat subsets* of the relator spaces $X(\mathcal{R})$ and $(X, Y)(\mathcal{R})$, respectively.

The origins of the relations Cl_{R} and $\operatorname{Int}_{\mathcal{R}}$ go back to Efremović's proximity δ [7] and Smirnov's strong inclusion $\Subset$ [30], respectively. The families $\tau_{\mathcal{R}}$ and $\mathcal{E}_{\mathcal{R}}$ were first explicitly used by the first author [38]. In particular, the practical notation $\tau\!\!\!-_{\mathcal{R}}$ has been suggested by János Kurdics.

Because of the above definitions, for any relator $\mathcal{R}$ on X to Y and $B \subseteq Y$, we have

$$\operatorname{Cl}_{\mathcal{R}}(B) = \mathcal{P}(X) \setminus \operatorname{Int}_{\mathcal{R}}\big(B^c\big) \qquad \text{and} \qquad \operatorname{cl}_{\mathcal{R}}(B) = X \setminus \operatorname{int}_{\mathcal{R}}\big(B^c\big),$$

and

$$\mathcal{D}_{\mathcal{R}} = \big\{ D \subset Y : \quad D^c \notin \mathcal{E}_{\mathcal{R}} \big\} = \big\{ D \subset Y : \quad \forall\ E \in \mathcal{E}_{\mathcal{R}} : \ E \cap D \neq \emptyset \big\}.$$

Moreover, if in particular, $\mathcal{R}$ is a relator on X, then we also have

$$\tau\!\!\!-_{\mathcal{R}} = \big\{ A \subset X : \quad A^c \in \tau_{\mathcal{R}} \big\} \quad \text{and} \quad \mathcal{F}_{\mathcal{R}} = \big\{ A \subset X : \quad A^c \in \mathcal{T}_{\mathcal{R}} \big\}.$$

In this respect, it is also worth mentioning that, for any relator $\mathcal{R}$ on X to Y we have

$$\operatorname{Cl}_{\mathcal{R}^{-1}} = \operatorname{Cl}_{\mathcal{R}}^{-1} \qquad \text{and} \qquad \operatorname{Int}_{\mathcal{R}^{-1}} = \mathcal{C}_Y \circ \operatorname{Int}_{\mathcal{R}}^{-1} \circ \mathcal{C}_X,$$

where $\mathcal{C}_X(A) = X \setminus A$ for all $A \subseteq X$. Moreover, in particular, for any relator $\mathcal{R}$ on X, we have $\tau\!\!\!-_{R} = \tau_{R^{-1}}$. Therefore, the proximal closures and proximally open sets are usually more convenient tools than the topological closures (proximal interiors) and topologically open sets, respectively.

The fat sets are frequently also more convenient tools than the topologically open sets [36]. For instance, if $\leq$ is a certain order relation on X, then $\mathcal{T}_{\leq}$ and $\mathcal{E}_{\leq}$ are just the families of all *ascending and residual subsets* of the ordered set $X(\leq)$, respectively.

To clarify the advantage of fat sets over the open ones, we can also note that if in particular $X = \mathbb{R}$, and R is a relation on X such that

$$R(x) = \{ x - 1 \} \cup [\, x\,,\, +\infty\, [$$

for all $x \in X$, then $\mathcal{T}_R = \{\emptyset, X\}$, but $\mathcal{E}_R$ is quite large family. Namely, the supersets of each $R(x)$, with $x \in X$, are also in $\mathcal{E}_R$.

If $\mathcal{R}$ is a relator on X to Y, and Φ and Ψ are relations on a relator space $\Gamma(\mathcal{U})$ to X and Y, respectively, then by using the relation $(\Phi \otimes \Psi)$, defined such that

$$(\Phi \otimes \Psi)(\gamma) = \Phi(\gamma) \times \Psi(\gamma)$$

for all $\gamma \in \Gamma$, we may also define

(11) $\Phi \in \operatorname{Lim}_{\mathcal{R}}(\Psi)$ if $(\Phi \otimes \Psi)^{-1}[R] \in \mathcal{E}_{\mathcal{U}}$ for all $R \in \mathcal{R}$,

(12) $\Phi \in \operatorname{Adh}_{\mathcal{R}}(\Psi)$ if $(\Phi \otimes \Psi)^{-1}[R] \in \mathcal{D}_{\mathcal{U}}$ for all $R \in \mathcal{R}$.

Now, for any $A \subseteq X$, we may also naturally write:

(13) $A \in \operatorname{lim}_{\mathcal{R}}(\Psi)$ if $A_\Gamma \in \operatorname{Lim}_{\mathcal{R}}(\Psi)$, (14) $A \in \operatorname{adh}_{\mathcal{R}}(\Psi)$ if $A_\Gamma \in \operatorname{Adh}_{\mathcal{R}}(\Psi)$,

where A_Γ is a relation on Γ to X such that $A_\Gamma(\gamma) = A$ for all $\gamma \in \Gamma$.

The *big limit relation* $\operatorname{Lim}_{\mathcal{R}}$, suggested by Efremović and Švarc [8], is, in general, a much stronger tool in the relator space $(X, Y)(\mathcal{R})$ than the *big closure and interior relations* $\operatorname{Cl}_{\mathcal{R}}$ and $\operatorname{Int}_{\mathcal{R}}$ suggested by Efremović [7] and Smirnov [30].

Namely, it can be shown that, for any $A \subseteq X$ and $B \subseteq Y$, we have $A \in \operatorname{Cl}_{\mathcal{R}}(B)$ if and only if there exist a preordered set $\Gamma(\leq)$ and functions φ and ψ of Γ to A and B, respectively, such that $\varphi \in \operatorname{Lim}_{\mathcal{R}}(\psi)$ $\big(\varphi \in \operatorname{Adh}_{\mathcal{R}}(\psi)\big)$.

To check the less obvious part of this statement, note that if $A \in \operatorname{Cl}_{\mathcal{R}}(B)$, then for each $R \in \mathcal{R}$ we have $R[A] \cap B \neq \emptyset$. Therefore, there exist $x_R \in A$ and $y_R \in B$ such that $y_R \in R(x_R)$.

Now, by defining $\varphi(R) = x_R$ and $\psi(R) = y_R$ for all $R \in \mathcal{R}$, and moreover $R_1 \leq R_2$ if $R_1, R_2 \in \mathcal{R}$ such that $R_2 \subseteq R_1$, we can easily see that $\mathcal{R}(\leq)$ is a partially ordered set, and in addition to $\varphi(R) = x_R \in A$ and $\psi(R) = y_R \in B$ we also have

$$(\varphi \otimes \psi)(R) = \big(\varphi(R),\ \psi(R)\big) = (x_R, y_R) \in R,$$

and thus $R \in (\varphi \otimes \psi)^{-1}[R]$ for all $R \in \mathcal{R}$.

Therefore, if $R \in \mathcal{R}$, then for every $S \in \mathcal{R}$, with $S \geq R$, i.e., $S \subseteq R$, we have

$$S \in (\varphi \otimes \psi)^{-1}[S] \subseteq (\varphi \otimes \psi)^{-1}[R],$$

and thus $[R, +\infty[\ \subseteq (\varphi \otimes \psi)^{-1}[R]$. This, shows that $(\varphi \otimes \psi)^{-1}[R]$, is a residual, i.e., a fat subset of $\mathcal{R}(\leq)$. Thus, by the definition of the relation $\operatorname{Lim}_{\mathcal{R}}$, we have $\varphi \in \operatorname{Lim}_{\mathcal{R}}(\psi)$.

Note that, to prove the corresponding statement for the relation $\operatorname{Adh}_{\mathcal{R}}$, we have to define $R_1 \leq R_1$ for all $R_1, R_2 \in \mathcal{R}$. Therefore, for our present purposes, partially ordered sets are not, in general, sufficient.

Finally, we note that if $\mathcal{R}$ is a relator on X to Y, then according to [44] for any $A \subseteq X$, $B \subseteq Y$, $x \in X$, and $y \in Y$ we may also naturally write:

(a) $B \in \operatorname{Ub}_{\mathcal{R}}(A)$ and $A \in \operatorname{Lb}_{\mathcal{R}}(B)$ if $A \times B \subseteq R$ for some $R \in \mathcal{R}$;

(b) $y \in \operatorname{ub}_{\mathcal{R}}(A)$ if $\{y\} \in \operatorname{Ub}_{\mathcal{R}}(B)$; (c) $x \in \operatorname{lb}_{\mathcal{R}}(B)$ if $\{x\} \in \operatorname{Lb}_{\mathcal{R}}(A)$,

(d) $A \in \mathcal{U}_{\mathcal{R}}$ if $\operatorname{ub}_{\mathcal{R}}(A) \neq \emptyset$, (e) $B \in \mathcal{L}_{\mathcal{R}}$ if $\operatorname{lb}_{\mathcal{R}}(B) \neq \emptyset$.

Moreover, in particular $\mathcal{R}$ is a relator on X, then for any $A \subseteq X$ we may also naturally define:

(f) $\max_{\mathcal{R}}(A) = A \cap \operatorname{ub}_{\mathcal{R}}(A)$; (g) $\min_{\mathcal{R}}(A) = A \cap \operatorname{lb}_{\mathcal{R}}(A)$;

(h) $\operatorname{Max}_{\mathcal{R}}(A) = \mathcal{P}(A) \cap \operatorname{Ub}_{\mathcal{R}}(A)$; (i) $\operatorname{Min}_{\mathcal{R}}(A) = \mathcal{P}(A) \cap \operatorname{Lb}_{\mathcal{R}}(A)$,

and thus also

(j) $\sup_{\mathcal{R}}(A) = \min_{\mathcal{R}}\big(\operatorname{ub}_{\mathcal{R}}(A)\big)$; (k) $\inf_{\mathcal{R}}(A) = \max_{\mathcal{R}}\big(\operatorname{lb}_{\mathcal{R}}(A)\big)$.

(l) $\operatorname{Sup}_{\mathcal{R}}(A) = \operatorname{Min}_{\mathcal{R}}\big(\operatorname{Lb}_{\mathcal{R}}(A)\big)$; (m) $\operatorname{Inf}_{\mathcal{R}}(A) = \operatorname{Max}_{\mathcal{R}}\big(\operatorname{Ub}_{\mathcal{R}}(A)\big)$.

Now, analogously to the families $\tau_{\mathcal{R}}$ and $\mathcal{T}_{\mathcal{R}}$, we may also naturally define:

(n) $A \in \mathfrak{u}_{\mathcal{R}}$ if $A \in \operatorname{Ub}_{\mathcal{R}}(A)$;

(o) $A \in \mathfrak{U}_{\mathcal{R}}$ if $A \subseteq \operatorname{ub}_{\mathcal{R}}(A)$; (p) $A \in \mathfrak{L}_{\mathcal{R}}$ if $A \subseteq \operatorname{lb}_{\mathcal{R}}(A)$.

Thus, for instance, it can be shown that

$$A \in \mathfrak{u}_{\mathcal{R}} \iff A \in \operatorname{Lb}_{\mathcal{R}}(A) \iff A \in \operatorname{Min}_{\mathcal{R}}(A) \iff A \in \operatorname{Inf}_{\mathcal{R}}(A),$$

and $\mathfrak{u}_{\mathcal{R}} = \operatorname{Min}_{\mathcal{R}}\big(\mathcal{P}(X)\big) = \operatorname{Max}_{\mathcal{R}}\big(\mathcal{P}(X)\big)$. Moreover, $\operatorname{Lb}_{\mathcal{R}} = \operatorname{Ub}_{\mathcal{R}^{-1}} = \operatorname{Ub}_{\mathcal{R}}^{-1}$.

However, the above algebraic structures are not independent of the former topological ones. Namely, if R is a relation on X to Y, then for any $A \subseteq X$ and $B \subseteq Y$ we have

$$\begin{aligned} A \times B \subseteq R \iff \forall\, a \in A: \ B \subseteq R(a) \iff \forall\, a \in A: \ R(a)^c \subseteq B^c \\ \iff \forall\, a \in A: \ R^c(a) \subseteq B^c \iff R^c[A] \subseteq B^c. \end{aligned}$$

Therefore, if $\mathcal{R}$ is a relator on X to Y, then by the corresponding definitions, for any $A \subseteq X$ and $B \subseteq Y$, we also have

$$A \in \operatorname{Lb}_{\mathcal{R}}(B) \iff A \in \operatorname{Int}_{\mathcal{R}^c}(B^c) \iff A \in \big(\operatorname{Int}_{\mathcal{R}^c} \circ \mathcal{C}_Y\big)(B).$$

Hence, we can already infer that

$$\operatorname{Lb}_{\mathcal{R}} = \operatorname{Int}_{\mathcal{R}^c} \circ \mathcal{C}_Y, \qquad \text{and} \qquad \operatorname{Int}_{\mathcal{R}} = \operatorname{Lb}_{\mathcal{R}^c} \circ \mathcal{C}_Y.$$

Therefore, in contrast to a common belief, some algebraic and topological structures are just as closely related to each other by the above equalities, and their particular cases

$$\operatorname{lb}_{\mathcal{R}} = \operatorname{int}_{\mathcal{R}^c} \circ \mathcal{C}_Y, \qquad \text{and} \qquad \operatorname{int}_{\mathcal{R}} = \operatorname{lb}_{\mathcal{R}^c} \circ \mathcal{C}_Y,$$

as the exponential and the trigonometric functions are by the celebrated Euler formulas [31, p. 227].

5. Increasingly regular structures for relators

According to [56], we shall also use the following

Definition 5.1. If $\mathfrak{F}$ is a structure and $\square$ is an unary operation for relators, then, we say that:

(1) $\mathfrak{F}$ is *increasingly upper $\square$–regular* if $\mathfrak{F}_{\mathcal{R}} \subseteq \mathfrak{F}_{\mathcal{S}}$ implies $\mathcal{R} \subseteq \mathcal{S}^{\square}$ for any two relators $\mathcal{R}$ and $\mathcal{S}$ on X to Y,

(2) $\mathfrak{F}$ is *increasingly lower $\square$–regular* if $\mathcal{R} \subseteq \mathcal{S}^{\square}$ implies $\mathfrak{F}_{\mathcal{R}} \subseteq \mathfrak{F}_{\mathcal{S}}$ for any two relators $\mathcal{R}$ and $\mathcal{S}$ on X to Y.

Remark 5.2. Now, the structure $\mathfrak{F}$ may be naturally called *increasingly* $\Box$*–regular* if it is increasingly both upper and lower $\Box$–regular.

Moreover, for instance, $\mathfrak{F}$ may also be naturally called *increasingly regular* if it is increasingly $\Box$–regular for some operation $\Box$ for relators.

Remark 5.3. If $\mathfrak{F}$ is an increasingly $\Box$–regular structure for relators, then because of the fundamental work of Pataki [26] we may also say that the pair $(\mathfrak{F}, \Box)$ is an *increasing Pataki connection for relators.*

In the theory of relators, increasing Pataki connections can also be most naturally obtained from the increasing Galois ones according to [47].

Definition 5.4. For any structure $\mathfrak{F}$ for relators, we define an operation $\Box_{\mathfrak{F}}$ for relators such that

$$\mathcal{R}^{\Box_{\mathfrak{F}}} = \left\{ S \subseteq X \times Y : \quad \mathfrak{F}_S \subseteq \mathfrak{F}_{\mathcal{R}} \right\}$$

for any relator $\mathcal{R}$ on X to Y.

Remark 5.5. Note that if in particular the structure $\mathfrak{F}$ is increasing, then by the above definition, the operation $\Box_{\mathfrak{F}}$ is also increasing.

The appropriateness of Definition 5.4, is also apparent from the following extensions and supplements of the corresponding results of Pataki [26], which were mainly proved in [56]. The proofs will only be included here for the reader's convenience.

Theorem 5.6. *If $\mathfrak{F}$ is a structure and $\Box$ is an operation for relators such that $\mathfrak{F}$ is increasingly $\Box$–regular, then $\Box = \Box_{\mathfrak{F}}$.*

Proof. By the corresponding definitions,

$$S \in \mathcal{R}^{\Box} \iff \{S\} \subseteq \mathcal{R}^{\Box} \iff \mathfrak{F}_{\{S\}} \subseteq \mathfrak{F}_{\mathcal{R}} \iff \mathfrak{F}_S \subseteq \mathfrak{F}_{\mathcal{R}} \iff S \in \mathcal{R}^{\Box_{\mathfrak{F}}}$$

for any relator $\mathcal{R}$ and relation S on X to Y.

From this theorem, we can immediately derive the following three corollaries.

Corollary 5.7. *If $\mathfrak{F}$ is a structure for relators, then there exists at most one operation $\Box$ for relators such that $\mathfrak{F}$ is increasingly $\Box$–regular.*

Corollary 5.8. *If $\mathfrak{F}$ is an increasingly regular structure for relators, then $\Box = \Box_{\mathfrak{F}}$ is the unique operation for relators such that $\mathfrak{F}$ is increasingly $\Box$–regular.*

Corollary 5.9. *A structure $\mathfrak{F}$ for relators is increasingly regular if and only if it is increasingly $\Box_{\mathfrak{F}}$–regular.*

Theorem 5.10. *If $\mathfrak{F}$ is a quasi-increasing structure for relators, then*

(1) $\mathfrak{F}$ is increasingly upper $\Box_{\mathfrak{F}}$–regular, (2) $\Box_{\mathfrak{F}}$ is extensive.

Proof. If $\mathcal{R}$ and $\mathcal{S}$ are relators on X to Y such that $\mathfrak{F}_{\mathcal{R}} \subseteq \mathfrak{F}_{\mathcal{S}}$, then by the quasi-increasingness of $\mathfrak{F}$, for any $R \in \mathcal{R}$, we also have $\mathfrak{F}_R \subseteq \mathfrak{F}_{\mathcal{S}}$. Hence, by Definition 5.4, we can already infer that $R \in \mathcal{S}^{\Box_{\mathfrak{F}}}$. Therefore, $\mathcal{R} \subseteq \mathcal{S}^{\Box_{\mathfrak{F}}}$. Thus, by Definition 5.1, assertion (1) is true.

Now, from the inclusion $\mathfrak{F}_{\mathcal{R}} \subseteq \mathfrak{F}_{\mathcal{R}}$, by using (1), we can infer that $\mathcal{R} \subseteq \mathcal{R}^{\Box_{\mathfrak{F}}}$. Therefore, (2) is also true.

From this theorem, by using Corollary 5.9 and Remark 5.5, we can immediately derive the following two corollaries.

Corollary 5.11. *A quasi-increasing structure $\mathfrak{F}$ for relators is increasingly regular if and only if it is increasingly lower $\Box_{\mathfrak{F}}$–regular.*

Corollary 5.12. *If in particular $\mathfrak{F}$ is an increasing structure for relators, then $\Box_{\mathfrak{F}}$ is already preclosure operation.*

Theorem 5.13. *If $\mathfrak{F}$ is an union-preserving structure for relators, then*

(1) $\mathfrak{F}$ is increasingly $\Box_{\mathfrak{F}}$–regular, *(2) $\Box_{\mathfrak{F}}$ is a closure operation.*

Proof. Suppose that $\mathcal{R}$ and $\mathcal{S}$ are relators on X to Y such that $\mathcal{R} \subseteq \mathcal{S}^{\Box_{\mathfrak{F}}}$, and $\Omega \in \mathfrak{F}_{\mathcal{R}}$. Then, since $\mathfrak{F}_{\mathcal{R}} = \bigcup_{R \in \mathcal{R}} \mathfrak{F}_R$, there exists $R \in \mathcal{R}$ such that $\Omega \in \mathfrak{F}_R$. Now, since $\mathcal{R} \subseteq \mathcal{S}^{\Box_{\mathfrak{F}}}$, we also have $R \in \mathcal{S}^{\Box_{\mathfrak{F}}}$. Hence, by Definition 5.4, we can infer that $\mathfrak{F}_R \subseteq \mathfrak{F}_{\mathcal{S}}$. Therefore, we also have $\Omega \in \mathfrak{F}_{\mathcal{S}}$. Consequently, $\mathfrak{F}_{\mathcal{R}} \subseteq \mathfrak{F}_{\mathcal{S}}$. This shows that $\mathfrak{F}$ is increasingly lower $\Box_{\mathfrak{F}}$–semiregular. Hence, by Theorem 5.10, we can see that (1) is true.

Now, assertion (2) will follow from (1) by the forthcoming Theorem 5.17.

Thus, in particular, we also have the following

Corollary 5.14. *Every union-preserving structure $\mathfrak{F}$ for relators is increasingly regular.*

In [56], by using the arguments of [49], we have also proved the following three theorems.

Theorem 5.15. *If $\mathfrak{F}$ is an increasingly $\Box$–regular structure for relators, then*

(1) $\Box$ is extensive, *(2) $\mathfrak{F}$ is increasing,*

(3) $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_{\mathcal{R}^{\Box}}$ for any relator $\mathcal{R}$ on X to Y.

Proof. If $\mathcal{R}$ is a relator on X to Y, then from the inclusion $\mathcal{R}^{\Box} \subseteq \mathcal{R}^{\Box}$, by using the increasing lower $\Box$–regularity of $\mathfrak{F}$, we can infer that $\mathfrak{F}_{\mathcal{R}^{\Box}} \subseteq \mathfrak{F}_{\mathcal{R}}$.

On the other hand, from the inclusion $\mathfrak{F}_{\mathcal{R}} \subseteq \mathfrak{F}_{\mathcal{R}}$, by using the increasing upper $\Box$–regularity of $\mathfrak{F}$, we can infer that $\mathcal{R} \subseteq \mathcal{R}^{\Box}$. Therefore, (1) is true.

Now, if $\mathcal{S}$ is a relator on X to Y such that $\mathcal{R} \subseteq \mathcal{S}$, then by using (1) we can see that $\mathcal{R} \subseteq \mathcal{S}^{\Box}$ also holds. Hence, by using the increasing lower $\Box$–regularity of $\mathfrak{F}$, we can infer that $\mathfrak{F}_{\mathcal{R}} \subseteq \mathfrak{F}_{\mathcal{S}}$. Therefore, assertion (2) is also true.

Now, from the inclusion $\mathcal{R} \subseteq \mathcal{R}^{\Box}$, by using (2), we can infer that $\mathfrak{F}_{\mathcal{R}} \subseteq \mathfrak{F}_{\mathcal{R}^{\Box}}$. Therefore, assertion (3) is also true.

From this theorem, by using Theorem 5.13, we can immediately derive

Corollary 5.16. *If $\mathfrak{F}$ is a union-preserving structure for relators, then $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_{\mathcal{R}^{\Box_{\mathfrak{F}}}}$ for any relator $\mathcal{R}$ on X to Y.*

Theorem 5.17. *For an operation $\Box$ for relators, the following assertions are equivalent:*

(1) $\Box$ is a closure operation *(2) $\Box$ is increasingly $\Box$–regular,*

(3) there exists an increasingly $\Box$–regular structure $\mathfrak{F}$ for relators.

Proof. To prove the implication (3) $\Longrightarrow$ (1), note that if (3) holds, then by Theorem 5.15 the operation $\Box$ is extensive. Moreover, for any relator $\mathcal{R}$ on X to Y, we have $\mathfrak{F}_{\mathcal{R}^{\Box}} = \mathfrak{F}_{\mathcal{R}}$. Hence, by taking $\mathcal{R}^{\Box}$ in place of $\mathcal{R}$, we can see that $\mathfrak{F}_{\mathcal{R}^{\Box\Box}} = \mathfrak{F}_{\mathcal{R}^{\Box}}$, and thus $\mathfrak{F}_{\mathcal{R}^{\Box\Box}} = \mathfrak{F}_{\mathcal{R}}$ also holds. Hence, by using the increasing upper $\square$–regularity of $\mathfrak{F}$, we can already infer that $\square$ is increasingly upper semi-idempotent in the sense that $\mathcal{R}^{\square\square} \subseteq \mathcal{R}^{\square}$. Now, by the extensivity of $\square$, it is clear that the corresponding equality is also true. That is, $\square$ is idempotent.

Thus, to obtain (1), it remains only to show that $\square$ is also increasing. For this, note that if $\mathcal{R}$ and $\mathcal{S}$ are relators on X to Y such that $\mathcal{R} \subseteq \mathcal{S}$, then by Theorem 5.15 we also have $\mathfrak{F}_{\mathcal{R}} \subseteq \mathfrak{F}_{\mathcal{S}}$. Moreover, we have $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_{\mathcal{R}^{\square}}$, and thus also $\mathfrak{F}_{\mathcal{R}^{\square}} \subseteq \mathfrak{F}_{\mathcal{S}}$. Hence, by using the increasing upper $\square$–regularity of $\mathfrak{F}$, we can already infer that $\mathcal{R}^{\square} \subseteq \mathcal{S}^{\square}$.

From this theorem, by Theorem 5.6, it is clear that in particular we also have

Corollary 5.18. *If $\Diamond$ is a closure operation for relators, then $\Diamond = \square_{\Diamond}$.*

Moreover, from Theorem 5.17, by using Definition 5.1, we can immediately derive

Corollary 5.19. *For a structure $\mathfrak{F}$ and an operation $\square$ for relators, the following assertions are equivalent;*

(1) $\mathfrak{F}$ is increasingly $\square$–regular,

(2) $\square$ is a closure operation, and for any two relators $\mathcal{R}$ and $\mathcal{S}$ on X to Y, we have $\mathfrak{F}_{\mathcal{R}} \subseteq \mathfrak{F}_{\mathcal{S}}$ if and only if $\mathcal{R}^{\square} \subseteq \mathcal{S}^{\square}$.

Theorem 5.20. *For a structure $\mathfrak{F}$ and an operation $\square$ for relators, the following assertions are equivalent:*

(1) $\mathfrak{F}$ is increasingly $\square$–regular,

(2) $\mathfrak{F}$ is increasing, and for every relator $\mathcal{R}$ on X to Y, $\mathcal{S} = \mathcal{R}^{\square}$ is the largest relator on X to Y such that $\mathfrak{F}_{\mathcal{S}} \subseteq \mathfrak{F}_{\mathcal{R}}$.

Proof. If (1) holds, then by Theorem 5.15 the structure $\mathfrak{F}$ is increasing, and for any relator $\mathcal{R}$ on X to Y we have $\mathfrak{F}_{\mathcal{R}^{\square}} = \mathfrak{F}_{\mathcal{R}}$. Moreover, if $\mathcal{S}$ is a relator on X to Y such that $\mathfrak{F}_{\mathcal{S}} \subseteq \mathfrak{F}_{\mathcal{R}}$, then by using the upper $\square$–regularity of $\mathfrak{F}$ we can see that $\mathcal{S} \subseteq \mathcal{R}^{\square}$. Thus, in particular, (2) also holds.

On the other hand, if (2) holds, and $\mathcal{R}$ and $\mathcal{S}$ are relators on X to Y such that $\mathfrak{F}_{\mathcal{S}} \subseteq \mathfrak{F}_{\mathcal{R}}$, then from the assumed maximality property of $\mathcal{R}^{\square}$ we can see that $\mathcal{S} \subseteq \mathcal{R}^{\square}$. Therefore, $\mathfrak{F}$ is upper $\square$–regular.

Conversely, if $\mathcal{R}$ and $\mathcal{S}$ are relators on X to Y such that $\mathcal{S} \subseteq \mathcal{R}^{\square}$, then by using the assumed increasingness of $\mathfrak{F}$ we can see that $\mathfrak{F}_{\mathcal{S}} \subseteq \mathfrak{F}_{\mathcal{R}^{\square}}$. Hence, by the assumed inclusion $\mathfrak{F}_{\mathcal{R}^{\square}} \subseteq \mathfrak{F}_{\mathcal{R}}$, it follows that $\mathfrak{F}_{\mathcal{S}} \subseteq \mathfrak{F}_{\mathcal{R}}$. Therefore, $\mathfrak{F}$ is also lower $\square$–regular, and thus (1) also holds.

From this theorem, by Theorem 5.15, it is clear that we also have

Corollary 5.21. *If $\mathfrak{F}$ is a $\square$–regular structure for relators, then for any relator $\mathcal{R}$ on X to Y, $\mathcal{S} = \mathcal{R}^{\square}$ is the largest relator on X to Y such that $\mathfrak{F}_{\mathcal{S}} = \mathfrak{F}_{\mathcal{R}}$.*

Now, by Theorem 5.6, it is clear that in particular we also have

Corollary 5.22. *If $\mathfrak{F}$ is a regular structure for relators, then for any relator $\mathcal{R}$ on X to Y $\mathcal{S} = \mathcal{R}^{\square_{\mathfrak{F}}}$ is the largest relator on X to Y such that $\mathfrak{F}_{\mathcal{S}} \subseteq \mathfrak{F}_{\mathcal{R}}$ $\big(\mathfrak{F}_{\mathcal{S}} = \mathfrak{F}_{\mathcal{R}}\big)$.*

Hence, by Theorem 5.13, it is clear that more specially we also have

Corollary 5.23. *If $\mathfrak{F}$ is a union-preserving structure for relators, then for any relator $\mathcal{R}$ on X to Y, $S = \mathcal{R}^{\square_{\mathfrak{F}}}$ is the largest relator on X to Y such that $\mathfrak{F}_{\mathcal{S}} \subseteq \mathfrak{F}_{\mathcal{R}}$ $\big(\mathfrak{F}_{\mathcal{S}} = \mathfrak{F}_{\mathcal{R}}\big)$.*

Finally, we note that, analogously to [26, Theorem 1.5], the following theorem is also true.

Theorem 5.24. *For an unary operation $\Box$ for relators, the following assertions are equivalent:*

(1) $\Box$ *is a semiclosure,*

(2) for every relator $\mathcal{R}$ *on* X *to* Y, $\mathcal{S} = \mathcal{R}^{\Box}$ *is the largest relator on* X *to* Y *such that* $\mathcal{R}^{\Box} = \mathcal{S}^{\Box}$,

(3) there exists a structure $\mathfrak{F}$ *for relators such that, for every relator* $\mathcal{R}$ *on* X *to* Y, $\mathcal{S} = \mathcal{R}^{\Box}$ *is the largest relator on* X *to* Y *such that* $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_{\mathcal{S}}$.

Remark 5.25. If $\mathfrak{F}$ is a structure for for relators, then two relators $\mathcal{R}$ and $\mathcal{S}$ on X to Y are called $\mathfrak{F}$*–equivalent* if $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_{\mathcal{S}}$.

Moreover, the relator $\mathcal{R}$ is called $\mathfrak{F}$*-simple* if it is $\mathfrak{F}$–equivalent to a singleton relator. And, in particular, $\mathcal{R}$ is called *properly simple* if it is $\mathfrak{F}$-simple with $\mathfrak{F}$ being the identity operation for relators.

6. Some important unary operations for relators

Definition 6.1. For any relator $\mathcal{R}$ on X to Y, the relators

$$\begin{aligned}
\mathcal{R}^{*} &= \big\{ S \subseteq X \times Y : \quad \exists\ R \in \mathcal{R} : \quad R \subseteq S \big\}, \\
\mathcal{R}^{\#} &= \big\{ S \subseteq X \times Y : \quad \forall\ A \subseteq X : \quad \exists\ R \in \mathcal{R} : \quad R[A] \subseteq S[A] \big\}, \\
\mathcal{R}^{\wedge} &= \big\{ S \subseteq X \times Y : \quad \forall\ x \in X : \quad \exists\ R \in \mathcal{R} : \quad R(x) \subseteq S(x) \big\},
\end{aligned}$$

and

$$\mathcal{R}^{\vartriangle} = \big\{ S \subseteq X \times Y : \quad \forall\ x \in X : \quad \exists\ u \in X : \quad \exists\ R \in \mathcal{R} : \quad R(u) \subseteq S(x) \big\}$$

are called the *uniform, proximal, topological, and paratopological closures (refinements)* of the relator $\mathcal{R}$, respectively.

Remark 6.2. Thus, we evidently have

$$\mathcal{R} \subseteq \mathcal{R}^{*} \subseteq \mathcal{R}^{\#} \subseteq \mathcal{R}^{\wedge} \subseteq \mathcal{R}^{\vartriangle}.$$

Moreover, if in particular $\mathcal{R}$ is a relator on X, then we can easily see that

$$\mathcal{R}^{\infty} \subseteq \mathcal{R}^{*\infty} \subseteq \mathcal{R}^{\infty *} \subseteq \mathcal{R}^{*}.$$

Remark 6.3. However, it is now more important to note that, because of the corresponding definitions of Section 4, we also have

$$\begin{aligned}
\mathcal{R}^{\#} &= \big\{ S \subseteq X \times Y : \quad \forall\ A \subseteq X : \quad A \in \operatorname{Int}_{R}\big(S[A] \big) \big\}, \\
\mathcal{R}^{\wedge} &= \big\{ S \subseteq X \times Y : \quad \forall\ x \in X : \quad x \in \operatorname{int}_{R}\big(S(x) \big) \big\}, \\
\mathcal{R}^{\vartriangle} &= \big\{ S \subseteq X \times Y : \quad \forall\ x \in X : \quad S(x) \in \mathcal{E}_{\mathcal{R}} \big\}.
\end{aligned}$$

Now, by using this remark and Definition 5.4, we can easily prove the following

Theorem 6.4. *For any relator* $\mathcal{R}$ *on* X *to* Y, *we have*

(1) $\mathcal{R}^{\#} = \mathcal{R}^{\Box_{\mathrm{Int}}}$, *(2)* $\mathcal{R}^{\wedge} = \mathcal{R}^{\Box_{\mathrm{int}}}$, *(3)* $\mathcal{R}^{\vartriangle} = \mathcal{R}^{\Box_{\mathcal{E}}}$,

Proof. We shall only prove that $\mathcal{R}^{\square_{\text{Int}}} \subseteq \mathcal{R}^{\#}$. The proof of the converse inclusion, and those of (2) and (3), will be left to the reader.

For this, we can note that if $S \in \mathcal{R}^{\square_{\text{Int}}}$, then by Definition 5.4 S is a relation on X to Y such that $\text{Int}_S \subseteq \text{Int}_{\mathcal{R}}$, and so $\text{Int}_S(B) \subseteq \text{Int}_{\mathcal{R}}(B)$ for all $B \subseteq Y$.

Thus, in particular, for any $A \subseteq X$, we have $\text{Int}_S\big(S[A]\big) \subseteq \text{Int}_{\mathcal{R}}\big(S[A]\big)$. Hence, by using that $A \in \text{Int}_S\big(S[A]\big)$, we can already infer that $A \in \text{Int}_{\mathcal{R}}\big(S[A]\big)$. Therefore, by Remark 6.3, $S \in \mathcal{R}^{\#}$ also holds.

From this theorem, by using Theorem 5.13, we can immediately derive

Theorem 6.5. $\#$, $\wedge$, *and* $\triangle$ *are closure operations for relators.*

Proof. By the corresponding definitions, it is clear that

$$\text{Int}_{\mathcal{R}} = \bigcup_{R \in \mathcal{R}} \text{Int}_R, \qquad \text{int}_{\mathcal{R}} = \bigcup_{R \in \mathcal{R}} \text{int}_R, \qquad \text{and} \qquad \mathcal{E}_{\mathcal{R}} = \bigcup_{R \in \mathcal{R}} \mathcal{E}_R$$

for any relator $\mathcal{R}$ on X to Y.

Therefore, the structures Int, int, and $\mathcal{E}$ are union-preserving. Thus, by Theorem 5.13, the operations $\square_{\text{Int}}$, $\square_{\text{int}}$, and $\square_{\mathcal{E}}$ are closures. Therefore, by Theorem 6.4, the required assertions are also true.

Remark 6.6. By using the definition of the operation $*$, we can easily see that $*$ is also a closure operation for relators.

It can actually be derived from the structures Lim and Adh. While, the structures lim and adh lead only to the operation $\wedge$.

Now, by using Remark 6.2 and Theorem 6.5, we can also easily prove

Theorem 6.7. *For any relator* $\mathcal{R}$ *on* X *to* Y*, we have*

(1) $\mathcal{R}^{\#} = \big(\mathcal{R}^{*}\big)^{\#} = \big(\mathcal{R}^{\#}\big)^{*}$,

(2) $\mathcal{R}^{\wedge} = \big(\mathcal{R}^{\Diamond}\big)^{\wedge} = \big(\mathcal{R}^{\wedge}\big)^{\Diamond}$ *with* $\Diamond = *$ *or* $\#$,

(3) $\mathcal{R}^{\triangle} = \big(\mathcal{R}^{\Diamond}\big)^{\triangle} = \big(\mathcal{R}^{\triangle}\big)^{\Diamond}$ *with* $\Diamond = *$, $\#$ *or* $\wedge$.

Proof. To prove (1), note that, by Remark 6.2 and Theorem 6.5, we have

$$\mathcal{R}^{\#} \subseteq \big(\mathcal{R}^{\#}\big)^{*} \subseteq \mathcal{R}^{\#\#} = \mathcal{R}^{\#} \qquad \text{and} \qquad \mathcal{R}^{\#} \subseteq \mathcal{R}^{*\#} \subseteq \mathcal{R}^{\#\#} = \mathcal{R}^{\#}.$$

Therefore, the corresponding equalities are also true.

Remark 6.8. By using Remark 6.2 and we can also easily prove that

(1) $\mathcal{R}^{*\infty} = \mathcal{R}^{\infty*\infty}$, (2) $\mathcal{R}^{\infty*} = \mathcal{R}^{*\infty*}$.

However, it is now more important to note that, by using Theorems 6.4, 5.13 and 5.17 and Corollary 5.23, we can also easily establish the following two theorems.

Theorem 6.9. *For any two relators* $\mathcal{R}$ *and* $\mathcal{S}$ *on* X *to* Y*, we have*

(1) $\mathcal{S} \subseteq \mathcal{R}^{\#} \iff \mathcal{S}^{\#} \subseteq \mathcal{R}^{\#} \iff \text{Int}_{\mathcal{S}} \subseteq \text{Int}_{\mathcal{R}} \iff \text{Cl}_{\mathcal{R}} \subseteq \text{Cl}_{\mathcal{S}}$,

(2) $\mathcal{S} \subseteq \mathcal{R}^{\wedge} \iff \mathcal{S}^{\wedge} \subseteq \mathcal{R}^{\wedge} \iff \text{int}_{\mathcal{S}} \subseteq \text{int}_{\mathcal{R}} \iff \text{cl}_{\mathcal{R}} \subseteq \text{cl}_{\mathcal{S}}$,

(3) $\mathcal{S} \subseteq \mathcal{R}^{\triangle} \iff \mathcal{S}^{\triangle} \subseteq \mathcal{R}^{\triangle} \iff \mathcal{E}_{\mathcal{S}} \subseteq \mathcal{E}_{\mathcal{R}} \iff \mathcal{D}_{\mathcal{R}} \subseteq \mathcal{D}_{\mathcal{S}}$.

Corollary 6.10. *For any two relators* $\mathcal{R}$ *and* $\mathcal{S}$ *on* X*,*

(1) $\mathcal{S} \subseteq \mathcal{R}^{\#} \implies \tau_{\mathcal{S}} \subseteq \tau_{\mathcal{R}} \iff \bar{\tau}_{\mathcal{S}} \subseteq \bar{\tau}_{\mathcal{R}}$,

(2) $\mathcal{S} \subseteq \mathcal{R}^{\wedge} \implies \mathcal{T}_{\mathcal{S}} \subseteq \mathcal{T}_{\mathcal{R}} \iff \mathcal{F}_{\mathcal{S}} \subseteq \mathcal{F}_{\mathcal{R}}$,

Theorem 6.11. *For any relator* $\mathcal{R}$ *on* X *to* Y,

(1) $\mathcal{S}=\mathcal{R}^{\#}$ *is the largest relator on* X *to* Y *such that* $\operatorname{Int}_{\mathcal{S}}\subseteq\operatorname{Int}_{\mathcal{R}}$ $\big(\operatorname{Int}_{\mathcal{S}}=\operatorname{Int}_{\mathcal{R}}\big)$, *or equivalently* $\operatorname{Cl}_{\mathcal{R}}\subseteq\operatorname{Cl}_{\mathcal{S}}$ $\big(\operatorname{Cl}_{\mathcal{R}}=\operatorname{Cl}_{\mathcal{S}}\big)$,

(2) $\mathcal{S}=\mathcal{R}^{\wedge}$ *is the largest relator on* X *to* Y *such that* $\operatorname{int}_{\mathcal{S}}\subseteq\operatorname{int}_{\mathcal{R}}$ $\big(\operatorname{int}_{\mathcal{S}}=\operatorname{int}_{\mathcal{R}}\big)$, *or equivalently* $\operatorname{cl}_{\mathcal{R}}\subseteq\operatorname{cl}_{\mathcal{S}}$ $\big(\operatorname{cl}_{\mathcal{R}}=\operatorname{cl}_{\mathcal{S}}\big)$,

(3) $\mathcal{S}=\mathcal{R}^{\triangle}$ *is the largest relator on* X *to* Y *such that* $\mathcal{E}_{\mathcal{S}}\subseteq\mathcal{E}_{\mathcal{R}}$ $\big(\mathcal{E}_{\mathcal{S}}=\mathcal{E}_{\mathcal{R}}\big)$, *or equivalently* $\mathcal{D}_{\mathcal{R}}\subseteq\mathcal{D}_{\mathcal{S}}$ $\big(\mathcal{D}_{\mathcal{R}}=\mathcal{D}_{\mathcal{S}}\big)$.

Corollary 6.12. *For any relator* $\mathcal{R}$ *on* X, *we have*

(1) $\tau_{\mathcal{R}}=\tau_{\mathcal{R}^{\#}}$, *(2)* $\mathcal{T}_{\mathcal{R}}=\mathcal{T}_{\mathcal{R}^{\wedge}}$.

Remark 6.13. To prove the above two theorems and their corollaries, recall that

$$\operatorname{Cl}_{\mathcal{R}}=\big(\operatorname{Int}_{\mathcal{R}}\circ\mathcal{C}_Y\big)^c,\quad \operatorname{cl}_{\mathcal{R}}=\big(\operatorname{int}_{\mathcal{R}}\circ\mathcal{C}_Y\big)^c,\quad\text{and}\quad \mathcal{D}_{\mathcal{R}}=\big\{\,B\subseteq Y:\ \ B^c\notin\mathcal{E}_{\mathcal{R}}\,\big\}$$

for any relator $\mathcal{R}$ on X to Y, and in particular

$$\overline{\tau}_{\mathcal{R}}=\big\{\,A\subseteq X:\ \ A^c\in\tau_{\mathcal{R}}\,\big\}\qquad\text{and}\qquad \mathcal{F}_{\mathcal{R}}=\big\{\,A\subseteq X:\ \ A^c\in\mathcal{T}_{\mathcal{R}}\,\big\}$$

for any relator $\mathcal{R}$ on X.

Concerning the operations $\wedge$ and $\triangle$, we can also prove the following straightforward extensions of [32, Theorem 6.7] and [27, Theorem 5.16].

Theorem 6.14. *If* $\mathcal{R}$ *is a nonvoid relator on* X *to* Y *and* $B\subseteq Y$, *then*

(1) $\operatorname{Int}_{\mathcal{R}^{\wedge}}(B)=\mathcal{P}\big(\operatorname{int}_{\mathcal{R}}(B)\big)$, *(2)* $\operatorname{Cl}_{\mathcal{R}^{\wedge}}(B)=\mathcal{P}\big(\operatorname{cl}_{\mathcal{R}}(B)^c\big)^c$.

Proof. To prove the less obvious part of (1), note that if $A\in\mathcal{P}\big(\operatorname{int}_{\mathcal{R}}(B)\big)$, i.e., $A\subseteq\operatorname{int}_{\mathcal{R}}(B)$, then for each $x\in A$ there exists $R_x\in\mathcal{R}$ such that $R_x(x)\subseteq B$. Hence, by defining $S(x)=R_x(x)$ if $x\in A$, and $S(x)=Y$ if $x\in A^c$, we can see that $S\in\mathcal{R}^{\wedge}$ and $S[A]\subseteq B$. Therefore, $A\in\operatorname{Int}_{\mathcal{R}^{\wedge}}(B)$ also holds.

Note that if $\mathcal{R}$ is not supposed to be nonvoid, then instead of (1) we can only prove that $\mathcal{P}\big(\operatorname{int}_{\mathcal{R}}(B)\big)=\operatorname{Int}_{\mathcal{R}^{\wedge}}(B)\cup\{\emptyset\}$.

Corollary 6.15. *If* $\mathcal{R}$ *is a nonvoid relator on* X, *then*

(1) $\tau_{\mathcal{R}^{\wedge}}=\mathcal{T}_{\mathcal{R}}$, *(2)* $\overline{\tau}_{\mathcal{R}^{\wedge}}=\mathcal{F}_{\mathcal{R}}$.

Corollary 6.16. *If* $\mathcal{R}$ *is a nonvoid relator on* X, *then*

(1) $\mathcal{T}_{\mathcal{R}}=\bigcup_{R\in\mathcal{R}^{\wedge}}\mathcal{T}_R$, *(2)* $\mathcal{F}_{\mathcal{R}}=\bigcup_{R\in\mathcal{R}^{\wedge}}\mathcal{F}_R$.

Remark 6.17. Note that if in particular $\mathcal{R}$ is a relator on X such that $\mathcal{R}=\emptyset$, then by the definition of $\mathcal{T}_{\mathcal{R}}$ we have $\mathcal{T}_{\mathcal{R}}=\{\emptyset\}$.

Moreover, if in addition $X\neq\emptyset$, then by the definition of $\mathcal{R}^{\wedge}$ we also have $\mathcal{R}^{\wedge}=\emptyset$. Thus, $\bigcup_{R\in\mathcal{R}^{\wedge}}\mathcal{T}_R=\emptyset$.

Therefore, if $\mathcal{R}=\emptyset$, but $X\neq\emptyset$, then the equalities stated in Corollary 6.16, and thus also those stated in Corollary 6.15 and Theorem 6.14, do not hold.

Theorem 6.18. *If* $\mathcal{R}$ *is a nonvoid relator on* X *to* Y *and* $B\subseteq Y$, *then*

(1) $\operatorname{Int}_{\mathcal{R}^{\triangle}}(B)=\{\emptyset\}$ *if* $B\notin\mathcal{E}_{\mathcal{R}}$ *and* $\operatorname{Int}_{\mathcal{R}^{\triangle}}(B)=\mathcal{P}(X)$ *if* $B\in\mathcal{E}_{\mathcal{R}}$;

(2) $\operatorname{Cl}_{\mathcal{R}^{\triangle}}(B)=\emptyset$ *if* $B\notin\mathcal{D}_{\mathcal{R}}$ *and* $\operatorname{Cl}_{\mathcal{R}^{\triangle}}(B)=\mathcal{P}(X)\setminus\{\emptyset\}$ *if* $B\in\mathcal{D}_{\mathcal{R}}$.

Proof. If $A \in \operatorname{Int}_{\mathcal{R}^{\vartriangle}}(B)$, then there exists $S \in \mathcal{R}^{\vartriangle}$ such that $S[A] \subseteq B$. Therefore, if $A \neq \emptyset$, then there exists $x \in X$ such that $S(x) \subseteq B$. Hence, since $S(x) \in \mathcal{E}_{\mathcal{R}}$, it is clear that $B \in \mathcal{E}_{\mathcal{R}}$. Therefore, the first part of (1) is true.

To prove the second part of (1), it is enough to note only that if $B \in \mathcal{E}_{\mathcal{R}}$, then $R = X \times B \in \mathcal{R}^{\vartriangle}$ such that $R[A] \subseteq B$, and thus $A \in \operatorname{Int}_{\mathcal{R}^{\vartriangle}}(B)$ for all $A \subseteq X$.

Corollary 6.19. *If $\mathcal{R}$ is a nonvoid relator on X to Y and $B \subseteq Y$, then*

(1) $\operatorname{cl}_{\mathcal{R}^{\vartriangle}}(B) = \emptyset$ *if* $B \notin \mathcal{D}_{\mathcal{R}}$ *and* $\operatorname{Cl}_{\mathcal{R}^{\vartriangle}}(B) = X$ *if* $B \in \mathcal{D}_{\mathcal{R}}$,

(2) $\operatorname{int}_{\mathcal{R}^{\vartriangle}}(B) = \emptyset$ *if* $B \notin \mathcal{E}_{\mathcal{R}}$ *and* $\operatorname{int}_{\mathcal{R}^{\vartriangle}}(B) = X$ *if* $B \in \mathcal{E}_{\mathcal{R}}$.

Corollary 6.20. *If $\mathcal{R}$ is a nonvoid relator on X, then*

(1) $\tau_{\mathcal{R}^{\vartriangle}} = \mathcal{T}_{\mathcal{R}^{\vartriangle}} = \mathcal{E}_{\mathcal{R}} \cup \{\emptyset\}$, *(2)* $\tau\!\!\!-_{\mathcal{R}^{\vartriangle}} = \mathcal{F}_{\mathcal{R}^{\vartriangle}} = \big(\mathcal{P}(X) \setminus \mathcal{D}_{\mathcal{R}}\big) \cup \{X\}$.

Proof. To check the first part of (1), because of Corollary 6.15 and Theorem 6.7, we can also note that $\tau_{\mathcal{R}^{\vartriangle}} = \tau_{\mathcal{R}^{\vartriangle\wedge}} = \mathcal{T}_{\mathcal{R}^{\vartriangle}}$.

Remark 6.21. Note that if in particular $\mathcal{R}$ is a relator on X such that $\mathcal{R} = \emptyset$, then by the definition of $\mathcal{E}_{\mathcal{R}}$ we have $\mathcal{E}_{\mathcal{R}} = \emptyset$. Moreover, if in addition $X \neq \emptyset$, then we also have $\mathcal{R}^{\vartriangle} = \emptyset$. Hence, by the definition of $\tau_{\mathcal{R}}$, we can again see that $\tau_{\mathcal{R}^{\wedge}} = \emptyset$.

Therefore, if $\mathcal{R} = \emptyset$, but $X \neq \emptyset$, then the assertions (1) and (2) of Corollary 6.20, and thus also those of Corollary 6.19 and Theorem 6.18 do not hold.

However, the second equalities in the assertions (1) and (2) of Corollary 6.20 do not require relator $\mathcal{R}$ to be nonvoid. Moreover, we can also prove the following

Theorem 6.22. *If $\mathcal{R}$ is a total relator on X, then*

(1) $\mathcal{E}_{\mathcal{R}} = \mathcal{T}_{\mathcal{R}^{\vartriangle}} \setminus \{\emptyset\}$, *(2)* $\mathcal{D}_{\mathcal{R}} = \mathcal{F}^{c}_{\mathcal{R}^{\vartriangle}} \cup \{X\}$.

Remark 6.23. A relator $\mathcal{R}$ on X to Y is called *total* if each member R of $\mathcal{R}$ is a total relation in the sense that the whole X is the domain of R.

By using the corresponding definitions, it can be easily seen that the relator $\mathcal{R}$ is total if and only if $\emptyset \notin \mathcal{E}_{\mathcal{R}}$ ($Y \in \mathcal{D}_{\mathcal{R}}$), or equivalently $\mathcal{D}_{\mathcal{R}} \neq \emptyset$ ($\mathcal{E}_{\mathcal{R}} \neq \mathcal{P}(Y)$).

In this respect, it is also noteworthy that conversely we have $\emptyset \notin \mathcal{D}_{\mathcal{R}}$ ($Y \in \mathcal{E}_{\mathcal{R}}$), or equivalently $\mathcal{E}_{\mathcal{R}} \neq \emptyset$ ($\mathcal{D}_{\mathcal{R}} \neq \mathcal{P}(Y)$), if and only if $X \neq \emptyset$ and $\mathcal{R} \neq \emptyset$.

7. Some further important unary operations for relators

In addition to Theorem 6.11, we can also prove the following

Theorem 7.1. *If R is a relation on X, then $S = R^{\infty}$ is the largest relation on X such that $\tau_R \subseteq \tau_S$ $\big(\tau_R = \tau_S\big)$, or equivalently $\tau\!\!\!-_R \subseteq \tau\!\!\!-_S$ $\big(\tau\!\!\!-_R = \tau\!\!\!-_S\big)$.*

Proof. If $A \in \tau_{R^{\infty}}$, then by the corresponding definitions we have $R^{\infty}[A] \subseteq A$. Hence, by using that $R \subseteq R^{\infty}$, and thus $R[A] \subseteq R^{\infty}[A]$, we can already infer that $R[A] \subseteq A$. Therefore, $A \in \tau_R$ also holds.

While, if $A \in \tau_R$ holds, then we have $R[A] \subseteq A$. Hence, by induction, we can see that $R^n[A] \subseteq A$ for all $n \in \mathbb{N}$. Now, since $R^0[A] = \Delta_X[A] = A$, we can already state that

$$R^{\infty}[A] = \left(\bigcup_{n=0}^{\infty} R^n\right)[A] = \bigcup_{n=0}^{\infty} R^n[A] \subseteq \bigcup_{n=0}^{\infty} A = A.$$

Therefore, $A \in \tau_{R^\infty}$ also holds.

The above arguments show that $\tau_{\mathcal{R}} = \tau_{\mathcal{R}^\infty}$. Therefore, to complete the proof of the first statement of the theorem, it remains to show only that if S is a relation on X such that $\tau_R \subseteq \tau_S$, then we necessarily have $S \subseteq R^\infty$.

For this, note that if $x \in X$, then because of the inclusion $R \subseteq R^\infty$ and the transitivity of R^∞ we have

$$R\,[\,R^\infty(x)\,] \subseteq R^\infty\,[\,R^\infty(x)\,] = \big(R^\infty \circ R^\infty\big)(x) \subseteq R^\infty(x)\,.$$

Therefore, $R^\infty(x) \in \tau_{\mathcal{R}}$. Hence, by using the assumption $\tau_R \subseteq \tau_S$, we can already infer that $R^\infty(x) \in \tau_S$, and thus $S\,[\,R^\infty(x)\,] \subseteq R^\infty(x)$. Now, by using the reflexivity of R^∞, we can see that $S(x) \subseteq R^\infty(x)$ also holds.

Remark 7.2. This theorem, and the fact that

$$R^\infty(x) = \bigcap\,\big\{\,A \in \tau_R : \quad x \in A\,\big\}$$

for all $x \in X$, was first proved by Mala [20].

Hence, we can immediately infer that

$$R^\infty = \bigcap\,\big\{\,R_A : \quad A \in \tau_R\,\big\}, \qquad \text{where} \qquad R_A = A^2 \cup A^c \times X\,.$$

Now, as an immediate consequence of Theorem 7.1, we can also state

Corollary 7.3. *For any relator $\mathcal{R}$ on X, we have*

(1) $\tau_{\mathcal{R}} = \tau_{\mathcal{R}^\infty}$, *(2)* $\bar{\tau}_{\mathcal{R}} = \bar{\tau}_{\mathcal{R}^\infty}$.

Proof. By the corresponding definitions, we have $\mathcal{R}^\infty = \{R^\infty : \ R \in \mathcal{R}\}$,

$$\tau_{\mathcal{R}} = \bigcup_{R \in \mathcal{R}} \tau_R\,, \qquad \text{and} \qquad \bar{\tau}_{\mathcal{R}} = \bigcup_{R \in \mathcal{R}} \bar{\tau}_R\,.$$

for any relator $\mathcal{R}$ on X. Thus, Theorem 7.1 can be applied to get the required equalities.

However, it now more important to note that, in addition to Theorem 6.4, we can also prove the following

Theorem 7.4. *For any relator $\mathcal{R}$ on X to Y, we have*

$$\mathcal{R}^{\square_\tau} = \mathcal{R}^{\#\partial}\,.$$

Proof. If $S \in \mathcal{R}^{\#\partial}$, then by the definition of the operation ∂ we have $S^\infty \in \mathcal{R}^{\#}$. Hence, by using Theorem 7.1 and Corollary 6.10, we can already see that $\tau_S = \tau_{S^\infty} \subseteq \tau_{\mathcal{R}}$. Therefore, by Definition 5.4, $S \in \mathcal{R}^{\square_\tau}$ also holds.

Conversely, if $S \in \mathcal{R}^{\square_\tau}$, then Definition 5.4 S is a relation on X to Y such that $\tau_S \subseteq \tau_{\mathcal{R}}$. Therefore, $A \in \tau_S$ implies $A \in \tau_{\mathcal{R}}$.

On the other hand, if $A \subseteq X$, then by using that $S \subseteq S^\infty$ and S^∞ is transitive, we can note that

$$S\left[\,S^\infty\,[\,A\,]\,\right] \subseteq S^\infty\left[\,S^\infty\,[\,A\,]\,\right] = \big(\mathcal{S}^\infty \circ S^\infty\big)\,[\,A\,] \subseteq S^\infty\,[\,A\,]\,,$$

and thus $S^\infty[\,A\,] \in \tau_S$.

Therefore, by the inclusion $\tau_S \subseteq \tau_{\mathcal{R}}$, for any $A \subseteq X$, we also have $S^\infty[\,A\,] \in \tau_{\mathcal{R}}$, and thus $\operatorname{Int}_{\mathcal{R}}\big(S^\infty[\,A\,]\big)$. Hence, by using that $A \subseteq \mathcal{S}^\infty\,[\,A\,]$, we can infer that $A \in \operatorname{Int}_{\mathcal{R}}\big(S^\infty[\,A\,]\big)$ also holds. Therefore, by Remark 6.3, $S^\infty \in \mathcal{R}^{\#}$, and thus $S \in \mathcal{R}^{\#\partial}$ also holds.

Now, by using Theorems 7.4, 5.13 and 5.17 and Corollary 5.25, we can easily establish the following counterparts of Theorems 6.5, 6.9 and 6.11.

Theorem 7.5. *The following assertions are true:*

(1) $\# \partial$ *is a closure operation for relators,*

(2) for any two relators $\mathcal{R}$ *and* $\mathcal{S}$ *on* X, *we have*

$$\mathcal{S} \subseteq \mathcal{R}^{\# \partial} \iff \mathcal{S}^{\# \partial} \subseteq \mathcal{R}^{\# \partial} \iff \tau_{\mathcal{S}} \subseteq \tau_{\mathcal{R}} \iff \mathcal{T}_{\mathcal{S}} \subseteq \mathcal{T}_{\mathcal{R}} ,$$

(3) for any relator $\mathcal{R}$ *on* X, $\mathcal{S} = \mathcal{R}^{\# \partial}$ *is the largest relator on* X *such that* $\tau_{\mathcal{S}} \subseteq \tau_{\mathcal{R}}$ $\big(\tau_{\mathcal{S}} = \tau_{\mathcal{R}} \big)$, *or equivalently* $\mathcal{T}_{\mathcal{S}} \subseteq \mathcal{T}_{\mathcal{R}}$ $\big(\mathcal{T}_{\mathcal{S}} = \mathcal{T}_{\mathcal{R}} \big)$.

Remark 7.6. The above two theorems and the next theorem were first proved by Pataki [26] and Mala [20], respectively, in somewhat different forms.

Theorem 7.7. *The following assertions are true:*

(1) $\# \infty$ *is a modification operation for relators,*

(2) for any two relators $\mathcal{R}$ *and* $\mathcal{S}$ *on* X, *we have*

$$\tau_{\mathcal{S}} \subseteq \tau_{\mathcal{R}} \iff \mathcal{T}_{\mathcal{S}} \subseteq \mathcal{T}_{\mathcal{R}} \iff \mathcal{S}^{\infty} \subseteq \mathcal{R}^{\#} \iff \mathcal{S}^{\# \infty} \subseteq \mathcal{R}^{\#} \iff \mathcal{S}^{\# \infty} \subseteq \mathcal{R}^{\# \infty}.$$

(3) for any relator $\mathcal{R}$ *on* X, $\mathcal{S} = \mathcal{R}^{\# \infty}$ *is the largest preorder relator on* X *such that* $\tau_{\mathcal{S}} \subseteq \tau_{\mathcal{R}}$ $\big(\tau_{\mathcal{S}} = \tau_{\mathcal{R}} \big)$, *or equivalently* $\mathcal{T}_{\mathcal{S}} \subseteq \mathcal{T}_{\mathcal{R}}$ $\big(\mathcal{T}_{\mathcal{S}} = \mathcal{T}_{\mathcal{R}} \big)$.

Proof. If $\mathcal{R}$ and $\mathcal{S}$ are relators on X, then by Theorem 7.5 and the definition of the operation ∂, we have

$$\tau_{\mathcal{S}} \subseteq \tau_{\mathcal{R}} \iff \mathcal{S} \subseteq \mathcal{R}^{\# \partial} \iff \mathcal{S}^{\infty} \subseteq \mathcal{R}^{\#} .$$

Moreover, from Corollary 6.12, we can see that $\tau_{\mathcal{S}} = \tau_{\mathcal{S}^{\#}}$. Therefore, we also have

$$\tau_{\mathcal{S}} \subseteq \tau_{\mathcal{R}} \iff \tau_{\mathcal{S}^{\#}} \subseteq \tau_{\mathcal{R}} \iff \mathcal{S}^{\# \infty} \subseteq \mathcal{R}^{\#} .$$

Furthermore, since ∞ is modification operation, we can note that

$$\mathcal{S}^{\# \infty} \subseteq \mathcal{R}^{\#} \implies \mathcal{S}^{\# \infty \infty} \subseteq \mathcal{R}^{\# \infty} \implies \mathcal{S}^{\# \infty} \subseteq \mathcal{R}^{\# \infty} .$$

Moreover, by using Remark 6.2 and Theorem 6.7, we can also easily that

$$\mathcal{S}^{\# \infty} \subseteq \mathcal{R}^{\# \infty} \implies \mathcal{S}^{\# \infty} \subseteq \mathcal{R}^{\# *} \implies \mathcal{S}^{\# \infty} \subseteq \mathcal{R}^{\#} .$$

Therefore, assertion (2) is true.

On the other hand, if $\mathcal{R}$ is a relator on X and $\mathcal{S} = \mathcal{R}^{\# \infty}$, then from Corollaries 7.3 and 6.12 we can see that $\tau_{\mathcal{S}} = \tau_{\mathcal{R}}$. Hence, by using assertion (2) we can infer that $\mathcal{S}^{\# \infty} = \mathcal{R}^{\# \infty}$, and thus $\big(\mathcal{R}^{\# \infty} \big)^{\# \infty} = \mathcal{R}^{\# \infty}$. Thus, since the operation $\#$ and ∞ are increasing, assertion (1) is also true.

Now, to prove the first part assertion (3), it remains only to note only that $\mathcal{R}$ is an arbitrary and $\mathcal{S}$ is a preorder relator on relator on X such that $\tau_{\mathcal{S}} \subseteq \tau_{\mathcal{R}}$, then by assertion (2) we have $\mathcal{S} = \mathcal{S}^{\infty} \subseteq \mathcal{R}^{\#}$, and thus also $\mathcal{S} = \mathcal{S}^{\infty} \subseteq \mathcal{R}^{\# \infty}$.

Remark 7.8. In this respect, it is worth noticing that, for any relator $\mathcal{R}$ on X, the following assertions are also equivalent:

(1) $S^{\# \partial} \subseteq \mathcal{R}^{\# \partial}$, (2) $S^{\# \infty} \subseteq \mathcal{R}^{\# \infty}$, (3) $\mathcal{S}^{\infty \#} \subseteq \mathcal{R}^{\infty \#}$.

The advantage of the modification operations $\# \infty$ and $\infty \#$ over the closure operation $\# \partial$ lies mainly in the fact that, in contrast to $\# \partial$, they are stable in the sense that they leave the relator $\{ X^2 \}$ fixed for any set X.

Now, in addition to Theorem 7.4, we can also prove the following

Theorem 7.9. *For any relator* $\mathcal{R}$ *on* X *to* Y, *we have*

$$\mathcal{R}^{\,\square\,\mathcal{T}} = \mathcal{R}^{\,\wedge\,\partial}\,.$$

Proof. If $\mathcal{R} \neq \emptyset$, then by the corresponding definitions, Corollary 6.18 and Theorems 7.5 and 6.7, it is clear that

$$S \in \mathcal{R}^{\,\square\,\mathcal{T}} \iff \mathcal{T}_S \subseteq \mathcal{T}_{\mathcal{R}} \iff \tau_S \subseteq \tau_{\mathcal{R}^{\wedge}} \iff S \in \mathcal{R}^{\,\wedge\,\#\,\partial} \iff S \in \mathcal{R}^{\,\wedge\,\partial}.$$

While, if $\mathcal{R} = \emptyset$, then by using the corresponding definitions we can see that

$$\mathcal{R}^{\,\square\,\mathcal{T}} = \emptyset \quad \text{if} \quad X \neq \emptyset \qquad \text{and} \qquad \mathcal{R}^{\,\square\,\mathcal{T}} = \{\emptyset\} \quad \text{if} \quad X = \emptyset$$

and

$$\mathcal{R}^{\,\wedge\,\partial} = \emptyset \quad \text{if} \quad X \neq \emptyset \qquad \text{and} \qquad \mathcal{R}^{\,\wedge\,\partial} = \{\emptyset\} \quad \text{if} \quad X = \emptyset\,.$$

Therefore, the required equality is again true.

Unfortunately, by the following example, the structures $\mathcal{T}$ and $\mathcal{F}$ are not union-preserving.

Example 7.10. For any set X, with $\operatorname{card}(X) > 2$, there exists an equivalence relator $\mathcal{R} = \{R_1, R_2\}$ on X such that $\mathcal{T}_{\mathcal{R}} \neq \mathcal{T}_{R_1} \cup \mathcal{T}_{R_2}$ and $\mathcal{F}_{\mathcal{R}} \neq \mathcal{F}_{R_1} \cup \mathcal{F}_{R_2}$.

Namely, if $x_1 \in X$ and $x_2 \in X \setminus \{x_1\}$, then by defining

$$R_i = \{x_i\}^2 \cup \big(X \setminus \{x_i\}\big)^2$$

for all $i = 1, 2$ we can see that $\{x_1, x_2\} \in \mathcal{T}_{\mathcal{R}} \setminus \big(\mathcal{T}_{R_1} \cup \mathcal{T}_{R_2}\big)$.

Therefore, because of Theorems 7.8 and 5.10 and Corollary 5.12, we can only state the following

Theorem 7.11. *The following assertions are true:*

(1) $\wedge\,\partial$ *is a preclosure operation for relators,*

(2) *for any two relators* $\mathcal{R}$ *and* $\mathcal{S}$ *on* X, *we have*
$$\mathcal{T}_{\mathcal{S}} \subseteq \mathcal{T}_{\mathcal{R}} \iff \mathcal{F}_{\mathcal{S}} \subseteq \mathcal{F}_{\mathcal{R}} \implies \mathcal{S}^{\wedge} \subseteq \mathcal{R}^{\,\wedge\,\partial} \implies \mathcal{S}^{\,\wedge\,\partial} \subseteq \mathcal{R}^{\,\wedge\,\partial}.$$

Remark 7.12. If X is a set with $\operatorname{card}(X) > 2$, then by using the equivalence relator $\mathcal{R} = \{X^2\}$, considered first by Mala [20, Example 5.3], it can be shown that the operation $\wedge\,\partial$ is not idempotent [26, Example 7.2].

Therefore, by Theorems 5.17 and 5.6, the structure $\mathcal{T}$ is not regular. Moreover, by Theorem 5.24, there does not exist a structure $\mathfrak{F}$ for relators such that, for every relator $\mathcal{R}$ on X, $\mathcal{S} = \mathcal{R}^{\,\wedge\,\partial}$ is the largest relator on X such that $\mathfrak{F}_{\mathcal{S}} = \mathfrak{F}_{\mathcal{R}}$.

However, from Theorem 7.6, by using Corollary 6.15, we can easily derive the following theorem of Mala [20].

Theorem 7.13. *The following assertions are true:*

(1) $\wedge\,\infty$ *is a modification operation for relators,*

(2) *for any two relators* $\mathcal{R}$ *and* $\mathcal{S}$ *on* X, *we have*
$$\mathcal{T}_{\mathcal{S}} \subseteq \mathcal{T}_{\mathcal{R}} \iff \mathcal{F}_{\mathcal{S}} \subseteq \mathcal{F}_{\mathcal{R}} \iff \mathcal{S}^{\,\wedge\,\infty} \subseteq \mathcal{R}^{\wedge} \iff \mathcal{S}^{\,\wedge\,\infty} \subseteq \mathcal{R}^{\,\wedge\,\infty}.$$

(3) *for any relator* $\mathcal{R}$ *on* X, $\mathcal{S} = \mathcal{R}^{\,\wedge\,\infty}$ *is the largest preorder relator on* X *such that* $\mathcal{T}_{\mathcal{S}} \subseteq \mathcal{T}_{\mathcal{R}}$ $\big(\mathcal{T}_{\mathcal{S}} = \mathcal{T}_{\mathcal{R}}\big)$, *or equivalently* $\mathcal{F}_{\mathcal{S}} \subseteq \mathcal{F}_{\mathcal{R}}$ $\big(\mathcal{F}_{\mathcal{S}} = \mathcal{F}_{\mathcal{R}}\big)$.

Remark 7.14. Note that if X and $\mathcal{R}$ are as in Remark 7.11, then by [20, Example 5.3] there does not exist a largest relator S on X such that $\mathcal{T}_{\mathcal{S}} = \mathcal{T}_{\mathcal{R}}$.

In the light of the several disadvantages of the structure $\mathcal{T}$, it is rather curious that most of the works in topology and analysis are based on open sets suggested by Tietze [60] and standardized by Bourbaki [2] and Kelley [14] .

Moreover, it also a striking fact that, despite the results of Pervin [28] , Fletcher and Lindgren [10] , and Szaz [48] , generalized topologies and minimal structures are still intensively investigated by a great number of mathematicians.

8. Some further results on unary operations for relators

In the sequel, we shall also need the following terminology of Pataki [26] .

Definition 8.1. For any two unary operations $\square$ and $\Diamond$ for relators, we say that $\square$ is *$\Diamond$–dominating, $\Diamond$–invariant, $\Diamond$–absorbing, and $\Diamond$–compatible* if, for any relator $\mathcal{R}$ on X to Y, we have

$$\mathcal{R}^{\Diamond} \subseteq \mathcal{R}^{\square}, \qquad \mathcal{R}^{\square} = \mathcal{R}^{\square\Diamond}, \qquad \mathcal{R}^{\square} = \mathcal{R}^{\Diamond\square}, \qquad \text{and} \qquad \mathcal{R}^{\square\Diamond} = \mathcal{R}^{\Diamond\square},$$

respectively.

Remark 8.2. Thus, the operation $\square$ is extensive if and only if it dominates the identity operation for relators. Moreover, $\square$ is idempotent if and only if it is $\square$–invariant ($\square$–absorbing).

In this respect, it is also worth mentioning that the operation $\square$ is $\Diamond$–invariant ($\square$–absorbing) if and only if $\mathcal{R}^{\square}$ is $\Diamond$–invariant ($\mathcal{R}$ and $\mathcal{R}^{\Diamond}$ are $\square$–equivalent) for every relator $\mathcal{R}$ on X to Y.

Remark 8.3. From Theorem 6.7, we can see that if $\Diamond, \square \in \{ *, \#, \wedge, \vartriangle \}$ such that $\Diamond$ precedes $\square$ in the above list, then $\square$ is both $\Diamond$–invariant and $\Diamond$–absorbing. Thus, in particular it is also $\Diamond$–compatible.

By using Definition 8.1, somewhat more generally, we can also state the following

Theorem 8.4. *If $\Diamond$ is an extensive and $\square$ is a $\Diamond$–dominating idempotent operation for relators, then $\square$ is $\Diamond$–invariant. Moreover, if in addition $\square$ is increasing, then $\square$ is $\Diamond$–absorbing and $\Diamond$–compatible.*

Remark 8.5. In this respect, it is also worth mentioning that if $\Diamond$ is an extensive and $\square$ is a $\Diamond$–dominating operation for relators, then $\square$ is also extensive.

Moreover, if $\Diamond$ is an increasing and $\square$ is an extensive operation for relators such that $\mathcal{R}^{\square\Diamond} \subseteq \mathcal{R}^{\square}$ for every relator $\mathcal{R}$ on X to Y, then $\square$ is $\Diamond$–dominating.

The importance of the compatibility property of operations lies mainly in

Theorem 8.6. *If $\square$ and $\Diamond$ are compatible closure (modification) operations for relators, then $\square\Diamond$ is also a closure (modification) operation for relators.*

Proof. By using the associativity of composition, and the idempotency and compatibility of the operations $\square$ and $\Diamond$, we can see that

$$\square\Diamond\square\Diamond = \Diamond\square\square\Diamond = \Diamond\square\Diamond = \square\Diamond\Diamond = \square\Diamond .$$

Therefore, the operation $\square\Diamond$ is also idempotent.

Remark 8.7. In this respect, it is also worth noticing that the composition of two union-preserving operation is also union-preserving.

It can be easily seen that the operations c, -1, ∞, ∂, and $*$ are union-preserving. Thus, their compositions are also union-preserving.

However, the important closure operations $\#$, $\wedge$, and $\vartriangle$ are not union-preserving. Concerning them, we can only prove the following

Theorem 8.8. *If $\square$ is a closure operation for relators, then for any family $\big(\mathcal{R}_i\big)_{i\in I}$ of relators on X to Y, we have*

$$\Big(\bigcup_{i\in I}\mathcal{R}_i\Big)^{\square}=\Big(\bigcup_{i\in I}\mathcal{R}_i^{\square}\Big)^{\square}.$$

Proof. If $\mathcal{R}=\bigcup_{i\in I}\mathcal{R}_i$, then for each $i\in I$ we have $\mathcal{R}_i\subseteq\mathcal{R}$. Hence, by using the increasingness of $\square$, we can infer that $\mathcal{R}_i^{\square}\subseteq\mathcal{R}^{\square}$. Therefore, we have $\bigcup_{i\in I}\mathcal{R}_i^{\square}\subseteq\mathcal{R}^{\square}$. Hence, by using the increasingness and the idempotency of $\square$, we can already infer that $\big(\bigcup_{i\in I}\mathcal{R}_i^{\square}\big)^{\square}\subseteq\mathcal{R}^{\square\square}=\mathcal{R}^{\square}$.

On the other hand, by the extensivity of $\square$, for each $i\in I$ we have $\mathcal{R}_i\subseteq\mathcal{R}_i^{\square}$, and hence also $\mathcal{R}_i\subseteq\bigcup_{i\in I}\mathcal{R}_i^{\square}$. Therefore, $\mathcal{R}=\bigcup_{i\in I}\mathcal{R}_i\subseteq\bigcup_{i\in I}\mathcal{R}_i^{\square}$. Hence, by using the increasingness of $\square$, we can already infer that $\mathcal{R}^{\square}\subseteq\big(\bigcup_{i\in I}\mathcal{R}_i^{\square}\big)^{\square}$. Therefore, the required equality is also true.

Remark 8.9. Hence, we can see that $\big(\bigcup_{i\in I}\mathcal{R}_i\big)^{\square}=\bigcup_{i\in I}\mathcal{R}_i^{\square}$ if and only the relator $\bigcup_{i\in I}\mathcal{R}_i^{\square}$ is $\square$–invariant.

Now, analogously to Theorem 8.8, we can also prove the following theorem of [56].

Theorem 8.10. *If $\square$ is a closure operation for relators, then for any family $\big(\mathcal{R}_i\big)_{i\in I}$ of relators on X to Y, we have*

$$\bigcap_{i\in I}\mathcal{R}_i^{\square}=\Big(\bigcap_{i\in I}\mathcal{R}_i^{\square}\Big)^{\square}.$$

Proof. If $\mathcal{R}=\bigcap_{i\in I}\mathcal{R}_i$, then for each $\in I$ we have $\mathcal{R}\subseteq\mathcal{R}_i$, and hence also $\mathcal{R}^{\square}\subseteq\mathcal{R}_i^{\square}$. Therefore, $\big(\bigcap_{i\in I}\mathcal{R}_i\big)^{\square}=\mathcal{R}^{\square}\subseteq\bigcap_{i\in I}\mathcal{R}_i^{\square}$.

Hence, by taking $\mathcal{R}_i^{\square}$ in place of $\mathcal{R}_i$, we can already infer that

$$\Big(\bigcap_{i\in I}\mathcal{R}_i^{\square}\Big)^{\square}\subseteq\bigcap_{i\in I}\mathcal{R}_i^{\square\square}=\bigcap_{i\in I}\mathcal{R}_i^{\square}\subseteq\Big(\bigcap_{i\in I}\mathcal{R}_i^{\square}\Big)^{\square}.$$

Therefore, the required equality is also true.

Remark 8.11. Hence, we can see that the relator $\bigcap_{i\in I}\mathcal{R}_i^{\square}$ is always $\square$–invariant. Moreover, if each $\mathcal{R}_i$ is $\square$-invariant, then the relator $\bigcap_{i\in I}\mathcal{R}_i$ is also $\square$–invariant.

Remark 8.12. Note that the proofs of the above three theorems also yield some useful statements for preclosure, semiclosure, and modification operations.

Analogously to the equivalence of the assertions (1) and (2) in Theorem 5.17, we can also prove the following

Theorem 8.13. *For a unary operation* $\square$ *for relators, the following assertions are equivalent:*

(1) $\square$ *is an increasing involution,*

(2) for any two relators $\mathcal{R}$ *and* $\mathcal{S}$ *on* X *to* Y*, we have*

$$\mathcal{R}^{\square} \subseteq \mathcal{S} \quad \Longleftrightarrow \quad \mathcal{R} \subseteq \mathcal{S}^{\square}.$$

Proof. If (1) holds, then for any two relators $\mathcal{R}$ and $\mathcal{S}$ on X to Y

$$\mathcal{R}^{\square} \subseteq \mathcal{S} \implies \mathcal{R}^{\square\,\square} \subseteq \mathcal{S}^{\square} \implies \mathcal{R} \subseteq \mathcal{S}^{\square} \implies \mathcal{R}^{\square} \subseteq \mathcal{S}^{\square\,\square} \implies \mathcal{R}^{\square} \subseteq \mathcal{S}.$$

Therefore, (2) also holds.

Conversely, if (2) holds, then for any relator $\mathcal{R}$ on X to Y

$$\mathcal{R}^{\square} \subseteq \mathcal{R}^{\square} \quad \Longrightarrow \quad \mathcal{R} \subseteq \mathcal{R}^{\square\,\square}, \quad \mathcal{R}^{\square\,\square} \subseteq \mathcal{R} \quad \Longrightarrow \quad \mathcal{R} = \mathcal{R}^{\square\,\square}.$$

Therefore, $\square$ is involutive. Thus, for any two relators $\mathcal{R}$ and $\mathcal{S}$ on X to Y

$$\mathcal{R} \subseteq \mathcal{S} \implies \mathcal{R}^{\square\,\square} \subseteq \mathcal{S}^{\square\,\square} \implies \mathcal{R}^{\square} \subseteq \mathcal{S}^{\square\,\square\,\square} \implies \mathcal{R}^{\square} \subseteq \mathcal{S}^{\square}.$$

Therefore, $\square$ is increasing, and thus (1) also holds.

Now, in addition Theorem 8.6, we can also prove the following

Theorem 8.14. *If* $\square$ *is a closure (modification) and* $\diamond$ *is an increasing involution operation for relators, then* $\Diamond = \diamond \square \diamond$ *is also a closure (modification) operation for relators.*

Proof. By using the associativity of composition, the involutiveness of $\diamond$, and the idempontency of $\square$, we can see that

$$\Diamond\,\Diamond = (\diamond\square\diamond)(\diamond\square\diamond) = (\diamond\square)\big((\diamond\diamond)(\square\diamond)\big)$$
$$= (\diamond\square)\big(\Delta(\square\diamond)\big) = (\diamond\square)\big(\square\diamond\big) = \diamond\big((\square\square)\diamond\big) = \diamond(\square\diamond) = \Diamond,$$

where Δ is the identity operation for relators. Therefore, $\Diamond$ is also idempotent.

Because of this theorem, we may also naturally introduce the following

Definition 8.15. For any unary operation $\square$ for relators, we write

$$\textcircled{$\square$} = c\,\square\,c \qquad \text{and} \qquad \boxbox = -1\,\square\,-1\,.$$

Remark 8.16. Thus, by Theorem 8.14, for instance $\circledast$ and $\boxast$ are also closure operations for relators.

However, this also quite obvious from the fact that, by the corresponding definitions, for any relator $\mathcal{R}$ on X to Y we have

(1) $\mathcal{R}^{\boxast} = \mathcal{R}^{*}$, (2) $\mathcal{R}^{\circledast} = \bigcup_{R \in \mathcal{R}} \mathcal{P}(R)$.

Namely, if for instance $S \in \mathcal{R}^{\circledast}$, then $S \in \mathcal{R}^{c*c}$, and thus $S^{c} \in \mathcal{R}^{c*}$. Therefore, there exists $R \in \mathcal{R}$ such that $R^{c} \subseteq S^{c}$. Hence, it follows that $S \subseteq R$, and thus $S \in \mathcal{P}(R)$. Therefore, $S \in \bigcup_{R \in \mathcal{R}} \mathcal{P}(R)$ also holds.

Now, to clear up the importance of Theorem 8.14 and Definition 8.15, we can also prove the following

Theorem 8.17. *For any relator* $\mathcal{R}$ *on* X *to* Y*, we have*

(1) $\mathcal{R}^{\textcircled{\#}} = \mathcal{R}^{\square_{\mathrm{Lb}}}$, *(2)* $\mathcal{R}^{\textcircled{$\wedge$}} = \mathcal{R}^{\square_{\mathrm{lb}}}$.

Proof. By using the corresponding definitions, Theorem 6.4, and the equality $\mathrm{Lb}_{\mathcal{R}} = \mathrm{Int}_{\mathcal{R}^c} \circ \mathcal{C}$, we can easily that, for any relation S on X to Y, we have

$$S \in \mathcal{R}^{\square_{\mathrm{Lb}}} \iff \mathrm{Lb}_S \subseteq \mathrm{Lb}_{\mathcal{R}} \iff \mathrm{Int}_{S^c} \circ \mathcal{C} \subseteq \mathrm{Int}_{\mathcal{R}^c} \circ \mathcal{C}$$
$$\iff \mathrm{Int}_{S^c} \subset \mathrm{Int}_{\mathcal{R}^c} \iff S^c \subset \mathcal{S}^{c\#} \iff \mathcal{R} \subset \mathcal{S}^{c\#c} \iff \mathcal{R} \subset \mathcal{S}^{\textcircled{\#}}.$$

Therefore, assertion (1) is true. The proof of assertion (2) is quite similar.

By the corresponding definitions, it is clear that

$$\mathrm{Lb}_{\mathcal{R}} = \bigcup_{R \in \mathcal{R}} \mathrm{Lb}_R \qquad \text{and} \qquad \mathrm{lb}_{\mathcal{R}} = \bigcup_{R \in \mathcal{R}} \mathrm{lb}_R$$

for any relator $\mathcal{R}$ on X to Y.

Therefore, analogously to Theorems 6.5, 6.9, and 6.10, we can also easily establish the following three theorems.

Theorem 8.18. *$\textcircled{\#}$ and $\owedge$ are closure operations for relators.*

Theorem 8.19. *For any two relators $\mathcal{R}$ and $\mathcal{S}$ on X to Y, we have*

(1) $\mathcal{S} \subseteq \mathcal{R}^{\textcircled{\#}} \iff \mathcal{S}^{\textcircled{\#}} \subseteq \mathcal{R}^{\textcircled{\#}} \iff \mathrm{Lb}_{\mathcal{S}} \subseteq \mathrm{Lb}_{\mathcal{R}}$,

(2) $\mathcal{S} \subseteq \mathcal{R}^{\owedge} \iff \mathcal{S}^{\owedge} \subseteq \mathcal{R}^{\owedge} \iff \mathrm{lb}_{\mathcal{S}} \subseteq \mathrm{lb}_{\mathcal{R}}$.

Theorem 8.20. *For any relator $\mathcal{R}$ on X to Y,*

(1) $\mathcal{S} = \mathcal{R}^{\textcircled{\#}}$ *is the largest relator on X to Y such that* $\mathrm{Lb}_{\mathcal{S}} \subseteq \mathrm{Lb}_{\mathcal{R}}$, $\big(\mathrm{Lb}_{\mathcal{S}} = \mathrm{Lb}_{\mathcal{R}}\big)$,

(2) $\mathcal{S} = \mathcal{R}^{\owedge}$ *is the largest relator on X to Y such that* $\mathrm{lb}_{\mathcal{S}} \subseteq \mathrm{lb}_{\mathcal{R}}$ $\big(\mathrm{lb}_{\mathcal{S}} = \mathrm{lb}_{\mathcal{R}}\big)$.

Remark 8.21. If $\mathcal{R}$ is a relator on X to Y, then in addition to Theorem 8.17 we can also state that

(1) $\mathcal{R}^{\square_{\mathrm{Ub}}} = \mathcal{R}^{\textcircled{\#}}$, (2) $\mathcal{R}^{\square_{\mathrm{ub}}} = \mathcal{R}^{\boxed{\owedge}}$.

Namely, if S is a relation on X to Y, then by using the equalities

$$\mathrm{Ub}_{\mathcal{R}} = \mathrm{Lb}_{\mathcal{R}^{-1}} = \mathrm{Lb}_{\mathcal{R}}^{-1} \qquad \text{and} \qquad \mathrm{ub}_{\mathcal{R}} = \mathrm{lb}_{\mathcal{R}^{-1}},$$

and Theorem 8.19, we can easily see that

$$S \in \mathcal{R}^{\square_{\mathrm{Ub}}} \iff \mathrm{Ub}_S \subseteq \mathrm{Ub}_{\mathcal{R}} \iff \mathrm{Lb}_S^{-1} \subseteq \mathrm{Lb}_{\mathcal{R}}^{-1}$$
$$\iff \mathrm{Lb}_S \subseteq \mathrm{Lb}_{\mathcal{R}} \iff S \in \mathcal{R}^{\textcircled{\#}}$$

and

$$S \in \mathcal{R}^{\square_{\mathrm{ub}}} \iff \mathrm{ub}_S \subseteq \mathrm{ub}_{\mathcal{R}} \iff \mathrm{lb}_{S^{-1}} \subseteq \mathrm{lb}_{\mathcal{R}^{-1}}$$
$$\iff S^{-1} \in \mathcal{R}^{-1\owedge} \iff S \in \mathcal{R}^{-1\owedge-1} \iff S \in \mathcal{R}^{\boxed{\owedge}}.$$

In this respect, it is also worth noticing that, by the associativity of composition and the inversion compatibility of c, we also have

$$\boxed{\owedge} = -1 \owedge -1 = -1\, c \wedge c\, -1 = c\, -1 \wedge -1\, c = c \boxed{\wedge} c = \textcircled{\boxed{\wedge}}.$$

9. Inversion compatible operations for relators

According to Definition 8.1, we may naturally have the following

Definition 9.1. An unary operation $\square$ for relators is called *inversion compatible* if for any relator $\mathcal{R}$ on X to Y we have

$$\big(\mathcal{R}^{\square}\big)^{-1}=\big(\mathcal{R}^{-1}\big)^{\square}.$$

Now, by using this definition, we can easily prove the following

Theorem 9.2. *For a unary operation $\square$ on relators, the following assertions are equivalent:*

(1) $\square$ *is inversion compatible,*

(2) $\big(\mathcal{R}^{\square}\big)^{-1}\subseteq\big(\mathcal{R}^{-1}\big)^{\square}$ *for any relator $\mathcal{R}$ on X to Y,*

(3) $\big(\mathcal{R}^{-1}\big)^{\square}\subseteq\big(\mathcal{R}^{\square}\big)^{-1}$ *for any relator $\mathcal{R}$ on X to Y.*

Proof. Note that if for instance (2) holds, then for any relator $\mathcal{R}$ on X to Y we also have $\Big(\big(\mathcal{R}^{-1}\big)^{\square}\Big)^{-1}\subseteq\mathcal{R}^{\square}$, and hence also $\big(\mathcal{R}^{-1}\big)^{\square}\subset\big(\mathcal{R}^{\square}\big)^{-1}$.

Hence, by using Definition 8.15, we can immediately derive the following

Theorem 9.3. *For a unary operation $\square$ for relators, the following assertions are equivalent:*

(1) $\square$ *is inversion compatible,*

(2) $\mathcal{R}^{\square}=\mathcal{R}^{\boxbox}$ *for any relator $\mathcal{R}$ on X to Y.*

(3) $\mathcal{R}^{\square}\subseteq\mathcal{R}^{\boxbox}$ $\big(\mathcal{R}^{\boxbox}\subseteq\mathcal{R}^{\square}\big)$ *for any relator $\mathcal{R}$ on X to Y.*

However, it is now more important to note that, by using the corresponding definitions, we can also easily prove the following

Theorem 9.4. *If $\square$ is a union-preserving operation for relators, then the following assertions are equivalent:*

(1) $\square$ *is inversion compatible,*

(2) $\big(\{R\}^{\square}\big)^{-1}=\big\{R^{-1}\big\}^{\square}$ *for any relation R on X to Y.*

Proof. To prove that (2) also implies (1), note that the operation -1 is union-preserving. Therefore, for any relator $\mathcal{R}$ on X to Y, we have

$$\big(\mathcal{R}^{\square}\big)^{-1}=\Big(\bigcup_{R\in\mathcal{R}}\{R\}^{\square}\Big)^{-1}=\bigcup_{R\in\mathcal{R}}\big(\{R\}^{\square}\big)^{-1}$$

$$=\bigcup_{R\in\mathcal{R}}\big\{R^{-1}\big\}^{\square}=\Big(\bigcup_{R\in\mathcal{R}}\big\{R^{-1}\big\}\Big)^{\square}=\big(\mathcal{R}^{-1}\big)^{\square}.$$

Now, analogously to Theorem 9.3, we can also state the following

Corollary 9.5. *If $\square$ is a union-preserving operation for relators, then the following assertions are equivalent:*

(1) $\square$ *is inversion compatible,*

(2) $\big(\{R\}^{\square}\big)^{-1}\subseteq\big\{R^{-1}\big\}^{\square}$ *for any relator $\mathcal{R}$ on X to Y,*

(3) $\{R^{-1}\}^{\Box} \subseteq \big(\{R\}^{\Box}\big)^{-1}$ *for any relator* $\mathcal{R}$ *on* X *to* Y.

However, this corollary cannot actually be used to simplify the proof of

Theorem 9.6. *The operations* c, ∞, ∂, *and* $*$ *are inversion compatible.*

Proof. By the corresponding definitions, it is clear that ∞ is a union-preserving operation for relators. Moreover, for any relation R on X we have

$$\big(R^{\infty}\big)^{-1} = \Big(\bigcup_{n=0}^{\infty} R^{n}\Big)^{-1} = \bigcup_{n=0}^{\infty} \big(R^{n}\big)^{-1} = \bigcup_{n=0}^{\infty} \big(R^{-1}\big)^{n} = \big(R^{-1}\big)^{\infty}.$$

Therefore, by Theorem 9.4, the operation ∞ is inversion compatible.

Now, to prove the inversion-compatibility of the operation ∂, it is enough to note only that, for any relator $\mathcal{R}$ and relation S on X, we have

$$S \in \big(\mathcal{R}^{-1}\big)^{\partial} \iff S^{\infty} \in \mathcal{R}^{-1} \iff \big(S^{\infty}\big)^{-1} \in \mathcal{R}$$
$$\iff \big(S^{-1}\big)^{\infty} \in \mathcal{R} \iff S^{-1} \in \mathcal{R}^{\partial} \iff S \in \big(\mathcal{R}^{\partial}\big)^{-1}.$$

Now, by using Theorem 9.2 and our former results on the structure Int, we can also prove the following

Theorem 9.7. *The operation* $\#$ *is also inversion compatible.*

Proof. If $\mathcal{R}$ is a relator on X to Y, then by [**?**, Theorem 0.0] and Theorem 6.11 we have

$$\mathrm{Cl}_{(\mathcal{R}^{\#})^{-1}} = \big(\mathrm{Cl}_{\mathcal{R}^{\#}}\big)^{-1} = \big(\mathrm{Cl}_{\mathcal{R}}\big)^{-1} = \mathrm{Cl}_{\mathcal{R}^{-1}} .$$

Hence, by using Theorem 6.9, we can already infer that $\big(\mathcal{R}^{\#}\big)^{-1} \subseteq \big(\mathcal{R}^{-1}\big)^{\#}$. Therefore, by Theorem 9.2, the corresponding equality is also true.

Remark 9.8. Unfortunately, the operations $\wedge$ and $\vartriangle$ are not inversion compatible. Therefore, we have also to consider the operations $\vee$ and $\triangledown$ defined by

$$\mathcal{R}^{\vee} = \big(\mathcal{R}^{\wedge}\big)^{-1} \qquad \text{amd} \qquad \mathcal{R}^{\triangledown} = \big(\mathcal{R}^{\vartriangle}\big)^{-1}$$

for every relator $\mathcal{R}$ on X to Y.

However, these operations have some very curious properties. For instance, the operations $\vee\vee$ and $\triangledown\triangledown$ already coincide with the extremal closure operations $\bullet$ and $\blacklozenge$, defined for any relator $\mathcal{R}$ on X to Y such that

$$\mathcal{R}^{\bullet} = \big\{\delta_{\mathcal{R}}\big\}^{*}, \qquad \text{where} \qquad \delta_{\mathcal{R}} = \bigcap \mathcal{R},$$

and

$$\mathcal{R}^{\blacklozenge} = \mathcal{R} \quad \text{if} \quad \mathcal{R} = \big\{X \times Y\big\} \qquad \text{and} \qquad \mathcal{R}^{\blacklozenge} = \mathcal{P}(X \times Y) \quad \text{if} \quad \mathcal{R} \neq \big\{X \times Y\big\}.$$

Note that $\blacklozenge$ is the ultimate stable unary operation for relators.

The usefulness of inversion compatible operations is apparent from the following simple theorems of [56].

Theorem 9.9. *If* $\Box$ *is an inversion compatible operation for relators, then for any relator* $\mathcal{R}$ *on* X *to* Y *the following assertions are equivalent:*

(1) $\mathcal{R}$ *is* $\Box$*–invariant,* *(2)* $\mathcal{R}^{-1}$ *is* $\Box$*–invariant.*

Definition 9.10. If $\square$ is an unary operation for relators, then a relator $\mathcal{R}$ on X is called *$\square$–symmetric* if

$$\left(\mathcal{R}^{\square}\right)^{-1}=\mathcal{R}^{\square}.$$

Remark 9.11. Now, the relator $\mathcal{R}$ may, for instance, be naturally called *properly, uniformly, proximally, topologically, and paratopologically symmetric* if it is $\square$–symmetric with $\square=\Delta\,,\,*\,,\,\#\,,\,\wedge\,,$ and $\triangle\,,$ respectively.

Theorem 9.12. *If $\mathcal{R}$ is a properly symmetric relator on X, then $\mathcal{R}$ is $\square$–symmetric for any inversion compatible operation $\square$ for relators.*

Theorem 9.13. *If $\square$ is an inversion compatible operation for relators, then for any relator $\mathcal{R}$ on X the following assertions are equivalent:*

(1) $\mathcal{R}$ is $\square$–symmetric,

(2) $\mathcal{R}^{-1}$ is $\square$–symmetric,

(3) $\mathcal{R}$ and $\mathcal{R}^{-1}$ are $\square$–equivalent.

Remark 9.14. In this respect, it is also worth noticing that if $\square$ is an unary operation for relators and $\mathcal{R}$ is a $\square$–symmetric relator on X to Y such that $\mathcal{R}$ and $\mathcal{R}^{-1}$ are $\square$–equivalent, then $\left(\mathcal{R}^{\square}\right)^{-1}=\mathcal{R}^{\square}=\left(\mathcal{R}^{-1}\right)^{\square}$.

However, it is now more important to note that, in addition to Theorem 9.13, we can also prove the following

Theorem 9.15. *If $\square$ is an inversion compatible closure operation for relators, then for any relator $\mathcal{R}$ on X the following assertions are equivalent:*

(1) $\mathcal{R}$ is $\square$–symmetric,

(2) $\mathcal{R}^{-1}\subseteq\mathcal{R}^{\square}$; *(3) $\mathcal{R}\subseteq\left(\mathcal{R}^{-1}\right)^{\square}$,*

(4) $\mathcal{R}$ is $\square$-equivalent to a properly symmetric relator $\mathcal{S}$ on X.

Proof. If (1) holds, then by the extensivity of $\square$, it is clear that $\mathcal{R}^{-1}\subseteq\left(\mathcal{R}^{\square}\right)^{-1}=\mathcal{R}^{\square}$. Therefore, (2) also holds.

Moreover, if (2) holds, then we can see that $\mathcal{R}\subseteq\left(\mathcal{R}^{\square}\right)^{-1}=\left(\mathcal{R}^{-1}\right)^{\square}$. Therefore, (3) also holds.

While, if (3) holds, then we can quite similarly see that (2) also holds. From (2) and (3), by using Theorem 0.0, we can infer that $\left(\mathcal{R}^{-1}\right)^{\square}\subseteq\mathcal{R}^{\square}\subset\left(\mathcal{R}^{-1}\right)^{\square}$, and thus $\mathcal{R}^{\square}=\left(\mathcal{R}^{-1}\right)^{\square}$. Therefore, by Theorem 9.0, (1) also holds.

On the other hand, if (1) holds, then $\mathcal{R}^{\square}$ is properly symmetric. Hence, since $\mathcal{R}^{\square}=\left(\mathcal{R}^{\square}\right)^{\square}$, we can already see that (4) holds with $\mathcal{S}=\mathcal{R}^{\square}$.

Conversely, if (4) holds, then it is clear that $\left(\mathcal{R}^{\square}\right)^{-1}=\left(\mathcal{S}^{\square}\right)^{-1}=\left(\mathcal{S}^{-1}\right)^{\square}=\mathcal{S}^{\square}=\mathcal{R}^{\square}$. Therefore, (1) also holds.

From this theorem, by using Theorem 5.13, we can immediately derive

Corollary 9.16. *If $\mathfrak{F}$ is a union-preserving structure for relators such that the induced operation $\square$ is inversion compatible, then for any relator $\mathcal{R}$ on X the following assertions are equivalent:*

(1) $\mathcal{R}$ is $\square_{\mathfrak{F}}$–symmetric, *(2) $\mathfrak{F}_{\mathcal{R}^{-1}}\subset\mathfrak{F}_{\mathcal{R}}$;* *(3) $\mathfrak{F}_{\mathcal{R}}\subset\mathfrak{F}_{\mathcal{R}^{-1}}$.*

Remark 9.17. Finally, we note that the theorems proved in this section can be generalized by using an arbitrary increasing involution operation $\diamond$ for relators instead of the inversion -1.

10. Composition compatible operations for relators

Composition compatibility properties of operations for relators have formerly been considered only in [51] and [56].

Definition 10.1. For a direct unary operation $\square$ for relators, we say that:

(1) $\square$ is left composition compatible if $\big(\mathcal{S}\circ\mathcal{R}\big)^{\square}=\big(\mathcal{S}\circ\mathcal{R}^{\square}\big)^{\square}$ for any two relators $\mathcal{R}$ on X to Y and $\mathcal{S}$ on Y to Z,

(2) $\square$ is right composition compatible if $\big(\mathcal{S}\circ\mathcal{R}\big)^{\square}=\big(\mathcal{S}^{\square}\circ\mathcal{R}\big)^{\square}$ for any two relators $\mathcal{R}$ on X to Y and $\mathcal{S}$ on Y to Z.

Remark 10.2. Now, the operation $\square$ may be naturally called composition compatible if it is both left and right composition compatible.

Note that, actually, this is also very weak composition compatibility property. However, by the following theorems, it will be sufficient for our subsequent purposes.

Theorem 10.3. *If $\square$ is a left (right) composition compatible unary operation for relators, then $\square$ is, in particular, idempotent.*

Proof. If $\square$ is left composition compatible, then for any relator $\mathcal{R}$ on X to Y we have

$$\mathcal{R}^{\square\,\square}=\big(\mathcal{R}^{\square}\big)^{\square}=\big(\{\Delta_Y\}\circ\mathcal{R}^{\square}\big)^{\square}=\big(\{\Delta_Y\}\circ\mathcal{R}\big)^{\square}=\mathcal{R}^{\square}.$$

Theorem 10.4. *If $\square$ is a composition compatible unary operation for relators, then for any two relators $\mathcal{R}$ on X to Y and $\mathcal{S}$ on Y to Z we have*

$$\big(\mathcal{S}\circ\mathcal{R}\big)^{\square}=\big(\mathcal{S}^{\square}\circ\mathcal{R}^{\square}\big)^{\square}.$$

Proof. Namely, now we evidently have $\big(\mathcal{S}\circ\mathcal{R}\big)^{\square}=\big(\mathcal{S}\circ\mathcal{R}^{\square}\big)^{\square}=\big(\mathcal{S}^{\square}\circ\mathcal{R}^{\square}\big)^{\square}$.

Remark 10.5. In this case, by Definition 10.1, we also have

$$\big(\mathcal{S}^{\square}\circ\mathcal{R}\big)^{\square}=\big(\mathcal{S}\circ\mathcal{R}^{\square}\big)^{\square}.$$

From Theorem 10.4, by using the associativity of composition, we can derive

Corollary 10.6. *If $\square$ is a composition compatible unary operation for relators, then for any three relators $\mathcal{R}$ on X to Y, $\mathcal{S}$ on Y to Z, and $\mathcal{T}$ on Z to W we have*

$$\big(\mathcal{T}\circ\mathcal{S}\circ\mathcal{R}\big)^{\square}=\big(\mathcal{T}^{\square}\circ\mathcal{S}^{\square}\circ\mathcal{R}^{\square}\big)^{\square}.$$

Proof. By using Theorem 10.4, we can see that

$$\big(\mathcal{T}\circ(\mathcal{S}\circ\mathcal{R})\big)^{\square}=\Big(\mathcal{T}^{\square}\circ\big(\mathcal{S}\circ\mathcal{R}\big)^{\square}\Big)^{\square}=\Big(\mathcal{T}^{\square}\circ\big(\mathcal{S}^{\square}\circ\mathcal{R}^{\square}\big)\Big)^{\square}.$$

Remark 10.7. In this case, by using Definition 10.1, we can also prove that

$$\big(\mathcal{T}\circ\mathcal{S}\circ\mathcal{R}\big)^{\square}=\big(\mathcal{T}^{\square}\circ\mathcal{S}\circ\mathcal{R}\big)^{\square}=\big(\mathcal{T}\circ\mathcal{S}^{\square}\circ\mathcal{R}\big)^{\square}=\big(\mathcal{T}\circ\mathcal{S}\circ\mathcal{R}^{\square}\big)^{\square}.$$

However, it is now more important to note that, that by using the corresponding definitions, we can also easily prove the following

Theorem 10.8. *If* $\square$ *is a preclosure operation for relators, then for any two relators* $\mathcal{R}$ *on* X *to* Y *and* $\mathcal{S}$ *on* Y *to* Z *we have*

(1) $\big(\mathcal{S}\circ\mathcal{R}\big)^{\square}\subseteq\big(\mathcal{S}\circ\mathcal{R}^{\square}\big)^{\square}\subseteq\big(\mathcal{S}^{\square}\circ\mathcal{R}^{\square}\big)^{\square}$,

(2) $\big(\mathcal{S}\circ\mathcal{R}\big)^{\square}\subseteq\big(\mathcal{S}^{\square}\circ\mathcal{R}\big)^{\square}\subseteq\big(\mathcal{S}^{\square}\circ\mathcal{R}^{\square}\big)^{\square}$.

Proof. By the extensivity $\square$, we have $\mathcal{R}\subseteq\mathcal{R}^{\square}$. Hence, by the increasingness of the elementwise composition of relators, we can see that $\mathcal{S}\circ\mathcal{R}\subseteq\mathcal{S}\circ\mathcal{R}^{\square}$. Thus, by the increasingness of $\square$, we also have $\big(\mathcal{S}\circ\mathcal{R}\big)^{\square}\subseteq\big(\mathcal{S}\circ\mathcal{R}^{\square}\big)^{\square}$. Hence, by writing $\mathcal{S}^{\square}$ in place of $\mathcal{S}$, we can see that $\big(\mathcal{S}^{\square}\circ\mathcal{R}\big)^{\square}\subseteq\big(\mathcal{S}^{\square}\circ\mathcal{R}^{\square}\big)^{\square}$. Therefore, the first part of (1) and the second part of (2) are true.

From this theorem, by using Definition 10.1, we can immediately derive

Corollary 10.9. *If* $\square$ *is a preclosure operation for relators, then*

(1) $\square$ *is left composition compatible if and only if* $\big(\mathcal{S}\circ\mathcal{R}^{\square}\big)^{\square}\subseteq\big(\mathcal{S}\circ\mathcal{R}\big)^{\square}$ *for any two relators* $\mathcal{R}$ *on* X *to* Y *and* $\mathcal{S}$ *on* Y *to* Z,

(2) $\square$ *is right composition compatible if and only if* $\big(\mathcal{S}^{\square}\circ\mathcal{R}\big)^{\square}\subseteq\big(\mathcal{S}\circ\mathcal{R}\big)^{\square}$ *for any two relators* R *on* X *to* Y *and* $\mathcal{S}$ *on* Y *to* Z.

Hence, by Theorem 5.17, it is clear that in particular we also have

Corollary 10.10. *If* $\square$ *is a closure operation for relators, then*

(1) $\square$ *is left composition compatible if and only if* $\mathcal{S}\circ\mathcal{R}^{\square}\subseteq\big(\mathcal{S}\circ\mathcal{R}\big)^{\square}$ *for any two relators* $\mathcal{R}$ *on* X *to* Y *and* $\mathcal{S}$ *on* Y *to* Z,

(2) $\square$ *is right composition compatible if and only if* $\mathcal{S}^{\square}\circ\mathcal{R}\subseteq\big(\mathcal{S}\circ\mathcal{R}\big)^{\square}$ *for any two relators* $\mathcal{R}$ *on* X *to* Y *and* $\mathcal{S}$ *on* Y *to* Z.

Remark 10.11. In addition to the above results, it is also worth noticing that an involution operation $\square$ for relators is left composition compatible if and only if $\mathcal{S}\circ\mathcal{R}=\mathcal{S}\circ\mathcal{R}^{\square}$ for any two relators $\mathcal{R}$ on X to Y and $\mathcal{S}$ on Y to Z.

Moreover, since $\mathcal{S}\circ\mathcal{R}=\bigcup_{S\in\mathcal{S}}S\circ\mathcal{R}$ holds, we can also at once state that an involution operation $\square$ for relators is left composition compatible if and only if $S\circ\mathcal{R}=S\circ\mathcal{R}^{\square}$ for any relator $\mathcal{R}$ on X to Y and relation S on Y to Z.

Now, by using Corollary 10.10 and Theorem 8.8, we can also prove the following

Theorem 10.12. *If* $\square$ *is a closure operation for relators, then*

(1) $\square$ *is left composition compatible if and only if* $S\circ\mathcal{R}^{\square}\subseteq\big(S\circ\mathcal{R}\big)^{\square}$ *for any relator* $\mathcal{R}$ *on* X *to* Y *and relation* S *on* Y *to* Z,

(2) $\square$ *is right composition compatible if and only if* $\mathcal{S}^{\square}\circ R\subseteq\big(\mathcal{S}\circ R\big)^{\square}$ *for any relation* R *on* X *to* Y *and relator* $\mathcal{S}$ *on* Y *to* Z.

Proof. If $\square$ is left composition compatible, then by Corollary 10.10, for any relator $\mathcal{R}$ and relation S on Y to Z, we have $\{S\}\circ\mathcal{R}^{\square}\subseteq\big(\{S\}\circ\mathcal{R}\big)^{\square}$, and thus $S\circ\mathcal{R}^{\square}\subseteq\big(S\circ\mathcal{R}\big)^{\square}$. Therefore, the "only if part" of (1) is true.

Conversely, if $\mathcal{R}$ is a relator on X to Y and $\mathcal{S}$ is a relator on Y to Z, and the inclusion $S\circ\mathcal{R}^{\square}\subseteq\big(S\circ\mathcal{R}\big)^{\square}$ holds for any relation S on Y to Z, then by using the corresponding definitions and Theorem 8.8 we can see that

$$\mathcal{S}\circ\mathcal{R}^{\Box}=\bigcup_{S\in\mathcal{S}}S\circ\mathcal{R}^{\Box}\subset\bigcup_{S\in\mathcal{S}}\big(S\circ\mathcal{R}\big)^{\Box}$$
$$\subseteq\Big(\bigcup\nolimits_{S\in\mathcal{S}}\big(S\circ\mathcal{R}\big)^{\Box}\Big)^{\Box}=\Big(\bigcup\nolimits_{S\in\mathcal{S}}S\circ\mathcal{R}\Big)^{\Box}=\big(\mathcal{S}\circ\mathcal{R}\big)^{\Box}.$$

Therefore, by Corollary 10.10, the "if part" of (1) is also true.

By using this theorem, we can somewhat more easily establish the composition compatibility properties of the basic closure operations considered in Section 6.

Theorem 10.13. *The operations $*$ and $\#$ are composition compatible.*

Proof. To prove right composition compatibility of $\#$, by Theorem 10.12, it is enough to prove only that, for any relation R on X to Y and relator $\mathcal{S}$ on Y to Z, we have $\mathcal{S}^{\#}\circ R\subseteq\big(\mathcal{S}\circ R\big)^{\#}$.

For this, suppose that $W\in\mathcal{S}^{\#}\circ R$ and $A\subset X$. Then, there exists $V\in\mathcal{S}^{\#}$ such that $W=V\circ R$. Moreover, there exists $S\in\mathcal{S}$ such that $S\big[\,R[\,A\,]\,\big]\subseteq V\big[\,R[\,A\,]\,\big]$, and thus $(S\circ R)[\,A\,]\subseteq(V\circ R)[\,A\,]=W[\,A\,]$. Hence, by taking $U=S\circ R$, we can see that $U\in\mathcal{S}\circ R$ such that $U[\,A\,]\subseteq W[\,A\,]$. Therefore, $W\in\big(\mathcal{S}\circ R\big)^{\#}$ also holds.

Theorem 10.14. *The operations $\wedge$ and $\vartriangle$ are left composition compatible.*

Proof. To prove left composition compatibility of $\vartriangle$, by Theorem 10.12, it is enough to prove only that, for any relator $\mathcal{R}$ on X to Y and relation S on Y to Z, we have $S\circ\mathcal{R}^{\vartriangle}\subseteq\big(S\circ\mathcal{R}\big)^{\vartriangle}$.

For this, suppose that $W\in S\circ\mathcal{R}^{\vartriangle}$ and $x\in X$. Then, there exists $V\in\mathcal{R}^{\vartriangle}$ such that $W=S\circ V$. Moreover, there exist $u\in X$ and $R\in\mathcal{R}$ such that $R(u)\subseteq V(x)$. Hence, we can infer that

$$(S\circ R)(u)=S\big[\,R(u)\,\big]\subseteq S\big[\,V(x)\,\big]=(S\circ V)(x)=W(x)\,.$$

Now, by taking $U=S\circ R$, we can see that $U\in S\circ\mathcal{R}$ such that $U(u)\subset W(x)$. Therefore, $W\in\big(S\circ R\big)^{\vartriangle}$ also holds.

Instead of the right composition compatibility of the operations $\wedge$ and $\vartriangle$, we can only prove the following

Theorem 10.15. *For any two relators $\mathcal{R}$ on X to Y and $\mathcal{S}$ on Y to Z, we have*

(1) $\big(\mathcal{S}\circ\mathcal{R}\big)^{\wedge}=\big(\mathcal{S}^{\#}\circ\mathcal{R}\big)^{\wedge}$, *(2)* $\big(\mathcal{S}\circ\mathcal{R}\big)^{\vartriangle}=\big(\mathcal{S}^{\#}\circ\mathcal{R}\big)^{\vartriangle}$.

Proof. By the extensivity $\#$, we have $\mathcal{S}\subset\mathcal{S}^{\#}$. Hence, by the elementwise definition of composition of relators, we can see that $\mathcal{S}\circ\mathcal{R}\subseteq\mathcal{S}^{\#}\circ\mathcal{R}$. Thus, by the increasingness of $\wedge$, we also have $\big(\mathcal{S}\circ\mathcal{R}\big)^{\wedge}\subseteq\big(\mathcal{S}^{\#}\circ\mathcal{R}\big)^{\wedge}$.

To get the converse inclusion, by Theorem 6.9, it is now enough to prove only that $S^{\#}\circ\mathcal{R}\subseteq\big(\mathcal{S}\circ\mathcal{R}\big)^{\wedge}$. For this, suppose that $W\in S^{\#}\circ\mathcal{R}$ and $x\in X$. Then, there exists $V\in\mathcal{S}^{\#}$ and $R\in\mathcal{R}$ such that $W=V\circ R$. Moreover, there exists $S\in\mathcal{S}$, such that $S\big[\,R(x)\,\big]\subseteq V\big[\,R(x)\,\big]$, and thus $(S\circ R)(x)\subseteq(V\circ R)(x)=W(x)$. Hence, by taking $U=S\circ R$, we can see that $U\in\mathcal{S}\circ\mathcal{R}$ such that $U(x)\subseteq W(x)$. Therefore, $W\in\big(\mathcal{S}\circ\mathcal{R}\big)^{\wedge}$ also holds.

Thus, we have proved (1). Assertion (2) can now be immediately derived from (1) by using that $\mathcal{U}^{\wedge\,\Delta}=\mathcal{U}^{\Delta}$ for any relator $\mathcal{U}$ on X to Z.

From this theorem, by using Theorem 10.14, we can immediately derive

Corollary 10.16. *For any two relators $\mathcal{R}$ on X to Y and $\mathcal{S}$ on Y to Z, we have*

(1) $\big(\mathcal{S}\circ\mathcal{R}\big)^{\wedge}=\big(\mathcal{S}^{\#}\circ\mathcal{R}^{\wedge}\big)^{\wedge}$, *(2)* $\big(\mathcal{S}\circ\mathcal{R}\big)^{\Delta}=\big(\mathcal{S}^{\#}\circ\mathcal{R}^{\Delta}\big)^{\Delta}$.

Remark 10.17. By using Theorem 10.12, we can also somewhat more easily prove that the operation $\circledast$, considered in Remark 8.16, is also composition compatible.

11. Reflexive, topological, and proximal relators

The subsequent definitions and theorems have been mainly taken from an unfinished doctoral dissertation of the first author [35]. (For some closely related results, see also [33] and [40].)

Definition 11.1. A relator $\mathcal{R}$ on X is called *reflexive* if each member R of $\mathcal{R}$ is a reflexive relation on X.

Remark 11.2. Thus, for a relator $\mathcal{R}$ on X, the following assertions are equivalent:

(1) $\mathcal{R}$ is reflexive,

(2) $x\in R(x)$ for all $x\in X$ and $R\in\mathcal{R}$,

(3) $A\subseteq R[A]$ for all $A\subseteq X$ and $R\in\mathcal{R}$.

The importance of reflexive relators is also apparent from the following two obvious theorems.

Theorem 11.3. *If $\mathcal{R}$ is relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is reflexive,

(2) $A\subseteq \operatorname{cl}_{\mathcal{R}}(A)$ for all $A\subseteq X$,

(3) $\operatorname{int}_{\mathcal{R}}(A)\subseteq A$ for all $A\subseteq X$.

Hint. To prove the implication (3) $\Longrightarrow$ (1), note that, for any $x\in X$ and $R\in\mathcal{R}$, we have $R(x)\subseteq R(x)$, and thus $x\in\operatorname{int}_{\mathcal{R}}\big(R(x)\big)$.

Remark 11.4. In addition to Remark 11.2 and Theorem 11.3, it is also worth mentioning that the relator $\mathcal{R}$ is reflexive if and only if the relation $\delta_{\mathcal{R}}=\bigcap\mathcal{R}$ is reflexive.

Namely, if $\mathcal{R}$ is a relator on X to Y, then for any $y\in Y$ we have

$$\operatorname{cl}_{\mathcal{R}}(y)=\operatorname{cl}_{\mathcal{R}}\big(\{y\}\big)=\bigcap_{R\in\mathcal{R}}R^{-1}\big[\{y\}\big]=\bigcap_{R\in\mathcal{R}}R^{-1}(y)=\Big(\bigcap_{R\in\mathcal{R}}R\Big)^{-1}(y)=\delta_{\mathcal{R}}^{-1}(y).$$

Theorem 11.5. *If $\mathcal{R}$ is relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is reflexive,

(2) $A\in\operatorname{Int}_{\mathcal{R}}(B)$ implies $A\subseteq B$ for all $A, B\subseteq X$,

(3) $A\cap B\neq\emptyset$ implies $A\in\operatorname{Cl}_{\mathcal{R}}(B)$ for all $A, B\subseteq X$.

Remark 11.6. In addition to the above two theorems , it is also worth mentioning that if $\mathcal{R}$ is a reflexive relator on X, then

(1) $\operatorname{Int}_{\mathcal{R}}$ is a transitive relation on $\mathcal{P}(X)$,

(2) $\operatorname{int}_{\mathcal{R}}\big(A \setminus \operatorname{int}_{\mathcal{R}}(A)\big) = \emptyset = \operatorname{int}_{\mathcal{R}}\big(\operatorname{cl}_{\mathcal{R}}(A) \setminus A\big)$ for all $A \subseteq X$.

Thus, for instance, for any $A \subseteq X$ we have $A \in \mathcal{F}_{\mathcal{R}}$ if and only if $\operatorname{cl}_{\mathcal{R}}(A) \setminus A \in \mathcal{T}_{\mathcal{R}}$.

Definition 11.7. For a relator $\mathcal{R}$ on X, we say that:

(1) $\mathcal{R}$ is *quasi-topological* if $x \in \operatorname{int}_{\mathcal{R}}\big(\operatorname{int}_{\mathcal{R}}\big(R(x)\big)\big)$ for all $x \in X$ and $R \in \mathcal{R}$,

(2) $\mathcal{R}$ is *topological* if for any $x \in X$ and $R \in \mathcal{R}$ there exists $V \in \mathcal{T}_{\mathcal{R}}$ such that $x \in V \subseteq R(x)$.

The appropriateness of these definitions is already quite obvious from the following three theorems.

Theorem 11.8. *If $\mathcal{R}$ is relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ *is quasi-topological,*

(2) $\operatorname{int}_{\mathcal{R}}\big(R(x)\big) \in \mathcal{T}_{\mathcal{R}}$ *for all* $x \in X$ *and* $R \in \mathcal{R}$*,*

(3) $\operatorname{int}_{\mathcal{R}}(A) \in \mathcal{T}_{\mathcal{R}}$ $\big(\operatorname{cl}_{\mathcal{R}}(A) \in \mathcal{F}_{\mathcal{R}}\big)$ *for all* $A \subseteq X$.

Theorem 11.9. *If $\mathcal{R}$ is relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ *is topological,* *(2)* $\mathcal{R}$ *is reflexive and quasi-topological.*

Theorem 11.10. *If $\mathcal{R}$ is relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ *is topological,*

(3) $\operatorname{int}_{\mathcal{R}}(A) = \bigcup \mathcal{T}_{\mathcal{R}} \cap \mathcal{P}(A)$ *for all* $A \subseteq X$*,*

(3) $\operatorname{cl}_{\mathcal{R}}(A) = \bigcap \mathcal{F}_{\mathcal{R}} \cap \mathcal{P}^{-1}(A)$ *for all* $A \subseteq X$.

Remark 11.11. By Theorem 11.8, the relator $\mathcal{R}$ may be called *weakly (strongly) quasi-topological* if $\operatorname{cl}_{\mathcal{R}}(x) \in \mathcal{F}_{\mathcal{R}}$ $\big(R(x) \in \mathcal{T}_{\mathcal{R}}\big)$ for all $x \in X$ and $R \in \mathcal{R}$.

Moreover, by Theorem 11.9, the relator $\mathcal{R}$ may be called *weakly (strongly) toplogical* if it is reflexive and weakly (strongly) quasi-topological.

However, it is now more important to note that, as a immediate consequence of the above theorems, we can also state

Corollary 11.12. *If $\mathcal{R}$ is a topological relator on X, then for any $A \subset X$*

(1) $A \in \mathcal{E}_{\mathcal{R}}$ *if and only if there exists* $V \in \mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\}$ *such that* $V \subseteq A$*,*

(2) $A \in \mathcal{D}_{\mathcal{R}}$ *if and only if for all* $W \in \mathcal{F}_{\mathcal{R}} \setminus \{X\}$ *we have* $A \setminus W \neq \emptyset$.

Theorem 11.13. *If $\mathcal{R}$ is relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ *is topological,*

(2) $\mathcal{R}$ *is topologically equivalent to* $\mathcal{R}^{\wedge\infty}$*,*

(3) $\mathcal{R}$ *is topologically equivalent to a preorder relator on* X.

Proof. For instance, we shall show that $(1) \Longrightarrow (3) \Longrightarrow (2)$. Namely, (2) trivially implies (3). Moreover, the proof of the implication $(3) \Longrightarrow (1)$ is quite straightforward.

For this, note that if (1) holds, then by Definition 11.7, for any $x \in X$ and $R \in \mathcal{R}$, there exists $V \in \mathcal{T}_{\mathcal{R}}$ such that $x \in V \subseteq R(x)$.

Hence, by considering the Pervin relator

$$\mathcal{S} = \mathcal{R}_{\mathcal{T}_{\mathcal{R}}} = \left\{ R_V : \quad V \in \mathcal{T}_{\mathcal{R}} \right\}, \qquad \text{where} \qquad R_V = V^{\,2} \cup V^{\,c} \times X\,,$$

we can note that $\mathcal{R} \subseteq \mathcal{S}^{\wedge}$, and thus $\mathcal{R}^{\wedge} \subseteq \mathcal{S}^{\wedge\wedge} = \mathcal{S}^{\wedge}$.

Moreover, since

$$R_V(x) = V \quad \text{if} \quad x \in V \qquad \text{and} \qquad R_V(x) = X \quad \text{if} \quad x \in V^{\,c},$$

we can also note that $\mathcal{S} \subseteq \mathcal{R}^{\wedge}$, and thus $\mathcal{S}^{\wedge} \subseteq \mathcal{R}^{\wedge\wedge} = \mathcal{R}^{\wedge}$.

Therefore, we actually have $\mathcal{R}^{\wedge} = \mathcal{S}^{\wedge}$, and thus $\mathcal{R}$ is topologically equivalent to $\mathcal{S}$. Hence, since $\mathcal{S}$ is a preorder relator on X, we can already see that (3) also holds.

On the other hand, if $\mathcal{S}$ is a preorder relator on X such that $\mathcal{R}^{\wedge} = \mathcal{S}^{\wedge}$, then we can easily see $\mathcal{S} \subseteq \mathcal{R}^{\wedge}$ and thus $\mathcal{S} = \mathcal{S}^{\infty} \subseteq \mathcal{R}^{\wedge\infty}$. Therefore, we also have $\mathcal{R}^{\wedge} = \mathcal{S}^{\wedge} \subseteq \left(\mathcal{R}^{\wedge\infty} \right)^{\wedge}$.

Moreover, by using Remark 6.2 and Theorem 6.7, we can can also easily see that $\mathcal{R}^{\wedge\infty} \subseteq \mathcal{R}^{\wedge *} = \mathcal{R}^{\wedge}$, and thus $\left(\mathcal{R}^{\wedge\infty} \right)^{\wedge} \subseteq \mathcal{R}^{\wedge\wedge} = \mathcal{R}^{\wedge}$. Therefore, we actually have $\mathcal{R}^{\wedge} = \left(\mathcal{R}^{\wedge\infty} \right)^{\wedge}$, and thus (2) also holds.

Definition 11.14. For any relator $\mathcal{R}$ on X, we say that:

(1) $\mathcal{R}$ is *quasi-proximal* if $A \in \operatorname{Int}_{\mathcal{R}} \left[\tau_{\mathcal{R}} \cap \operatorname{Int}_{\mathcal{R}} \left(R[\,A\,] \right) \right]$ for all $A \subseteq X$ and $R \in \mathcal{R}$,

(2) $\mathcal{R}$ is *proximal* if for any $A \subseteq X$ and $R \in \mathcal{R}$ there exists $V \in \tau_{\mathcal{R}}$ such that $A \subseteq V \subseteq R[\,A\,]$.

Remark 11.15. Note that thus, for any relator $\mathcal{R}$ on X, the following assertions are equivalent:

(1) $\mathcal{R}$ is quasi-proximal,

(2) for any $A \subseteq X$ and $R \in \mathcal{R}$, there exists $V \in \tau_{\mathcal{R}}$ such that $A \in \operatorname{Int}_{\mathcal{R}}(V)$ and $V \in \operatorname{Int}_{\mathcal{R}} \left(R[\,A\,] \right)$.

The appropriateness of the above definitions is already quite obvious from the following analogues of Theorems 11.9, 11.10 and 11.13.

Theorem 11.16. *If $\mathcal{R}$ is relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is proximal, *(2) $\mathcal{R}$ is reflexive and quasi-proximal.*

Theorem 11.17. *If $\mathcal{R}$ is relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is proximal,

(2) $Int_{\mathcal{R}}(A) = \mathcal{P}\left[\tau_{\mathcal{R}} \cap \mathcal{P}(A) \right]$ for all $A \subseteq X$,

(3) for any $B \in \operatorname{Int}_{\mathcal{R}}(A)$, there exists $V \in \tau_{\mathcal{R}}$ such that $B \subseteq V \subseteq A$.

Theorem 11.18. *If $\mathcal{R}$ is relator on X, then the following assertions are equivalent:*

(1) *$\mathcal{R}$ is proximal,*

(2) *$\mathcal{R}$ is proximally equivalent to $\mathcal{R}^{\infty}$ or $\mathcal{R}^{\#\infty}$,*

(3) *$\mathcal{R}$ is proximally equivalent to a preorder relator on X.*

In principle, each theorem on topological and quasi-topological relators can be immediately derived from a corresponding theorem on proximal and quasi-proximal relators by using the following two theorems.

Theorem 11.19. *For a nonvoid relator $\mathcal{R}$ on X, the following assertions are equivalent:*

(1) *$\mathcal{R}$ is quasi-topological,* *(2)* *$\mathcal{R}^{\wedge}$ is quasi-proximal.*

Proof. Note that if (2) holds, then in particular, for any $x \in X$ and $R \in \mathcal{R}$, we have

$$\{x\} \in \operatorname{Int}_{\mathcal{R}^{\wedge}} \big[\, \tau_{\mathcal{R}^{\wedge}} \cap \operatorname{Int}_{\mathcal{R}^{\wedge}} \big(R\,[\,\{x\}\,]\,\big)\,\big] \subseteq \operatorname{Int}_{\mathcal{R}^{\wedge}} \big[\operatorname{Int}_{\mathcal{R}^{\wedge}} \big(R(x) \big) \big]\,.$$

Therefore, there exists $V \in \operatorname{Int}_{\mathcal{R}^{\wedge}} \big(R(x) \big)$ such that $\{x\} \in \operatorname{Int}_{\mathcal{R}^{\wedge}}(V)$. Hence, by using Theorem 6.14, we can infer that $x \in \operatorname{int}_{\mathcal{R}}(V)$ and $V \subseteq \operatorname{int}_{\mathcal{R}}\big(R(x) \big)$. Therefore, $x \in \operatorname{int}_{\mathcal{R}} \big(\operatorname{int}_{\mathcal{R}}\big(R(x) \big) \big)$ also holds, and thus $\mathcal{R}$ is quasi-topological.

To prove the converse implication, assume now that (1) holds and $A \subseteq X$ and $S \in \mathcal{R}^{\wedge}$. Define $V = \operatorname{int}_{\mathcal{R}}\big(S\,[\,A\,] \big)$. Then, by Theorem 11.8 and Corollary 6.15, we have $V \in \mathcal{T}_{\mathcal{R}} = \tau_{\mathcal{R}^{\wedge}}$. Moreover, since $V \subseteq \operatorname{int}_{\mathcal{R}}\big(S\,[\,A\,] \big)$, by Theorem 6.14 we also have $V \in \operatorname{Int}_{\mathcal{R}^{\wedge}}\big(S\,[\,A\,] \big)$. Therefore, $V \in \tau_{\mathcal{R}^{\wedge}} \cap \operatorname{Int}_{\mathcal{R}^{\wedge}}\big(S\,[\,A\,] \big)$.

On the other hand, since $S \in \mathcal{R}^{\wedge}$ and $S\,[\,A\,] \subseteq S[\,A\,]$, we can also note that $A \in \operatorname{Int}_{\mathcal{R}^{\wedge}}\big(S\,[\,A\,] \big)$. Hence, by using Theorem 6.14, we can infer that $A \subseteq \operatorname{int}_{\mathcal{R}}\big(S\,[\,A\,] \big) = V$. Moreover, since $V \in \tau_{\mathcal{R}^{\wedge}}$, we can also note that $V \in \operatorname{Int}_{\mathcal{R}^{\wedge}}(V)$. Hence, since $A \subseteq V$, we can infer that $A \in \operatorname{Int}_{\mathcal{R}^{\wedge}}(V)$. Therefore, since $V \in \tau_{\mathcal{R}^{\wedge}} \cap \operatorname{Int}_{\mathcal{R}^{\wedge}}\big(S\,[\,A\,] \big)$, we also have $A \in \operatorname{Int}_{\mathcal{R}} \big[\, \tau_{\mathcal{R}} \cap \operatorname{Int}_{\mathcal{R}} \big(R\,[\,A\,] \big) \big]$. This shows that (2) also holds.

Remark 11.20. From the above proof, we can see that, for a relator $\mathcal{R}$ on X, the following assertions are also equivalent:

(1) $\mathcal{R}$ is quasi-proximal,

(2) $\{x\} \in \operatorname{Int}_{\mathcal{R}^{\wedge}} \big[\operatorname{Int}_{\mathcal{R}^{\wedge}} \big(R(x) \big) \big]$ for all $x \in X$ and $R \in \mathcal{R}$,

(3) $A \in \operatorname{Int}_{\mathcal{R}^{\wedge}} \big[\operatorname{Int}_{\mathcal{R}^{\wedge}} \big(S\,[\,A\,] \big) \big]$ for all $A \subseteq X$ and $S \in \mathcal{R}^{\wedge}$.

Theorem 11.21. *For any relator $\mathcal{R}$ on X, the following assertions are equivalent:*

(1) *$\mathcal{R}$ is topological,* *(2)* *$\mathcal{R}^{\wedge}$ is proximal.*

Proof. If (1) holds, then by Theorem 11.9 the relator $\mathcal{R}$ is reflexive and quasi-topological. Hence, by the corresponding definitions, it is clear that the relator $\mathcal{R}^{\wedge}$ is also reflexive. Moreover, if $\mathcal{R} \neq \emptyset$, then from Theorem 11.18 we can see that $\mathcal{R}^{\wedge}$ is quasi-proximal. Thus, by Theorem 11.16, assertion (2) also holds.

Quite similarly, we can also see that (2) implies (1) whenever $\mathcal{R} \neq \emptyset$. The case $\mathcal{R} = \emptyset$ has to be treated separately by using that $\mathcal{R}^{\wedge} = \emptyset$ if $\mathcal{R} = \emptyset$ and $X \neq \emptyset$.

12. The main definitions of mild continuities

Notation 12.1. In the sequel, we shall assume that:

(1) $\Box$ *is a direct unary operation for relators,*

(2) $(X, Y)(\mathcal{R})$ *and* $(Z, W)(\mathcal{S})$ *are relator spaces,*

(3) $\mathcal{F}$ *is a relator on* X *to* Z *and* $\mathcal{G}$ *is a relator on* Y *to* W*.*

Remark 12.2. Now, to keep in mind the above assumptions, for any $R \in \mathcal{R}$, $S \in \mathcal{S}$, $F \in \mathcal{F}$ and $G \in \mathcal{G}$, we may again use the diagram:

$$\begin{array}{ccc} X & \xrightarrow{F} & Z \\ {\scriptstyle R}\big\downarrow & & \big\downarrow{\scriptstyle S} \\ Y & \xrightarrow{G} & W \end{array}$$

Moreover, by Definition 1.8, we may, for instance, naturally consider the next

Definition 12.3. Under the above assumptions, we say that the pair $(\mathcal{F}, \mathcal{G})$ of relators is *mildly* $\Box$*–continuous*, with respect to the relators $\mathcal{R}$ and $\mathcal{S}$, if

$$\Big(\big(\mathcal{G}^{\Box} \big)^{-1} \circ \mathcal{S}^{\Box} \circ \mathcal{F}^{\Box} \Big)^{\Box} \subseteq \mathcal{R}^{\Box\Box}.$$

Remark 12.4. Thus, the pair $(\mathcal{F}, \mathcal{G})$ may be naturally called *properly mildly continuous* if it is mildly $\Box$–continuous with $\Box$ being the identity operation for relators. That is, $\mathcal{G}^{-1} \circ \mathcal{S} \circ \mathcal{F} \subseteq \mathcal{R}$.

Remark 12.5. Moreover, the pair $(\mathcal{F}, \mathcal{G})$ may, for instance, be naturally called *uniformly, proximally, topologically, and paratopologically mildly continuous* if it is mildly $\Box$–continuous with $\Box = \ast$, $\#$, $\wedge$, and $\vartriangle$, respectively.

Remark 12.6. And, the pair $(\mathcal{F}, \mathcal{G})$ may, for instance, be naturally called *quasi-topologically and ultra-topologically* mildly continuous if its is mildly $\Box$–continuous with $\Box = \wedge\infty$ and $\wedge\partial$, respectively.

Remark 12.7. Moreover, the pair $(\mathcal{F}, \mathcal{G})$ may, for instance, be naturally called *infinitesimally and ultimately mildly continuous* if it is $\Box$–mildly continuous with $\Box = \bullet$ and $\blacklozenge$, respectively.

Now, by specializing Definition 12.3, we may also naturally have the following

Definition 12.8. For any $F \in \mathcal{F}$ and $G \in \mathcal{G}$, the pair (F, G) of relations is called *mildly* $\Box$*–continuous*, with respect to the relators $\mathcal{R}$ and $\mathcal{S}$, if the pair $\big(\{F\}, \{G\}\big)$ of relators has the same property.

Remark 12.9. To apply this definition, recall that if in particular $\Box = \#$ or $\wedge$, then for any $F \in \mathcal{F}$ and $G \in \mathcal{G}$ we have

$$\{F\}^{\Box} = \{F\}^{\ast} \qquad \text{and} \qquad \big(\{G\}^{\Box}\big)^{-1} = \big(\{G\}^{\ast}\big)^{-1} = \big\{G^{-1}\big\}^{\ast}.$$

However, in contrast to the above equalities, for instance we already have

$$\{F\}^{\vartriangle} = \big(F \circ X^{X}\big)^{\ast} \quad \text{and} \quad \big(\{G\}^{\vartriangle}\big)^{-1} = \Big(\big(G \circ Y^{Y}\big)^{\ast}\Big)^{-1} = \Big(\big(Y^{Y}\big)^{-1} \circ G^{-1}\Big)^{\ast}.$$

Now, by using Definition 12.8, we may also naturally introduce the following

Definition 12.10. Under the assumptions of Notation 11.1, we say that the pair $(\mathcal{F}, \mathcal{G})$ of relators is *elementwise mildly $\square$–continuous*, with respect to the relators $\mathcal{R}$ and $\mathcal{S}$, if for any $F \in \mathcal{F}$ and $G \in \mathcal{G}$ the pair (F, G) of relations is mildly $\square$–continuous with respect to the relators $\mathcal{R}$ and $\mathcal{S}$.

Remark 12.11. Thus, the pair $(\mathcal{F}, \mathcal{G})$ may, for instance, be naturally called *elementwise topologically mildly continous* if it is elementwise mildly $\square$–continuous with $\square = \wedge$.

Now, as a natural extension of [43, Definition 4.6], we may also naturally have

Definition 12.12. Under the assumptions of Notation 12.1, we say that the pair

(1) $(\mathcal{F}, \mathcal{G})$ is *lower selectionally mildly $\square$–continuous* if for any $F \in \mathcal{F}$ and $G \in \mathcal{G}$ and any selection f of F the pair (f, G) is mildly $\square$–continuous,

(2) $(\mathcal{F}, \mathcal{G})$ is *upper selectionally mildly $\square$–continuous* if for any $F \in \mathcal{F}$ and $G \in \mathcal{G}$ and any selection g of G the pair (F, g) is mildly $\square$–continuous.

Remark 12.13. Now, the pair $(\mathcal{F}, \mathcal{G})$ may also be naturally called *selectionally mildly $\square$–continuous* if it is both lower and upper selectionally mildly $\square$–continuous.

Remark 12.14. Moreover, the pair $(\mathcal{F}, \mathcal{G})$ may also be naturally called *doubly selectionally mildly $\square$–continuous* if for any $F \in \mathcal{F}$ and $G \in \mathcal{G}$ and for any selections f of F and g of G, the pair (f, g) is mildly $\square$–continuous.

Remark 12.15. Finally, we note that, in the $X = Y$ and $Z = W$ particular case, the relator $\mathcal{F}$ and a relation $F \in \mathcal{F}$ may, for instance, be naturally called mildly $\square$–continuous if the pairs $(\mathcal{F}, \mathcal{F})$ and (F, F), respectively, have the same property.

13. Reduction theorems for mild continuities

Remark 13.1. If in particular the operation $\square$ is idempotent, then by the corresponding definitions it is clear that the following assertions are equivalent:

(1) $(\mathcal{F}, \mathcal{G})$ is mildly $\square$–continuous, (2) $\Big(\big(\mathcal{G}^{\square} \big)^{-1} \circ \mathcal{S}^{\square} \circ \mathcal{F}^{\square} \Big)^{\square} \subseteq \mathcal{R}^{\square}$.

Moreover, by using Theorems 5.17 and 8.13, we can easily prove the following two theorems.

Theorem 13.2. *If in particular $\square$ is a closure operation, then the following assertions are equivalent:*

(1) $(\mathcal{F}, \mathcal{G})$ is mildly $\square$–continuous, *(2) $\big(\mathcal{G}^{\square} \big)^{-1} \circ \mathcal{S}^{\square} \circ \mathcal{F}^{\square} \subseteq \mathcal{R}^{\square}$.*

Proof. Note that now, by the idempotency of $\square$ and Theorem 5.17, for any two relators $\mathcal{U}$ and $\mathcal{V}$ on X to Y we have

$$\mathcal{V}^{\square} \subseteq \mathcal{U}^{\square\square} \iff \mathcal{V}^{\square} \subseteq \mathcal{U}^{\square} \iff \mathcal{V} \subseteq \mathcal{U}^{\square}.$$

Theorem 13.3. *If in particular* $\square$ *is an increasing involution, then the following assertions are equivalent:*

(1) $(\mathcal{F},\, \mathcal{G})$ *is mildly* $\square$*–continuous,*

(2) $\big(\mathcal{G}^{\square}\big)^{-1} \circ \mathcal{S}^{\square} \circ \mathcal{F}^{\square} \subseteq \mathcal{R}^{\square}$, (3) $\Big(\big(\mathcal{G}^{\square}\big)^{-1} \circ \mathcal{S}^{\square} \circ \mathcal{F}^{\square}\Big)^{\square} \subseteq \mathcal{R}$.

Proof. Note that now, by the involutiveness of $\square$ and Theorem 8.13, for any two relators $\mathcal{U}$ and $\mathcal{V}$ on X to Y we have

$$\mathcal{V}^{\square} \subseteq \mathcal{U}^{\square\square} \iff \mathcal{V}^{\square} \subseteq \mathcal{U} \iff \mathcal{V} \subseteq \mathcal{U}^{\square}.$$

Now, as an immediate consequence of the above two theorems and Remark 12.4, we can also state

Corollary 13.4. *If in particular* $\square$ *is either a closure operation or an increasing involution, then the following assertions are equivalent:*

(1) $(\mathcal{F},\, \mathcal{G})$ *is mildly* $\square$*–continuous with respect to* $\mathcal{R}$ *and* $\mathcal{S}$*,*

(2) $\big(\mathcal{F}^{\square},\, \mathcal{G}^{\square}\big)$ *is properly mildly continuous with respect to* $\mathcal{R}^{\square}$ *and* $\mathcal{S}^{\square}$*.*

However, it is now more important to note that, by using Theorem 10.3 and Corollary 10.6, we can easily prove the following

Theorem 13.5. *If in particular* $\square$ *is inversion and composition compatible, then the following assertions are equivalent:*

(1) $(\mathcal{F},\, \mathcal{G})$ *is mildly* $\square$*–continuous,* (2) $\big(\mathcal{G}^{-1} \circ \mathcal{S} \circ \mathcal{F}\big)^{\square} \subseteq \mathcal{R}^{\square}$.

Proof. Note that now, by the assumed compatibilities of $\square$ and Corollary 10.6, we have

$$\Big(\big(\mathcal{G}^{\square}\big)^{-1} \circ \mathcal{S}^{\square} \circ \mathcal{F}^{\square}\Big)^{\square} = \Big(\big(\mathcal{G}^{-1}\big)^{\square} \circ \mathcal{S}^{\square} \circ \mathcal{F}^{\square}\Big)^{\square} = \big(\mathcal{G}^{-1} \circ \mathcal{S} \circ \mathcal{F}\big)^{\square}.$$

Moreover, by Theorem 10.3, we now also have $\mathcal{R}^{\square\square} = \mathcal{R}^{\square}$.

From this theorem, by Theorem 5.17, it is clear that in particular we also have

Theorem 13.6. *If in particular* $\square$ *is an inversion and composition compatible closure operation, then the following assertions are equivalent:*

(1) $(\mathcal{F},\, \mathcal{G})$ *is mildly* $\square$*–continuous,* (2) $\mathcal{G}^{-1} \circ \mathcal{S} \circ \mathcal{F} \subseteq \mathcal{R}^{\square}$.

Hence, by using Remark 12.4, we can immediately derive the following

Corollary 13.7. *Under the above assumptions on* $\square$*, the following assertions are also equivalent:*

(1) $(\mathcal{F},\, \mathcal{G})$ *is mildly* $\square$*–continuous with respect to* $\mathcal{R}$ *and* $\mathcal{S}$*,*

(2) $\big(\mathcal{F},\, \mathcal{G}\big)$ *is properly mildly continuous with respect to* $\mathcal{R}^{\square}$ *and* $\mathcal{S}$*.*

However, it is now more important to note that, by using Theorem 13.6, we can also easily prove the following two theorems.

Theorem 13.8. *If in particular* $\square$ *is an inversion and composition compatible closure operation, dominating another such operation* $\Diamond$ *for relators, then the mild* $\Diamond$*–continuity of* $(\mathcal{F},\, \mathcal{G})$ *implies the mild* $\square$*–continuity of* $(\mathcal{F},\, \mathcal{G})$*.*

Proof. If the pair $(\mathcal{F}, \mathcal{G})$ is mildly $\Diamond$–continuous, then by Theorem 13.6 we have $\mathcal{G}^{-1} \circ \mathcal{S} \circ \mathcal{F} \subseteq \mathcal{R}^{\Diamond}$. Hence, by using the inclusion $\mathcal{R}^{\Diamond} \subseteq \mathcal{R}^{\Box}$, we can infer that $G^{-1} \circ \mathcal{S} \circ F \subseteq \mathcal{R}^{\Box}$. Therefore, by Theorem 13.6, the pair $(\mathcal{F}, \mathcal{G})$ is also mildly $\Box$–continuous.

Remark 13.9. From this theorem, by Theorems 6.5, 9.6 and 10.13, it is clear that, for instance, the "proper mild continuity of $(\mathcal{F}, \mathcal{G})$" implies the "uniform mild continuity of $(\mathcal{F}, \mathcal{G})$ "implies the "proximal mild continuity of $(\mathcal{F}, \mathcal{G})$".

Theorem 13.10. *If in particular $\Box$ is an inversion and composition compatible closure operation, then the following assertions are equivalent:*

(1) *$(\mathcal{F}, \mathcal{G})$ is mildly $\Box$–continuous,*

(2) *$(\mathcal{F}, \mathcal{G})$ is elementwise mildly $\Box$–continuous.*

Proof. By Theorem 13.6 and the corresponding definitions, it is clear that

$$(1) \iff \mathcal{G}^{-1} \circ \mathcal{S} \circ \mathcal{F} \subseteq \mathcal{R}^{\Box}$$
$$\iff \forall\, F \in \mathcal{F} : \ \forall\, G \in \mathcal{G} : \quad G^{-1} \circ \mathcal{S} \circ F \subseteq \mathcal{R}^{\Box} \iff (2).$$

Remark 13.11. Unfortunately, this theorem cannot also be applied to the operations $\wedge$ and $\vartriangle$.

Therefore, it is worth noticing that the implication $(1) \Longrightarrow (2)$ is already true if $\Box$ is only increasing.

For this, it is convenient to prove first a more general theorem.

Theorem 13.12. *If in particular $\Box$ is increasing and $(\mathcal{F}, \mathcal{G})$ is mildly $\Box$–continuous, then for any $\mathcal{F}_1 \subseteq \mathcal{F}$ and $\mathcal{G}_1 \subseteq \mathcal{G}$ the pair $(\mathcal{F}_1, \mathcal{G}_1)$ is also mildly $\Box$–continuous.*

Proof. Because of the assumed increasingness of $\Box$, we have

$$\mathcal{F}_1^{\Box} \subseteq \mathcal{F}^{\Box}, \qquad \text{and thus also} \qquad \left(\mathcal{G}_1^{\Box}\right)^{-1} \subseteq \left(\mathcal{G}^{\Box}\right)^{-1}.$$

Hence, by using the increasingness of composition of relators, we can infer that

$$\left(\mathcal{G}_1^{\Box}\right)^{-1} \circ \mathcal{S}^{\Box} \circ \mathcal{F}_1^{\Box} \subseteq \left(\mathcal{G}^{\Box}\right)^{-1} \circ \mathcal{S}^{\Box} \circ \mathcal{F}^{\Box}.$$

Thus, again by the increasingness $\Box$, we also have

$$\Big(\left(\mathcal{G}_1^{\Box}\right)^{-1} \circ \mathcal{S}^{\Box} \circ \mathcal{F}_1^{\Box}\Big)^{\Box} \subseteq \Big(\left(\mathcal{G}^{\Box}\right)^{-1} \circ \mathcal{S}^{\Box} \circ \mathcal{F}^{\Box}\Big)^{\Box}.$$

Therefore, by Definition 12.3, the mild $\Box$–continuity of $(\mathcal{F}, \mathcal{G})$ implies that of $(\mathcal{F}_1, \mathcal{G}_1)$.

Hence, by letting $\mathcal{F}_1$ and $\mathcal{G}_1$ to be singletons, we can immediately derive

Corollary 13.13. *If in particular $\Box$ is increasing and $(\mathcal{F}, \mathcal{G})$ is mildly $\Box$–continuous, then $(\mathcal{F}, \mathcal{G})$ is elementwise mildly $\Box$–continuous.*

14. Some further theorems on mild continuities

By using the corresponding definitions, we can also easily prove the following

Theorem 14.1. *If in particular $\Box$ is $\Diamond$–absorbing, for some direct unary operation $\Diamond$ for relators, then the following assertions are equivalent:*

(1) $(\mathcal{F},\, \mathcal{G})$ is mildly $\Box$–continuous with respect to $\mathcal{R}$ and $\mathcal{S}$,

(2) $\big(\mathcal{F}^{\Diamond},\, \mathcal{G}^{\Diamond}\big)$ is mildly $\Box$–continuous with respect to $\mathcal{R}^{\Diamond}$ and $\mathcal{S}^{\Diamond}$.

Proof. By the corresponding definitions, it is clear that

$$(1) \iff \Big(\,(\mathcal{G}^{\Box})^{-1}\circ\mathcal{S}^{\Box}\circ\mathcal{F}^{\Box}\Big)^{\Box}\subseteq\mathcal{R}^{\Box\,\Box}$$
$$\iff \bigg(\Big(\big(\mathcal{G}^{\Diamond}\big)^{\Box}\Big)^{-1}\circ\big(\mathcal{S}^{\Diamond}\big)^{\Box}\circ\big(\mathcal{F}^{\Diamond}\big)^{\Box}\bigg)^{\Box}\subseteq\big(\mathcal{R}^{\Diamond}\big)^{\Box\,\Box} \iff (2)\,.$$

From this theorem, by letting $\Diamond=\Box$, we can immediately derive

Corollary 14.2. *If in particular $\Box$ is idempotent, then the following assertions are equivalent:*

(1) $(\mathcal{F},\, \mathcal{G})$ is mildly $\Box$–continuous with respect to $\mathcal{R}$ and $\mathcal{S}$,

(2) $\big(\mathcal{F}^{\Box},\, \mathcal{G}^{\Box}\big)$ is mildly $\Box$–continuous with respect to $\mathcal{R}^{\Box}$ and $\mathcal{S}^{\Box}$.

Now, as an extension of Theorem 13.12, we can also easily prove the following

Theorem 14.3. *If in particular $\Box$ is increasing and $\Diamond$–absorbing, for some direct unary operation $\Diamond$ for relators, and the pair $(\mathcal{F},\, \mathcal{G})$ is mildly $\Box$–continuous, then for any $\mathcal{F}_1\subseteq\mathcal{F}^{\Diamond}$ and $\mathcal{G}_1\subseteq\mathcal{G}^{\Diamond}$ the pair $\big(\mathcal{F}_1,\, \mathcal{G}_1\big)$ is also mildly $\Box$–continuous.*

Proof. Because of the above assumptions, we have

$$\mathcal{F}_1^{\Box}\subseteq\mathcal{F}^{\Diamond\,\Box}=\mathcal{F}^{\Box},\qquad\text{and thus also}\qquad\big(\mathcal{G}_1^{\Box}\big)^{-1}\subseteq\big(\mathcal{G}^{\Box}\big)^{-1}.$$

Hence, as in the proof of Theorem 13.12, we can already infer that

$$\Big(\,(G_1^{\Box})^{-1}\circ\mathcal{S}^{\Box}\circ\mathcal{F}_1^{\Box}\Big)^{\Box}\subseteq\Big(\,(\mathcal{G}^{\Box})^{-1}\circ\mathcal{S}^{\Box}\circ\mathcal{F}^{\Box}\Big)^{\Box}.$$

Therefore, by Definition 12.3, the mild $\Box$–continuity of $(\mathcal{F},\, \mathcal{G})$ implies that of $\big(\mathcal{F}_1,\, \mathcal{G}_1\big)$.

Remark 14.4. In this theorem, instead of the $\Diamond$–absorbingness of $\Box$, it is enough to assume only that $\mathcal{U}^{\Diamond\,\Box}\subseteq\mathcal{U}^{\Box}$ for every relator $\mathcal{U}$.

However, if in particular $\Diamond$ is extensive, then because of the assumed increasingness of $\Box$ the corresponding equality is also true.

A simple application of the $\Diamond=*$ particular cease of Theorem 14.3 to singleton relators gives the following

Corollary 14.5. *If in particular $\Box$ is an increasing, $*$–absorbing operation, and $F\in\mathcal{F}$ and $G\in\mathcal{G}$ such that the pair $(F,\, G)$ is mildly $\Box$–continuous, then for any $F_1\in\mathcal{F}$ and $G_1\in\mathcal{G}$, with $F\subseteq F_1$ and $G\subseteq G_1$, the pair $(F_1,\, G_1)$ is also mildly $\Box$–continuous.*

Proof. By the above assumptions on F_1 and G_1, and the definition of $*$, we have $\{F_1\} \subseteq \{F\}^*$ and $\{G_1\} \subseteq \{G\}^*$. Therefore, by Theorem 14.3, the mild $\Box$–continuity of $\big(\{F\}, \{G\}\big)$ implies that of $\big(\{F_1\}, \{G_1\}\big)$. Thus, by Definition 12.8, the mild $\Box$–continuity of (F, G) also implies that of (F_1, G_1).

By using Theorem 13.2, we can also easily prove the following

Theorem 14.6. *If in particular* $\Box$ *is a* $\Diamond$*–invariant closure operation for some closure operation* $\Diamond$ *for relators, then the following assertions are equivalent:*

(1) $(\mathcal{F}, \mathcal{G})$ *is mildly* $\Box$*–continuous with respect to* $\mathcal{R}$ *and* $\mathcal{S}$,

(2) $\big(\mathcal{G}^{\Box}, \mathcal{F}^{\Box}\big)$ *is mildly* $\Diamond$*–continuous with respect to* $\mathcal{R}^{\Box}$ *and* $\mathcal{S}^{\Box}$.

Proof. By Theorem 13.2 and the corresponding definitions, it is clear that

$$(1) \iff \big(\mathcal{G}^{\Box}\big)^{-1} \circ \mathcal{S}^{\Box} \circ \mathcal{F}^{\Box} \subseteq \mathcal{R}^{\Box}$$
$$\iff \Big(\big(\mathcal{G}^{\Box}\big)^{\Diamond}\Big)^{-1} \circ \big(\mathcal{S}^{\Box}\big)^{\Diamond} \circ \big(\mathcal{F}^{\Box}\big)^{\Diamond} \subseteq \big(\mathcal{R}^{\Box}\big)^{\Diamond} \iff (2)\,.$$

Now, in additions to Theorems 14.6 and 14.1, we can also easily prove

Theorem 14.7. *If in particular* $\Box$ *is a* $\Diamond$*–compatible closure operation, for some closure operation* $\Diamond$ *for relators, then the following assertions are equivalent:*

(1) $(\mathcal{F}, \mathcal{G})$ *is mildly* $\Box\Diamond$*–continuous with respect to* $\mathcal{R}$ *and* $\mathcal{S}$,

(2) $\big(\mathcal{F}^{\Box}, \mathcal{G}^{\Box}\big)$ *is mildly* $\Diamond$*–continuous with respect to* $\mathcal{R}^{\Box}$ *and* $\mathcal{S}^{\Box}$,

(3) $\big(\mathcal{G}^{\Diamond}, \mathcal{F}^{\Diamond}\big)$ *is mildly* $\Box$*–continuous with respect to* $\mathcal{R}^{\Diamond}$ *and* $\mathcal{S}^{\Diamond}$.

Proof. From Theorem 8.6, we know that $\Box\Diamond$ is also a closure operation for relators. Hence, by Theorem 13.2 and the corresponding definitions, it is clear that

$$(1) \iff \big(\mathcal{G}^{\Box\Diamond}\big)^{-1} \circ \mathcal{S}^{\Box\Diamond} \circ \mathcal{F}^{\Box\Diamond} \subseteq \mathcal{R}^{\Box\Diamond}$$
$$\iff \Big(\big(\mathcal{G}^{\Box}\big)^{\Diamond}\Big)^{-1} \circ \big(\mathcal{S}^{\Box}\big)^{\Diamond} \circ \big(\mathcal{F}^{\Box}\big)^{\Diamond} \subseteq \big(\mathcal{R}^{\Box}\big)^{\Diamond} \iff (2)\,.$$

Now, since $\Box\Diamond = \Diamond\Box$, it is clear that assertions (1) and (3) are also equivalent.

Remark 14.8. From the latter two theorems, by letting $\Diamond$ to be the identity operation for relators, we can also immediately derive the "closure operation part" of Corollary 14.4.

However, it is now more important to note that by using the corresponding definitions, we can also easily prove the following

Theorem 14.9. *If in particular* $\Box$ *is inversion compatible, then the following assertions are equivalent:*

(1) $(\mathcal{F}, \mathcal{G})$ *is mildly* $\Box$*–continuous with respect to* $\mathcal{R}$ *and* $\mathcal{S}$,

(2) $(\mathcal{G}, \mathcal{F})$ *is mildly* $\Box$*–continuous with respect to* $\mathcal{R}^{-1}$ *and* $\mathcal{S}^{-1}$.

Proof. By Definition 12.3 and an inversion property of composition, it is clear that

$$
\begin{aligned}
(1) \iff & \Big((\mathcal{G}^{\Box})^{-1} \circ \mathcal{S}^{\Box} \circ \mathcal{F}^{\Box} \Big)^{\Box} \subseteq (\mathcal{R}^{\Box})^{\Box} \\
\iff & \bigg(\Big((\mathcal{G}^{\Box})^{-1} \circ \mathcal{S}^{\Box} \circ \mathcal{F}^{\Box} \Big)^{\Box} \bigg)^{-1} \subseteq \Big((\mathcal{R}^{\Box})^{\Box} \Big)^{-1} \\
\iff & \bigg(\Big((\mathcal{G}^{\Box})^{-1} \circ \mathcal{S}^{\Box} \circ \mathcal{F}^{\Box} \Big)^{-1} \bigg)^{\Box} \subseteq \Big((\mathcal{R}^{\Box})^{-1} \Big)^{\Box} \\
\iff & \Big((\mathcal{F}^{\Box})^{-1} \circ (\mathcal{S}^{\Box})^{-1} \circ \mathcal{G}^{\Box} \Big)^{\Box} \subseteq \Big((\mathcal{R}^{\Box})^{-1} \Big)^{\Box} \\
\iff & \Big((\mathcal{F}^{\Box})^{-1} \circ (\mathcal{S}^{-1})^{\Box} \circ \mathcal{G}^{\Box} \Big)^{\Box} \subseteq \Big((\mathcal{R}^{-1})^{\Box} \Big)^{\Box} \iff (2) .
\end{aligned}
$$

Remark 14.10. Unfortunately, concerning the elementwise complementation of relators, we cannot prove a similar theorem.

15. Detailed reformulations of proper, uniform, and proximal mild continuities

Recall that if in particular $\Box$ is an inversion and composition compatible closure operation, then by Theorem 13.10, instead of the mild $\Box$–continuity of the pair $(\mathcal{F}, \mathcal{G})$ it is enough to investigate only that of the pair (F, G) for all $F \in \mathcal{F}$ and $G \in \mathcal{G}$.

Therefore, concerning proper, uniform, and proximal mild continuities, we shall only prove here some very particular theorems.

Theorem 15.1. *For any $F \in \mathcal{F}$ and $G \in \mathcal{G}$, the following assertions are equivalent:*

(1) (F, G) *is properly mildly continuous,*

(2) $G^{-1} \circ \mathcal{S} \circ F \subseteq \mathcal{R}$, *(3)* $\operatorname{cl}_{F \boxtimes G}(\mathcal{S}) \subseteq \mathcal{R}$.

Proof. From Remark 12.4, we know that (1) and (2) are equivalent. Moreover, by [53, Theorem 4.3], for any $F \in \mathcal{F}$, $G \in \mathcal{G}$ and $S \in \mathcal{S}$ we have

$$G^{-1} \circ S \circ F = \big(F^{-1} \boxtimes G^{-1} \big)[\, S \,] = (F \boxtimes G)^{-1} [\, S \,] = \operatorname{cl}_{F \boxtimes G}(S) .$$

Therefore, because of the plausible notation

$$\operatorname{cl}_{F \boxtimes G}(\mathcal{S}) = \big\{ \operatorname{cl}_{F \boxtimes G}(S) : \quad S \in \mathcal{S} \big\} ,$$

assertions (2) and (3) are also equivalent.

Now, by using this theorem, we can also easily prove the following

Theorem 15.2. *For any $F \in \mathcal{F}$ and $G \in \mathcal{G}$, the following assertions are equivalent:*

(1) (F, G) *is properly mildly continuous,*

(2) for each $S \in \mathcal{S}$ *there exists* $R \in \mathcal{R}$ *such that* $R = G^{-1} \circ S \circ F$,

(3) for each $S \in \mathcal{S}$ *there exists* $R \in \mathcal{R}$ *such that for every* $x \in X$ *and* $y \in Y$ *we have* $y \in R(x)$ *if and only if* $G(y) \cap S\,[\,F(x)\,] \neq \emptyset$,

(4) for each $S \in \mathcal{S}$ there exists $R \in \mathcal{R}$ such that for every $x \in X$ and $y \in Y$ we have $y \in R(x)$ if and only if there exist $z \in F(x)$ and $w \in G(y)$ such that $w \in S(z)$.

Proof. By Theorem 15.1 and the corresponding definitions, it is clear that

$$(1) \iff G^{-1} \circ \mathcal{S} \circ F \subseteq \mathcal{R} \iff \forall\ S \in \mathcal{S}: \quad G^{-1} \circ S \circ F \in \mathcal{R}$$
$$\iff \forall\ S \in \mathcal{S}: \ \exists\ R \in \mathcal{R}: \quad R = G^{-1} \circ S \circ F.$$

Moreover, we can note that

$$R = G^{-1} \circ S \circ F \iff \forall\ x \in X: \ \forall\ y \in Y:$$
$$\Big(y \in R(x) \iff y \in \big(G^{-1} \circ S \circ F \big)(x) \Big).$$

Furthermore, by using some basic facts on relations, we can also easily see that

$$y \in \big(G^{-1} \circ S \circ F \big)(x) \iff y \in G^{-1}\big[S[F(x)] \big] \iff G(y) \cap S[F(x)] \neq \emptyset$$
$$\iff \exists\ w \in G(y): \ w \in S[F(x)] \iff \exists\ w \in G(y): \ \exists\ z \in F(x): \ w \in S(z).$$

Therefore, assertions (1)–(4) are also equivalent.

Analogously to Theorem 15.1, we can also easily prove the following

Theorem 15.3. *For any $F \in \mathcal{F}$ and $G \in \mathcal{G}$, the following assertions are equivalent:*

(1) (F, G) is uniformly mildly continuous,

(2) $G^{-1} \circ \mathcal{S} \circ F \subseteq \mathcal{R}^*$, *(3) $\operatorname{cl}_{F \boxtimes G}(\mathcal{S}) \subseteq \mathcal{R}^*$.*

Proof. By Remark 6.6, Theorems 9.6 and 10.13, $*$ is an inversion and composition compatible closure operation for relators. Therefore, by Theorem 13.6 and Remark 12.5, (1) and (2) are equivalent. Moreover, from the proof of Theorem 15.1, it is clear that (2) and (3) are also equivalent.

From this theorem, by Theorem 15.1, it is clear that we also have

Corollary 15.4. *For any $F \in \mathcal{F}$ and $G \in \mathcal{G}$, the following assertions are equivalent:*

(1) (F, G) is uniformly mildly continuous with respect to $\mathcal{R}$ and $\mathcal{S}$,

(2) (F, G) is properly mildly continuous with respect to $\mathcal{R}^*$ and $\mathcal{S}$.

Moreover, by using Theorem 15.3, we can also easily prove the following

Theorem 15.5. *For any $F \in \mathcal{F}$ and $G \in \mathcal{G}$, the following assertions are equivalent:*

(1) (F, G) is uniformly mildly continuous,

(2) for each $S \in \mathcal{S}$ there exists $R \in \mathcal{R}$ such that $R \subseteq G^{-1} \circ S \circ F$,

(3) for each $S \in \mathcal{S}$ there exists $R \in \mathcal{R}$ such that for every $x \in X$ and $y \in R(x)$ we have $G(y) \cap S[F(x)] \neq \emptyset$,

(4) for each $S \in \mathcal{S}$ there exists $R \in \mathcal{R}$ such that for every $x \in X$ and $y \in R(x)$ there exist $z \in F(x)$ and $w \in G(y)$ such that $w \in S(z)$.

Proof. By Theorem 15.3 and the corresponding definitions, it is clear that

$$(1) \iff G^{-1} \circ \mathcal{S} \circ F \subseteq \mathcal{R}^* \iff \forall\ S \in \mathcal{S}: \ G^{-1} \circ S \circ F \in \mathcal{R}^*$$
$$\iff \forall\ S \in \mathcal{S}: \ \exists\ R \in \mathcal{R}: \ R \subseteq G^{-1} \circ S \circ F\,.$$

Moreover, as in the proof of Theorem 15.2, we can note that

$$R \subseteq G^{-1} \circ S \circ F \iff \forall\ x \in X: \ R(x) \subseteq \big(G^{-1} \circ S \circ F \big)(x)$$
$$\iff \forall\ x \in X: \ \forall\ y \in R(x): \ y \in \big(G^{-1} \circ S \circ F \big)(x)\,.$$

Furthermore, from the proof of Theorem 15.2 we know that

$$y \in \big(G^{-1} \circ S \circ F \big)(x) \iff G(y) \cap S\,[\,F(x)\,] \neq \emptyset$$
$$\iff \exists\ w \in G(y): \ \exists\ z \in F(x): \ w \in S(z)\,.$$

Therefore, assertions (1)–(4) are also equivalent.

Analogously to Theorem 15.3, we can also easily prove the following

Theorem 15.6. *For any $F \in \mathcal{F}$ and $G \in \mathcal{G}$, the following assertions are equivalent:*

(1) $(F,\ G)$ *is proximally mildly continuous,*

(2) $G^{-1} \circ \mathcal{S} \circ F \subseteq \mathcal{R}^{\#}$, *(3)* $\mathrm{cl}_{F \boxtimes G}(\mathcal{S}) \subseteq \mathcal{R}^{\#}$.

Remark 15.7. From Theorems 15.1, 15.3 and 15.6, because of the inclusions $\mathcal{R} \subseteq \mathcal{R}^* \subseteq \mathcal{R}^{\#}$, it is also clear that "the proper mild continuity of $(F,\ G)$" implies "the uniform mild continuity of $(F,\ G)$" implies "the proximal mild continuity of $(F,\ G)$".

Moreover, from the above mentioned theorems and the equality $\mathcal{R}^{\#} = \big(\mathcal{R}^{\#}\big)^*$, it is clear that we also have

Corollary 15.8. *For any $F \in \mathcal{F}$ and $G \in \mathcal{G}$, the following assertions are equivalent:*

(1) $(F,\ G)$ *is proximally mildly continuous with respect to* $\mathcal{R}$ *and* $\mathcal{S}$,

(2) $(F,\ G)$ *is properly or uniformly mildly continuous with respect to* $\mathcal{R}^{\#}$ *and* $\mathcal{S}$.

Now, by using Theorem 15.6, we can also easily prove the following

Theorem 15.9. *For any $F \in \mathcal{F}$ and $G \in \mathcal{G}$, the following assertions are equivalent:*

(1) $(F,\ G)$ *is proximally mildly continuous,*

(2) for each $A \subseteq X$ *and* $S \in \mathcal{S}$ *there exists* $R \in \mathcal{R}$ *such that* $R\,[\,A\,] \subseteq G^{-1}\big[\,S\big[\,F\,[\,A\,]\,\big]\,\big]$,

(3) for each $A \subseteq X$ *and* $S \in \mathcal{S}$ *there exists* $R \in \mathcal{R}$ *such that for every* $x \in A$ *and* $y \in R(x)$ *we have* $G(y) \cap S\big[\,F\,[\,A\,]\,\big] \neq \emptyset$,

(4) for each $A \subseteq X$ *and* $S \in \mathcal{S}$ *there exists* $R \in \mathcal{R}$ *such that for every* $x \in X$ *and* $y \in R(x)$ *there exist* $u \in A$ *and* $z \in F(u)$ *and* $w \in G(y)$ *such that* $w \in S(z)$.

Proof. By Theorem 15.6 and the corresponding definitions, it is clear that

$$(1) \iff G^{-1}\circ \mathcal{S}\circ F \subseteq \mathcal{R}^{\#} \iff \forall\ S\in\mathcal{S}:\ \ G^{-1}\circ S\circ F\in\mathcal{R}^{\#}$$
$$\iff \forall\ S\in\mathcal{S}:\ \ \forall\ A\subseteq X:\ \ \exists\ R\in\mathcal{R}:\quad R\,[\,A\,]\subseteq\big(G^{-1}\circ S\circ F\big)\,[\,A\,]\,.$$

Moreover, since $R\,[\,A\,]=\bigcup_{x\in A} R(x)$, we can also note that

$$R\,[\,A\,]\subseteq\big(G^{-1}\circ S\circ F\big)\,[\,A\,] \iff \forall\ x\in A:\ \ R(x)\subseteq\big(G^{-1}\circ S\circ F\big)\,[\,A\,]$$
$$\iff \forall\ x\in A:\ \ \forall\ y\in R(x):\quad y\in\big(G^{-1}\circ S\circ F\big)\,[\,A\,]\,.$$

Furthermore, by using some basic facts on relations, we can also easily see that

$$y\in\big(G^{-1}\circ S\circ F\big)\,[\,A\,] \iff y\in G^{-1}\big[\,S\big[\,F\,[\,A\,]\,\big]\,\big]$$
$$\iff G(y)\cap S\big[\,F\,[\,A\,]\,\big]\neq\emptyset \iff \exists\ w\in G(y):\ \ w\in S\big[\,F\,[\,A\,]\,\big]$$
$$\iff \exists\ w\in G(y):\ \ \exists\ z\in F\,[\,A\,]:\ \ w\in S(z)$$
$$\iff \exists\ w\in G(y):\ \ \exists\ u\in A:\ \ \exists\ z\in F(u):\ \ w\in S(z)\,.$$

Therefore, assertions (1)–(4) are also equivalent.

16. Detailed reformulations of topological mild continuity

By using Theorem 13.6, we can easily establish the following

Theorem 16.1. *The following assertions are equivalent:*

(1) $(\mathcal{F},\ \mathcal{G})$ *is topologically mildly continuous with respect to* $\mathcal{R}$ *and* $\mathcal{S}$ *,*

(2) $(\mathcal{F}^{\wedge},\ \mathcal{G}^{\wedge})$ *is properly, uniformly, or proximally mildly continuous with respect to* $\mathcal{R}^{\wedge}$ *and* $\mathcal{S}^{\wedge}$ *.*

Proof. From Remark 6.6 and Theorems 6.5, 6.7 and 10.13, we know that $*$, $\#$, and $\wedge$ are closure operations for relators such that $\wedge$ is both $*-$ and $\#-$invariant. Therefore, by Theorem 13.6, assertion (1) is equivalent to both "the uniformly and proximally part" of (2). Moreover, from Corollary 13.4, we know that (1) is also equivalent to "the properly part" of (2).

From this theorem, by using Theorem 13.10, we can immediately derive

Corollary 16.2. *The following assertions are equivalent:*

(1) $(\mathcal{F},\ \mathcal{G})$ *is topologically mildly continuous with respect to* $\mathcal{R}$ *and* $\mathcal{S}$ *,*

(2) $(\mathcal{F}^{\wedge},\ \mathcal{G}^{\wedge})$ *is elementwise properly, uniformly, or proximally mildly continuous with respect to* $\mathcal{R}^{\wedge}$ *and* $\mathcal{S}^{\wedge}$ *.*

Remark 16.3. This corollary shows that, instead of the topological mild continuity of the pair $(\mathcal{F},\ \mathcal{G})$ with respect to the relators $\mathcal{R}$ and $\mathcal{S}$, it is enough to investigate only the proper, uniform, or proximal mild continuity of the pair (F,G) with respect to the relators $\mathcal{R}^{\wedge}$ and $\mathcal{S}^{\wedge}$ for all $F\in\mathcal{F}^{\wedge}$ and $G\in\mathcal{G}^{\wedge}$.

Thus, since $\mathcal{F}$ and $\mathcal{G}$ were quite arbitrary relators in Notation 12.1, it is actually enough to prove the following

Theorem 16.4. *For any* $F \in \mathcal{F}$ *and* $G \in \mathcal{G}$ *, the following assertions are equivalent :*

(1) $(F\,,\; G)$ *is topologically mildly continuous with respect to* $\mathcal{R}$ *and* $\mathcal{S}$ *,*

(2) $(F\,,\; G)$ *is properly, uniformly, or proximally mildly continuous with respect to* $\mathcal{R}^{\wedge}$ *and* $\mathcal{S}^{\wedge}$ *.*

Proof. By Definition 12.8, assertion (1) is equivalent to the statement that :

(a) $\big(\{F\},\; \{G\}\big)$ is topologically mildly continuous with respect to $\mathcal{R}$ and $\mathcal{S}$.

Moreover, by Theorem 16.1, assertion (a) is equivalent to the statement that :

(b) $\big(\{F\}^{\wedge},\; \{G\}^{\wedge}\big)$ is properly, uniformly, or proximally mildly continuous with respect to $\mathcal{R}^{\wedge}$ and $\mathcal{S}^{\wedge}$.

Now, by Remark 12.9, we can see that assertion (b) is equivalent to the statement that :

(c) $\big(\{F\}^{*},\; \{G\}^{*}\big)$ is properly, uniformly, or proximally mildly continuous with respect to $\mathcal{R}^{\wedge}$ and $\mathcal{S}^{\wedge}$.

Moreover, since $\wedge$ is $*$–invariant, we can see that assertion (c) is equivalent to the statement that :

(d) $\big(\{F\}^{*},\; \{G\}^{*}\big)$ is properly, uniformly, or proximally mildly continuous with respect to $\big(\mathcal{R}^{\wedge}\big)^{*}$ and $\big(\mathcal{S}^{\wedge}\big)^{*}$.

Now, we may recall that the operations $*$ and $\#$ are $*$–absorbing, Therefore, by Theorem 14.1, "the uniformly or proximally part" of assertion (d) is equivalent to the statement that :

(e) $\big(\{F\},\; \{G\}\big)$ is uniformly or proximally mildly continuous with respect to $\mathcal{R}^{\wedge}$ and $\mathcal{S}^{\wedge}$.

Moreover, we can note that $*$ is an inversion and composition compatible closure operation. Therefore, by Corollary 13.7, "the uniformly part" of assertion (e) is equivalent to the statement that :

(f) $\big(\{F\},\, \{G\}\big)$ is properly mildly continuous with respect to $\big(\mathcal{R}^{\wedge}\big)^{*}$ and $\mathcal{S}^{\wedge}$.

However, since $\big(\mathcal{R}^{\wedge}\big)^{*} = \mathcal{R}^{\wedge}$, assertion (f) is equivalent to the statement that :

(g) $\big(\{F\},\, \{G\}\big)$ is properly mildly continuous with respect to $\mathcal{R}^{\wedge}$ and $\mathcal{S}^{\wedge}$.

Now, by Definition 12.8, we can see that assertion (e) is equivalent to the statement that :

(h) $(F\,,\; G)$ is uniformly or proximally mildly continuous with respect to $\mathcal{R}^{\wedge}$ and $\mathcal{S}^{\wedge}$.

Moreover, assertion (g) is equivalent to the statement that :

(i) $(F\,,\; G)$ is properly mildly continuous with respect to $\mathcal{R}^{\wedge}$ and $\mathcal{S}^{\wedge}$.

Hence, it is clear that assertions (1) and (2) are also equivalent.

Now, from the "properly part" of this theorem, by using Theorem 15.1, we can immediately derive the following

Theorem 16.5. *For any $F \in \mathcal{F}$ and $G \in \mathcal{G}$, the following assertions are equivalent:*

(1) (F, G) *is topologically mildly continuous,*

(2) $G^{-1} \circ \mathcal{S}^\wedge \circ F \subseteq \mathcal{R}^\wedge$, *(3)* $\mathrm{cl}_{F \boxtimes G}(\mathcal{S}^\wedge) \subseteq \mathcal{R}^\wedge$.

Moreover, from the "uniform part" of Theorem 16.4, by using Theorem 15.5, we can easily derive the following

Theorem 16.6. *For any $F \in \mathcal{F}$ and $G \in \mathcal{G}$, the following assertions are equivalent:*

(1) (F, G) *is topologically mildly continuous,*

(2) for each $x \in X$ and $V \in \mathcal{S}^\wedge$ there exists $R \in \mathcal{R}$ such that $R(x) \subseteq G^{-1}\big[V[F(x)]\big]$,

(3) for each $x \in X$ and $V \in \mathcal{S}^\wedge$ there exists $R \in \mathcal{R}$ such that for every $y \in R(x)$ we have $G(y) \cap V[F(x)] \neq \emptyset$,

(4) for each $x \in X$ and $V \in \mathcal{S}^\wedge$ there exists $R \in \mathcal{R}$ such that for every $y \in R(x)$ there exist $z \in F(u)$ and $w \in G(y)$ such that $w \in V(z)$.

Proof. By the "uniform part" of Theorem 16.4, assertion (1) is equivalent to the statement that:

(a) $\big(F, G)$ is uniformly mildly continuous with respect to $\mathcal{R}^\wedge$ and $\mathcal{S}^\wedge$.

Moreover, by Theorem 15.5, assertion (a) is equivalent to the statements that:

(b) for each $V \in \mathcal{S}^\wedge$ there exists $U \in \mathcal{R}^\wedge$ such that $U \subseteq G^{-1} \circ V \circ F$,

(c) for each $V \in \mathcal{S}^\wedge$ there exists $U \in \mathcal{R}^\wedge$ such that for every $x \in X$ and $y \in U(x)$ we have $G(y) \cap V[F(x)] \neq \emptyset$,

(d) for each $V \in \mathcal{S}^\wedge$ there exists $U \in \mathcal{R}^\wedge$ such that for every $x \in X$ and $y \in U(x)$ there exist $z \in F(x)$ and $w \in G(y)$ such that $w \in V(z)$.

Therefore, to complete the proof, we need only show that the each of assertions (b)–(d) is equivalent to the corresponding assertion of the theorem. For this, for instance, we shall show that assertions (c) and (3) are equivalent.

Note that if (c) holds, then for each $V \in \mathcal{S}^\wedge$ there exists $U \in \mathcal{R}^\wedge$ such that $U \subseteq G^{-1} \circ V \circ F$, and thus also $U(x) \subseteq G^{-1}\big[V[F(x)]\big]$ for all $x \in X$. Moreover, by the definition of $\mathcal{R}^\wedge$, for each $x \in X$ there exists $R \in \mathcal{R}$ such that $R(x) \subseteq U(x)$, and thus also $R(x) \subseteq G^{-1}\big[V[F(x)]\big]$. Therefore, (3) also holds.

Conversely, if (3) holds, then for each $x \in X$ and $V \in \mathcal{S}^\wedge$ there exists $R_x \in \mathcal{R}$ such that $R_x(x) \subseteq G^{-1}\big[V[F(x)]\big]$. Hence, by defining a relation U on X to Y such that $U(x) = R_x(x)$ for all $x \in X$, we can at once see that $U \in \mathcal{R}^\wedge$ such that $U(x) = R_x(x) \subseteq G^{-1}\big[V[F(x)]\big]$ for all $x \in X$. Therefore, $U \subset G^{-1} \circ V \circ F$, and thus (c) also holds.

Now, from Theorem 16.5, by letting F to be a function, we can easily derive

Theorem 16.7. *If $f \in \mathcal{F}$ and $G \in \mathcal{G}$ such that f is a function, then the following assertions are equivalent:*

(1) (f, G) *is topologically mildly continuous,*

(2) $G^{-1} \circ \mathcal{S} \circ f \subseteq \mathcal{R}^\wedge$, *(3)* $\mathrm{cl}_{f \boxtimes G}(\mathcal{S}) \subseteq \mathcal{R}^\wedge$.

Proof. From Theorem 16.5, we know that assertion (1) is equivalent to the statement that: (a) $G^{-1} \circ \mathcal{S}^{\wedge} \circ f \subseteq \mathcal{R}^{\wedge}$.

Moreover, by the inclusion $\mathcal{S} \subseteq \mathcal{S}^{\wedge}$ and the increasingness of composition, it is clear that (a) implies (2). Therefore, to prove the equivalence of (1) and (2), we need only show that (2) also implies (a).

For this, we can note that if $x \in X$ and $V \in \mathcal{S}^{\wedge}$, then by the assumption that $\operatorname{card}\big(f(x)\big) \leq 1$ and the definition of $\mathcal{S}^{\wedge}$ there exists $S \in \mathcal{S}$ such that $S\,[\,f(x)\,] \subseteq V\,[\,f(x)\,]$. Hence, we can already infer that

$$\big(G^{-1} \circ S \circ f\,\big)(x) = G^{-1}\big[\,S\,[\,f(x)\,]\,\big] \subseteq G^{-1}\big[\,V\,[\,f(x)\,]\,\big] = \big(G^{-1} \circ V \circ f\,\big)(x)\,.$$

Moreover, if (2) holds, then $G^{-1} \circ S \circ f \in \mathcal{R}^{\wedge}$. Therefore, there exists $R \in \mathcal{R}$ such that $R(x) \subseteq \big(G^{-1} \circ S \circ f\,\big)(x)$, and thus $R(x) \subseteq \big(G^{-1} \circ V \circ f\,\big)(x)$. This shows that $G^{-1} \circ V \circ f \in \mathcal{R}^{\wedge}$, and thus (a) also holds.

Now, to complete the proof, it remains to note only that, by Theorem 15.1, assertions (2) and (3) are also equivalent.

Remark 16.8. From this theorem, by using the inclusion $\mathcal{R}^{\#} \subseteq \mathcal{R}^{\wedge}$ and Theorem 14.6, we can at once see that "the proximal mild continuity of $(\,f\,,\,G)$" implies "the topological mild continuity of $(\,f\,,\,G)$".

Moreover, from Theorem 16.7, by using Theorems 15.1, 15.3 and 15.6, we can also easily derive

Corollary 16.9. *Under the assumptions of Theorem 16.7, the following assertions are also equivalent:*

(1) $(\,f\,,\,G)$ *is topologically mildly continuous with respect to* $\mathcal{R}$ *and* $\mathcal{S}$*,*

(2) $(\,f\,,\,G)$ *is properly, uniformly or proximally mildly continuous with respect to* $\mathcal{R}^{\wedge}$ *and* $\mathcal{S}$*.*

On the other hand, from the proof of Theorem 16.7, it is clear that, as a consequence of Theorem 16.6, we can also state the following

Theorem 16.10. *If* $f \in \mathcal{F}$ *and* $G \in \mathcal{G}$ *such that* f *is a function, then the following assertions are equivalent:*

(1) $(\,f\,,\,G)$ *is topologically mildly continuous,*

(2) for each $x \in X$ *and* $S \in \mathcal{S}$ *there exists* $R \in \mathcal{R}$ *such that* $R(x) \subseteq G^{-1}\big[\,S\,[\,f(x)\,]\,\big]$*,*

(3) for each $x \in X$ *and* $S \in \mathcal{S}$ *there exists* $R \in \mathcal{R}$ *such that for every* $y \in R(x)$ *we have* $G(y) \cap S[\,f(x)\,] \neq \emptyset$*,*

(4) for each $x \in X$ *and* $S \in \mathcal{S}$ *there exists* $R \in \mathcal{R}$ *such that for every* $y \in R(x)$ *there exists* $w \in G(y)$ *such that* $w \in S\,[\,f(x)\,]$*.*

Remark 16.11. Note that if in particular the whole X is the domain of f, then in the above assertions we may write $S\big(f(x)\big)$ instead of $S\,[\,f(x)\,]$.

17. Detailed reformulations of paratopological mild continuity

By using Theorem 14.6 and Corollary 13.4, analogously to Theorem 16.1, we can also easily prove the following

Theorem 17.1. *The following assertions are equivalent:*

(1) $(\mathcal{F}, \mathcal{G})$ *is paratopologically mildly continuous with respect to* $\mathcal{R}$ *and* $\mathcal{S}$*,*

(2) $(\mathcal{F}^{\vartriangle}, \mathcal{G}^{\vartriangle})$ *is properly, uniformly, proximally, or topologically mildly continuous with respect to* $\mathcal{R}^{\vartriangle}$ *and* $\mathcal{S}^{\vartriangle}$*.*

Hence, by using Theorem 13.10, we can only derive the following

Corollary 17.2. *The following assertions are equivalent:*

(1) $(\mathcal{F}, \mathcal{G})$ *is paratopologically mildly continuous with respect to* $\mathcal{R}$ *and* $\mathcal{S}$*,*

(2) $(\mathcal{F}^{\vartriangle}, \mathcal{G}^{\vartriangle})$ *is elementwise properly, uniformly, or proximally mildly continuous with respect to* $\mathcal{R}^{\vartriangle}$ *and* $\mathcal{S}^{\vartriangle}$*.*

Remark 17.3. This corollary shows that, instead of the paratopological mild continuity of the pair $(\mathcal{F}, \mathcal{G})$ with respect to the relators $\mathcal{R}$ and $\mathcal{S}$, it is enough to investigate only the proper, uniform, or proximal mild continuity of the pair (F, G) with respect to the relators $\mathcal{R}^{\vartriangle}$ and $\mathcal{S}^{\vartriangle}$ for all $F \in \mathcal{F}^{\vartriangle}$ and $G \in \mathcal{G}^{\vartriangle}$.

However, instead of an analogue of Theorem 16.4, we can now only prove

Theorem 17.4. *For any* $F \in \mathcal{F}$ *and* $G \in \mathcal{G}$*, the following assertions are equivalent:*

(1) (F, G) *is paratopologically mildly continuous with respect to* $\mathcal{R}$ *and* $\mathcal{S}$*,*

(2) $\big(F \circ X^X, \ G \circ Y^Y\big)$ *is properly, uniformly, proximally, or topologically mildly continuous with respect to* $\mathcal{R}^{\vartriangle}$ *and* $\mathcal{S}^{\vartriangle}$*.*

Proof. By Definition 12.8, assertion (1) is equivalent to the statement that:

(a) $\big(\{F\}, \{G\}\big)$ is paratopologically mildly continuous with respect to $\mathcal{R}$ and $\mathcal{S}$.

Moreover, by Theorem 17.1, assertion (a) is equivalent to the statement that:

(b) $\big(\{F\}^{\vartriangle}, \{G\}^{\vartriangle}\big)$ is properly, uniformly, proximally, or topologically mildly continuous with respect to $\mathcal{R}^{\vartriangle}$ and $\mathcal{S}^{\vartriangle}$.

Now, by Remark 12.9, we can see that assertion (b) is equivalent to the statement that:

(c) $\big(\big(F \circ X^X\big)^*, \ \big(G \circ Y^Y\big)^*\big)$ is properly, uniformly, proximally, or topologically mildly continuous with respect to $\mathcal{R}^{\vartriangle}$ and $\mathcal{S}^{\vartriangle}$.

Moreover, since $\vartriangle$ is $*$–invariant, we can see that assertion (c) is equivalent to the statement that:

(d) $\big(\big(F \circ X^X\big)^*, \ \big(G \circ Y^Y\big)^*\big)$ is properly, uniformly, proximally, or topologically mildly continuous with respect to $\big(\mathcal{R}^{\vartriangle}\big)^*$ and $\big(\mathcal{S}^{\vartriangle}\big)^*$.

Now, we may recall that the operations $*$, $\#$, and $\wedge$ are $*$–absorbing. Therefore, by Theorem 14.1, the "uniformly, proximally, or topologically part" of assertion (d) is equivalent to the statement that:

(e) $\big(F \circ X^X, \ G \circ Y^Y\big)$ is uniformly, proximally, or topologically mildly continuous with respect to $\mathcal{R}^{\vartriangle}$ and $\mathcal{S}^{\vartriangle}$.

Moreover, we may recall that $*$ is an inversion and composition compatible closure operation. Therefore, by Corollary 13.7, "the uniformly part" of assertion (e) is equivalent to the statement that:

(f) $\big(F\circ X^X,\ G\circ Y^Y\big)$ is properly mildly continuous with respect to $\big(\mathcal{R}^{\vartriangle}\big)^{*}$ and $\mathcal{S}^{\vartriangle}$.

However, since $\big(\mathcal{R}^{\vartriangle}\big)^{*}=\mathcal{R}^{\vartriangle}$, assertion (f) is equivalent to the statement that:

(g) $\big(F\circ X^X,\ G\circ Y^Y\big)$ is properly mildly continuous with respect to $\mathcal{R}^{\vartriangle}$ and $\mathcal{S}^{\vartriangle}$.

Hence, it is clear that assertions (1) and (2) are also equivalent.

From this theorem, by using Theorem 13.10, we can immediately derive

Corollary 17.5. *For any $F\in\mathcal{F}$ and $G\in\mathcal{G}$, the following assertions are equivalent:*

(1) $(F,\ G)$ is paratopologically mildly continuous with respect to $\mathcal{R}$ and $\mathcal{S}$,

(2) $\big(F\circ\varphi,\ G\circ\psi\big)$ is properly, uniformly, or proximally mildly continuous with respect to $\mathcal{R}^{\vartriangle}$ and $\mathcal{S}^{\vartriangle}$ for all $\varphi\in X^X$ and $\psi\in Y^Y$.

Remark 17.6. Hence, it is clear that if in particular $X=Z$ and $Y=W$, then the following assertions are equivalent:

(1) $(\Delta_X,\ \Delta_Y)$ is paratopologically mildly continuous with respect to $\mathcal{R}$ and $\mathcal{S}$,

(2) (φ,ψ) is properly, uniformly, or proximally mildly continuous with respect to $\mathcal{R}^{\vartriangle}$ and $\mathcal{S}^{\vartriangle}$ for all $\varphi\in X^X$ and $\psi\in Y^Y$.

Moreover, from Corollary 17.5, by using Theorem 13.6, we can also easily derive the following

Theorem 17.7. *For any $F\in\mathcal{F}$ and $G\in\mathcal{G}$, the following assertions are equivalent:*

(1) $(F,\ G)$ is paratopologically mildly continuous with respect to $\mathcal{R}$ and $\mathcal{S}$,

(2) (φ,ψ) is properly, uniformly, or proximally mildly continuous with respect to $\mathcal{R}^{\vartriangle}$ and $G^{-1}\circ\mathcal{S}^{\vartriangle}\circ F$ for all $\varphi\in X^X$ and $\psi\in Y^Y$.

Proof. By " the properly part" of Corollary 17.5, assertion (1) is equivalent the statement that:

(a) $(G\circ\psi)^{-1}\circ\mathcal{S}^{\vartriangle}\circ(F\circ\varphi)\subseteq\mathcal{R}^{\vartriangle}$ for all $\varphi\in X^X$ and $\psi\in Y^Y$.

However, by the corresponding properties of composition, we have

$$\psi^{-1}\circ\big(G^{-1}\circ\mathcal{S}^{\vartriangle}\circ F\big)\circ\varphi=(G\circ\psi)^{-1}\circ\mathcal{S}^{\vartriangle}\circ(F\circ\varphi)$$

for all $\varphi\in X^X$ and $\psi\in Y^Y$.

Therefore, assertion (a) is equivalent to the statement that:

(b) $\psi^{-1}\circ\big(G^{-1}\circ\mathcal{S}^{\vartriangle}\circ F\big)\circ\varphi\subseteq\mathcal{R}^{\vartriangle}$ for all $\varphi\in X^X$ and $\psi\in Y^Y$.

However, by the corresponding definitions, this means only that:

(c) (φ,ψ) is properly mildly continuous with respect to $\mathcal{R}^{\vartriangle}$ and $G^{-1}\circ\mathcal{S}^{\vartriangle}\circ F$ for all $\varphi\in X^X$ and $\psi\in Y^Y$.

Moreover, we can note that $\mathcal{R}^{\vartriangle}=\big(\mathcal{R}^{\vartriangle}\big)^{\Diamond}$ for $\Diamond=*$ and $\#$. Therefore, assertion (b) is equivalent to the statement that:

(d) $\psi^{-1}\circ\big(G^{-1}\circ\mathcal{S}^{\vartriangle}\circ F\big)\circ\varphi\subseteq\big(\mathcal{R}^{\vartriangle}\big)^{\Diamond}$ for all $\varphi\in X^X$ and $\psi\in Y^Y$.

However, by Theorem 13.6, assertion (d) is equivalent to the statement that:

(e) (φ, ψ) is mildly $\Diamond$–continuous with respect to $\mathcal{R}^{\vartriangle}$ and $G^{-1} \circ \mathcal{S}^{\vartriangle} \circ F$ for all $\varphi \in X^X$ and $\psi \in Y^Y$.

Hence, it is clear that assertions (1) and (2) are also equivalent.

Because of Corollary 17.5, it is also worth proving here the following analogue of [43, Theorem 6.1].

Theorem 17.8. *For any $F \in \mathcal{F}$ and $G \in \mathcal{G}$, the following assertions are equivalent:*

(1) (F, G) is properly, uniformly, or proximally mildly continuous with respect to $\mathcal{R}^{\vartriangle}$ and $\mathcal{S}$,

(2) $G^{-1} \circ \mathcal{S} \circ F \subseteq \mathcal{R}^{\vartriangle}$, *(3) $G^{-1}\big[S[F(x)]\big] \in \mathcal{E}_{\mathcal{R}}$ for all $x \in X$ and $S \in \mathcal{S}$,*

(4) for each $x \in X$ and $S \in \mathcal{S}$ there exist $u \in X$ and $R \in \mathcal{R}$ such that $R(u) \subseteq G^{-1}\big[S[F(x)]\big]$,

(5) for each $x \in X$ and $S \in \mathcal{S}$ there exist $u \in X$ and $R \in \mathcal{R}$ such that for any $y \in R(u)$ we have $G(y) \cap S[F(x)] \neq \emptyset$,

(6) for each $x \in X$ and $S \in \mathcal{S}$ there exist $u \in X$ and $R \in \mathcal{R}$ such that for any $y \in R(u)$ there exist $z \in F(x)$ and $w \in G(y)$ such that $w \in S(z)$.

Proof. By Remark 12.4, it is clear that the "properly part" of (1) is equivalent to (2). Moreover, we can note that $\mathcal{R}^{\vartriangle} = \big(\mathcal{R}^{\vartriangle}\big)^{\Diamond}$ for $\Diamond = *$ and $\#$. Therefore, by Theorem 13.6, "the uniformly or proximally parts" of (1) are also equivalent to (2).

Furthermore, if (1) holds, by the corresponding definitions, for each $S \in \mathcal{S}$ we have $G^{-1} \circ S \circ F \in \mathcal{R}^{\vartriangle}$. Thus, by the definition of $\mathcal{R}^{\vartriangle}$, for each $x \in X$ there exist $u \in X$ and $R \in \mathcal{R}$ such that $R(u) \subseteq \big(G^{-1} \circ S \circ F\big)(x)$, and thus also $R(u) \subseteq G^{-1}\big[S[F(x)]\big]$. Therefore, (4), and thus (3) also holds. Now, by using the latter argument, we can also easily see that (3) also implies (1).

Moreover, from the proofs of Theorems 15.2 and 15.5, it is clear that the equivalences $(4) \Longleftrightarrow (5) \Longleftrightarrow (6)$ are also true.

Remark 17.9. Note that, by Theorem 15.5, assertion (1) is, for instance, also equivalent to the statement that:

(3') for each $S \in \mathcal{S}$ there exists $U \in \mathcal{R}^{\vartriangle}$ such that $U \subseteq G^{-1} \circ \mathcal{S} \circ F$.

However, it is now, more important to note that, analogously to [43, Theorem 6.3], now we can now also prove the following

Theorem 17.10. *If f is a function on X onto Z and $G \in \mathcal{G}$, then under the addition assumption $\mathcal{S} \neq \emptyset$ the following assertions are equivalent:*

(1) (f, G) is properly, uniformly, or proximally mildly continuous with respect to $\mathcal{R}^{\vartriangle}$ and $\mathcal{S}$,

(2) (f, G) is properly, uniformly, or proximally mildly continuous with respect to $\mathcal{R}^{\vartriangle}$ and $\mathcal{S}^{\vartriangle}$.

Proof. From Theorem 17.8, we can see that assertion (1) is equivalent to the statement that:

(a) for each $x \in X$ and $S \in \mathcal{S}$ there exist $u \in X$ and $R \in \mathcal{R}$ such that $R(u) \subseteq G^{-1}\big[\, S\,[\, f(x)\,]\,\big]$.

Moreover, assertion (2) is equivalent to the statement that:

(b) for each $x \in X$ and $V \in \mathcal{S}^{\Delta}$ there exist $u \in X$ and $R \in \mathcal{R}$ such that for $R(u) \subseteq G^{-1}\big[\, V\,[\, f(x)\,]\,\big]$.

Hence, since $\mathcal{S} \subseteq \mathcal{S}^{\Delta}$, it is clear that (b) implies (a), and thus (2) implies (1). Therefore, we need only show that (a) also implies (b), and thus (1) also implies (2).

For this, assume that (a) holds, and $x \in X$ and $V \in \mathcal{S}^{\Delta}$. Now, if in particular $x \in f^{-1}[\, Z\,]$, then since f is a function we can see that $f(x) \in Z$. Therefore, by the definition of $\mathcal{S}^{\Delta}$, there exists $z \in Z$ and $S \in \mathcal{S}$ such that $S(z) \subseteq V\big(f(x)\big)$. Moreover, since $Z = f[\, X\,]$, there exists $t \in X$ such that $z = f(t)$. Thus, we also have $S\big(f(t)\big) \subseteq V\big(f(x)\big)$. Hence, we can see that $S\,[\, f(t)\,] \subseteq V\,[\, f(x)\,]$, and thus also $G^{-1}\big[\, S\,[\, f(t)\,]\,\big] \subseteq G^{-1}\big[\, V\,[\, f(x)\,]\,\big]$. Moreover, by using (a) we can see that there exist $u \in X$ and $R \in \mathcal{R}$ such that $R(u) \subseteq G^{-1}\big[\, S\,[\, f(t)\,]\,\big]$. Thus, we also have $R(u) \subseteq G^{-1}\big[\, V\,[\, f(t)]\,\big]$.

On the other hand, if $x \in X \setminus f^{-1}[\, Z\,]$, then by taking $S \in \mathcal{S}$ and using (a) we can see that there exist $u \in X$ and $R \in \mathcal{R}$ such that $R(u) \subseteq G^{-1}[\, S\,[\, f(x)\,]\,\big]$. Hence, since $f(x) = \emptyset$, we can infer that $R(u) \subseteq \emptyset$. Therefore, $R(u) \subseteq G^{-1}\big[\, V\,[\, f(x)\,]\,\big]$ also holds.

18. Characterizations of proximal mild continuity

The subsequent theorems have also been mainly taken from an unfinished work of the first author [35].

Theorem 18.1. *For any $F \in \mathcal{F}$ and $G \in \mathcal{G}$, the following assertions are equivalent:*

(1) $(F,\, G)$ is proximally mildly continuous,

(2) $A \in \operatorname{Cl}_{\mathcal{R}}(B)$ implies $F\,[\, A\,] \in \operatorname{Cl}_{\mathcal{S}}\,(\, G\,[\, B\,]\,)$ for all $A \subseteq X$ and $B \subseteq Y$,

(3) $F\,[\, A\,] \in \operatorname{Int}_{\mathcal{S}}\big(D\big)$ implies $A \in \operatorname{Int}_{\mathcal{R}}\big(\, G^{-1}\,[\, D\,]\big)$ for all $A \subseteq X$ and $D \subseteq W$.

Proof. Define $\mathcal{U} = G^{-1} \circ \mathcal{S} \circ F$. Then, by Theorems 15.6 and 6.9, we have

$$(1) \iff \mathcal{U} \subseteq \mathcal{R}^{\#} \iff \operatorname{Cl}_{\mathcal{R}} \subseteq \operatorname{Cl}_{\mathcal{U}} \iff \big(A \in \operatorname{Cl}_{\mathcal{R}}(B) \implies A \in \operatorname{Cl}_{\mathcal{U}}(B)\big).$$

Therefore, to prove the equivalence of (1) and (2), we need only show that, for any $A \subseteq X$ and $B \subseteq Y$, we have

$$A \in \operatorname{Cl}_{\mathcal{U}}(B) \iff F\,[\, A\,] \in \operatorname{Cl}_{\mathcal{S}}\,(\, G\,[\, B\,]\,)\,.$$

For this, by the corresponding definitions and some basic theorems on relations, it is enough to note only that, for any $S \in \mathcal{S}$, we have

$$\big(G^{-1} \circ S \circ F\big)\,[\, A\,] \cap B \neq \emptyset \iff G^{-1}\big[\, S\,[\, F\,[\, A\,]\,\big] \cap B \neq \emptyset \iff S\,\big[\, F\,[\, A\,]\,\big] \cap G\,[\, B\,] \neq \emptyset\,.$$

Finally, to complete the proof, we note that the equivalence of (2) and (3) can be proved by using the relationships between the structures Cl and Int, and the inclusions

$$G^{-1}\big[\,G\,[\,B\,]^{\,c}\,\big] \subseteq B^{c} \qquad \text{and} \qquad G\big[\,G^{-1}\,[\,D\,]^{\,c}\,\big] \subseteq D^{c}$$

with $B \subseteq Y$ and $D \subseteq W$. (To check the latter one, instead of a direct proof, one can note that $G^{-1}\,[\,D\,]^{\,c} = \mathrm{cl}_G(D)^{c} = \mathrm{int}_G(D^{c})$ and $G\,[\,\mathrm{int}_G(D^{c})\,] \subseteq D^{c}$.)

Namely, if (2) holds, then by the above mentioned results, it is clear that for any $A \subseteq X$ and $D \subseteq W$

$$F\,[\,A\,] \in \mathrm{Int}_{\mathcal{S}}(D) \implies F\,[\,A\,] \notin \mathrm{Cl}_{\mathcal{S}}\big(D^{c}\big) \implies F\,[\,A\,] \notin \mathrm{Cl}_{\mathcal{S}}\big(G\big[\,G^{-1}\,[\,D\,]^{\,c}\,\big]\big)$$
$$\implies A \notin \mathrm{Cl}_{\mathcal{R}}\big(G^{-1}\,[\,D\,]^{\,c}\big) \implies A \in \mathrm{Int}_{\mathcal{R}}\big(G^{-1}\,[\,D\,]\big).$$

Thus, (3) also holds.

While, if (3) holds, then again by the above mentioned results, it is clear that for any $A \subseteq X$ and $B \subseteq Y$

$$F\,[\,A\,] \notin \mathrm{Cl}_{\mathcal{S}}\big(G\,[\,B\,]\big) \implies F\,[\,A\,] \in \mathrm{Int}_{\mathcal{S}}\big(G\,[\,B\,]^{\,c}\big) \implies A \in \mathrm{Int}_{\mathcal{R}}\big(G^{-1}\big[\,G\,[\,B^{c}\,]\,\big]\big)$$
$$\implies A \in \mathrm{Int}_{\mathcal{R}}\big(B^{c}\big) \implies A \notin \mathrm{Cl}_{\mathcal{R}}(B).$$

Therefore, (2) also holds.

Now, by using the above theorem, we can also easily prove the following

Theorem 18.2. *If $F \in \mathcal{F}$ and $g \in \mathcal{G}$ such that g is a function and $Y = g^{-1}\,[\,W\,]$, then the following assertions are equivalent:*

(1) (F, g) is proximally mildly continuous,

(2) $A \in \mathrm{Cl}_{\mathcal{R}}\big(g^{-1}\,[\,D\,]\big)$ implies $F\,[\,A\,] \in \mathrm{Cl}_{\mathcal{S}}(D)$ for all $A \subseteq X$ and $D \subseteq W$.

Proof. If (1) holds and $A \subseteq X$ and $D \subseteq W$, then by Theorem 17.1 and the inclusion $g\big[\,g^{-1}\,[\,D\,]\,\big] \subseteq D$ it is clear that

$$A \in \mathrm{Cl}_{\mathcal{R}}\big(g^{-1}\,[\,D\,]\big) \implies F\,[\,A\,] \in \mathrm{Cl}_{\mathcal{S}}\big(g\big[\,g^{-1}\,[\,D\,]\,\big]\big) \implies F\,[\,A\,] \in \mathrm{Cl}_{\mathcal{S}}(D).$$

Therefore, (2) also holds.

While, if (2) holds, and $A \subseteq X$ and $B \subseteq Y$, then by using the inclusion $B \subseteq g^{-1}\big[\,g\,[\,B\,]\,\big]$ and assertion (2) we can see that

$$A \in \mathrm{Cl}_{\mathcal{R}}(B) \implies A \in \mathrm{Cl}_{\mathcal{R}}\big(g^{-1}\big[\,g\,[\,B\,]\,\big]\big) \implies F\,[\,A\,] \in \mathrm{Cl}_{\mathcal{S}}\big(g\,[\,B\,]\big).$$

Therefore, by Theorem 18.1, assertion (1) also holds.

Remark 18.3. More exactly, we can also state that (1) implies (2) if g is a function, and (2) implies (1) if $Y = g^{-1}\,[\,W\,]$.

Moreover, by using Theorem 18.1, we can also easily prove the following two theorems.

Theorem 18.4. *If $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that G^{-1} is a function and $G\,[\,Y\,] = W$, then the following assertions are equivalent:*

(1) (F, G) is proximally mildly continuous,

(2) $F\,[\,A\,] \in \mathrm{Int}_{\mathcal{S}}\,(G\,[\,B\,])$ implies $A \in Int_{\mathcal{R}}(B)$ for all $A \subseteq X$ and $B \subseteq Y$.

Remark 18.5. More exactly, we can also state that (1) implies (2) if G^{-1} is a function, and (2) implies (1) if $G[Y] = W$.

Theorem 18.6. *If $f \in \mathcal{F}$ and $G \in \mathcal{G}$ such that f is a function and $X = f^{-1}[Z]$, then the following assertions are equivalent:*

(1) (f, G) is proximally mildly continuous,

(2) $C \in \operatorname{Int}_{\mathcal{S}}(D)$ implies $f^{-1}[C] \in \operatorname{Int}_{\mathcal{R}}\big(G^{-1}[D]\big)$ for all $C \subseteq Z$ and $D \subseteq W$.

Remark 18.7. More exactly, we can also state that (1) implies (2) if f is a function, and (2) implies (1) if $X = f^{-1}[Y]$.

Now, by using Theorem 18.6, we can also easily prove the following

Theorem 18.8. *If $f \in \mathcal{F}$ and $g \in \mathcal{G}$ such that f and g are functions and $X = f^{-1}[Z]$ and $Y = g^{-1}[W]$, then the following assertions are equivalent:*

(1) (f, g) is proximally mildly continuous,

(2) $f^{-1}[C] \in \operatorname{Cl}_{\mathcal{R}}\big(g^{-1}[D]\big)$ implies $C \in \operatorname{Cl}_{\mathcal{S}}(D)$ for all $C \subseteq Z$ and $D \subseteq W$.

Remark 18.9. More exactly, we can also state that (1) implies (2) if f and g are functions, and (2) implies (1) if $X = f^{-1}[Z]$ and $Y = g^{-1}[W]$.

Moreover, by using Theorems 18.4 and 18.6, we can also easily prove

Theorem 18.10. *If $f \in \mathcal{F}$ and $G \in \mathcal{G}$ such that f and G^{-1} are functions and $X = f^{-1}[Z]$ and $G[Y] = W$, then the following assertions are equivalent:*

(1) (f, G) is proximally mildly continuous,

(2) $C \in \operatorname{Int}_{\mathcal{S}}(G[B])$ implies $f^{-1}[C] \in \operatorname{Int}_{\mathcal{R}}(B)$ for all $B \subseteq Y$ and $C \subseteq Z$.

Remark 18.11. More exactly, we can also state that (1) implies (2) if f and G^{-1} are functions, and (2) implies (1) if $X = f^{-1}[Z]$ and $G[Y] = W$.

19. Characterizations of topological mild continuity

From the results of Section 18, by using Theorems 16.4 and 6.14, we can easily derive several criteria for topological mild continuity.

However, for this, we have to assume tacitly throughout this section that *the relators $\mathcal{R}$ and $\mathcal{S}$, considered in Notation 12.1, are nonvoid.*

Theorem 19.1. *If $F \in \mathcal{F}$ and $G \in \mathcal{G}$, then the following assertions are equivalent:*

(1) (F, G) is topologically mildly continuous,

(2) $F[A] \subseteq \operatorname{int}_{\mathcal{S}}(D)$ implies $A \subseteq \operatorname{int}_{\mathcal{R}}\big(G^{-1}[D]\big)$ for all $A \subseteq X$ and $D \subseteq W$,

(3) $A \cap \operatorname{cl}_{\mathcal{R}}(B) \neq \emptyset$ implies $F[A] \cap \operatorname{cl}_{\mathcal{S}}(G[B]) \neq \emptyset$ for all $A \subseteq X$ and $B \subseteq Y$.

Proof. From Theorem 16.4, we can see that assertion (1) is equivalent to the statement that:

(a) (F, G) is proximally mildly continuous with respect to the relators $\mathcal{R}^\wedge$ and $\mathcal{S}^\wedge$.

Moreover, from Theorem 18.1, we can see that assertion (a) is equivalent to the statement that:

(b) $F[A] \in \operatorname{Int}_{\mathcal{S}^\wedge}(D)$ implies $A \in \operatorname{Int}_{\mathcal{R}^\wedge}\big(G^{-1}[D]\big)$ for all $A \subseteq X$ and $D \subseteq W$.

Furthermore, from Theorem 6.14, we can see that assertion (b) is equivalent to assertion (2).

Finally, we note that the equivalence of assertions (2) and (3) can be proved by using the relationship between the structures cl and int.

Remark 19.2. Note that if (1) holds, then by Theorem 16.5 we have

$$G^{-1} \circ \mathcal{S}^\wedge \circ F \subseteq \mathcal{R}^\wedge .$$

Moreover, if for instance $\mathcal{R} = \emptyset$, but $X \neq \emptyset$, then by the definition of $\mathcal{R}^\wedge$ we have $\mathcal{R}^\wedge = \emptyset$. Therefore, we necessarily have $\mathcal{S}^\wedge = \emptyset$, and thus also $\mathcal{S} = \emptyset$.

Hence, by the definition of int, we can see that $\operatorname{int}_{\mathcal{S}}(D) = \emptyset$ and quite similarly $\operatorname{int}_{\mathcal{R}}\big(G^{-1}[D]\big) = \emptyset$ for all $D \subseteq W$. Therefore, assertion (2) fails to hold if $X \neq F^{-1}[Z]$.

However, it is now more important to note that the above theorem can be reformulated in the following more concise form.

Corollary 19.3. *Under the conditions of Theorem 19.1, the following assertions are equivalent:*

(1) (F, G) *is topologically mildly continuous,*

(2) $\operatorname{cl}_{\mathcal{R}}(B) \subseteq F^{-1}\big[\operatorname{cl}_{\mathcal{S}}(G[B])\big]$ *for all* $B \subseteq Y$,

(3) $\operatorname{int}_{\mathcal{R}}\big(G^{-1}[D]\big)^c \subseteq F^{-1}\big[\operatorname{int}_{\mathcal{S}}(D)^c\big]$ *for all* $D \subseteq W$.

Proof. By using that $F[A] = \bigcup_{x \in A} F(x)$ for all $A \subseteq X$, we can see that assertion (2) of Theorem 19.1 is equivalent to the statement that:

(a) $F(x) \subseteq \operatorname{int}_{\mathcal{S}}(D)$ implies $x \in \operatorname{int}_{\mathcal{R}}\big(G^{-1}[D]\big)$ for all $x \in X$ and $D \subseteq W$.

Moreover, we can note that, for any $x \in X$, we have

$$F(x) \subseteq \operatorname{int}_{\mathcal{S}}(D) \iff F(x) \cap \operatorname{int}_{\mathcal{S}}(D)^c = \emptyset$$
$$\iff x \notin F^{-1}\big[\operatorname{int}_{\mathcal{S}}(D)^c\big] \iff x \in F^{-1}\big[\operatorname{int}_{\mathcal{S}}(D)^c\big]^c.$$

Therefore, the assertion (a) is equivalent to the statement that:

(b) $F^{-1}\big[\operatorname{int}_{\mathcal{S}}(D)^c\big]^c \subseteq \operatorname{int}_{\mathcal{R}}\big(G^{-1}[D]\big)$ for all $D \subseteq W$.

And, this is evidently equivalent to the assertion (3) of the present corollary.

Now analogously to Theorem 19.1 and Corollary 19.3, we can also easily prove the following theorems.

Theorem 19.4. *If* $F \in \mathcal{F}$ *and* $g \in \mathcal{G}$ *such that* g *is a function and* $Y = g^{-1}[W]$, *then the following assertions are equivalent:*

(1) (F, g) *is topologically mildly continuous,*

(2) $\operatorname{cl}_{\mathcal{R}}\left(g^{-1}[D]\right) \subseteq F^{-1}\left[\operatorname{cl}_{\mathcal{S}}(D)\right]$ *for all* $D \subset W$,

(3) $A \cap \operatorname{cl}_{\mathcal{R}}\left(g^{-1}[D]\right) \neq \emptyset$ *implies* $F[A] \cap \operatorname{cl}_{\mathcal{S}}(D) \neq \emptyset$ *for all* $A \subseteq X$ *and* $D \subseteq W$.

Theorem 19.5. *If* $F \in \mathcal{F}$ *and* $G \in \mathcal{G}$ *such that* G^{-1} *is a function and* $G[Y] = W$, *then the following assertions are equivalent:*

(1) (F, G) *is topologically mildly continuous,*

(2) $\operatorname{int}_{\mathcal{R}}(B)^c \subseteq F^{-1}\left[\operatorname{int}_{\mathcal{S}}(G[B])^c\right]$ *for all* $B \subseteq Y$,

(3) $F[A] \subseteq \operatorname{int}_{\mathcal{S}}(G[B])$ *implies* $A \subseteq \operatorname{int}_{\mathcal{R}}(B)$ *for all* $A \subseteq X$ *and* $B \subseteq Y$.

Theorem 19.6. *If* $f \in \mathcal{F}$ *and* $G \in \mathcal{G}$ *such that* f *is a function and* $X = f^{-1}[Z]$, *then the following assertions are equivalent:*

(1) (f, G) *is topologically mildly continuous,*

(2) $f^{-1}[\operatorname{int}_{\mathcal{S}}(D)] \subseteq \operatorname{int}_{\mathcal{R}}\left(G^{-1}[D]\right)$ *for all* $D \subseteq W$,

(3) $C \subseteq \operatorname{int}_{\mathcal{S}}(D)$ *implies* $f^{-1}[C] \subseteq \operatorname{int}_{\mathcal{R}}\left(G^{-1}[D]\right)$ *for all* $C \subseteq Z$ *and* $D \subseteq W$.

Theorem 19.7. *If* $f \in \mathcal{F}$ *and* $g \in \mathcal{G}$ *such that* f *and* g *are functions and* $f^{-1}[Z] = X$ *and* $Y = g^{-1}[W]$, *then the following assertions are equivalent:*

(1) (f, g) *is topologically mildly continuous,*

(2) $f\left[\operatorname{Cl}_{\mathcal{R}}\left(g^{-1}[D]\right)\right] \subseteq \operatorname{Cl}_{\mathcal{S}}(D)$ *for all* $D \subseteq W$,

(3) $f^{-1}[C] \cap \operatorname{Cl}_{\mathcal{R}}\left(g^{-1}[D]\right) \neq \emptyset$ *implies* $C \subseteq \operatorname{Cl}_{\mathcal{S}}(D)$ *for all* $C \subseteq Z$ *and* $D \subseteq W$.

Theorem 19.8. *If* $f \in \mathcal{F}$ *and* $G \in \mathcal{G}$ *such that* f *and* G^{-1} *are functions and* $X = f^{-1}[Z]$ *and* $G[Y] = W$, *then the following assertions are equivalent:*

(1) (f, G) *is topologically mildly continuous,*

(2) $f^{-1}[\operatorname{int}_{\mathcal{S}}(G[B])] \subseteq \operatorname{int}_{\mathcal{R}}(B)$ *for all* $B \subseteq Y$,

(3) $C \subseteq \operatorname{int}_{\mathcal{S}}(G[B])$ *implies* $f^{-1}[C]) \subseteq \operatorname{int}_{\mathcal{R}}(B)$ *for all* $C \subseteq Z$ *and* $B \subseteq Y$.

20. Fatness and denseness preserving and reversing relations

The following definition has been first introduced in [43].

Definition 20.1. For any $G \in \mathcal{G}$, we say that the relation

(1) G is *fatness preserving*, with respect to the relators $\mathcal{R}$ and $\mathcal{S}$, if $E \in \mathcal{E}_{\mathcal{R}}$ implies $G[E] \in \mathcal{E}_{\mathcal{S}}$,

(2) G is *denseness preserving*, with respect to the relators $\mathcal{R}$ and $\mathcal{S}$, if $D \in \mathcal{D}_{\mathcal{R}}$ implies $G[D] \in \mathcal{D}_{\mathcal{S}}$.

Remark 20.2. Recall that, by the corresponding definitions and some basic theorems on fat and dense sets, we have

(1) $\mathcal{D}_{\mathcal{R}} = \{ D \subseteq Y : \quad \forall\ R \in \mathcal{R} : \quad X = R^{-1}[D] \}$,

(2) $\mathcal{E}_{\mathcal{R}} = \{ E \subseteq Y : \quad \exists\ x \in X : \quad \exists\ R \in \mathcal{R} : \quad R(x) \subseteq E \}$,

(3) $\mathcal{D}_{\mathcal{R}} = \{ D \subseteq Y : \quad D^c \notin \mathcal{E}_{\mathcal{R}} \} = \{ D \subseteq Y : \quad \forall\ E \in \mathcal{E}_{\mathcal{R}} : \quad D \cap E \neq \emptyset \}$,

(4) $\mathcal{E}_{\mathcal{R}} = \{ E \subseteq Y : \quad E^c \notin \mathcal{D}_{\mathcal{R}} \} = \{ E \subseteq Y : \quad \forall\ D \in \mathcal{D}_{\mathcal{R}} : \quad D \cap E \neq \emptyset \}$.

Now, by using Definition 20.1 and Remark 20.2, we can easily prove the following theorem of [43].

Theorem 20.3. *For any $G \in \mathcal{G}$, the following assertions are equivalent:*

(1) G is denseness preserving; *(2) G^{-1} is fatness preserving.*

Proof. Suppose that (1) holds and $E \in \mathcal{E}_{\mathcal{S}}$. Then, by Definition 20.1, for any $D \in \mathcal{D}_{\mathcal{R}}$ we have $G[D] \in \mathcal{D}_{\mathcal{S}}$. Moreover, by Remark 20.2, we can state that $E \cap G[D] \neq \emptyset$, and thus $G^{-1}(E) \cap D \neq \emptyset$. Hence, by Remark 20.2, we can already infer that $G^{-1}[E] \in \mathcal{E}_{\mathcal{R}}$. Thus, assertion (2) also holds.

The converse implication $(2) \Longrightarrow (1)$ can be proved quite similarly.

Moreover, by using this theorem, we can also easily prove the following reformulation of [52, Theorem 11.3].

Theorem 20.4. *For any $G \in \mathcal{G}$, the following assertions are equivalent:*

(1) G is denseness preserving,

(2) $G^{-1}[S(z)] \in \mathcal{E}_{\mathcal{R}}$ for all $z \in Z$ and $S \in \mathcal{S}$,

(3) for any $z \in Z$ and $S \in \mathcal{S}$ there exist $x \in X$ and $R \in \mathcal{R}$ such that $R(x) \subseteq G^{-1}[S(z)]$,

(4) for any $z \in Z$ and $S \in \mathcal{S}$ there exist $x \in X$ and $R \in \mathcal{R}$ such that for any $y \in R(x)$ we have $G(y) \cap S(z) \neq \emptyset$.

Proof. If $z \in Z$ and $S \in \mathcal{S}$, then $S(z) \in \mathcal{E}_{\mathcal{S}}$. Hence, if (1) holds, by using Theorem 20.3, we can infer that $G^{-1}[S(z)] \in \mathcal{E}_{\mathcal{R}}$. Therefore, there exist $x \in X$ and $R \in \mathcal{R}$ such that $R(x) \subseteq G^{-1}[S(z)]$. Thus, for any $y \in R(x)$, we have $y \in G^{-1}[S(z)]$, and so $G(y) \cap S(z) \neq \emptyset$.

Hence, it is clear that $(1) \Longrightarrow (2) \Longrightarrow (3) \Longrightarrow (4)$. The converse implications can be proved quite similarly.

From the above theorem, it is clear that more specially we also have

Corollary 20.5. *If in particular* $X = Z$ *, then the following assertions are equivalent :*

(1) G *is denseness preserving with respect to* $\mathcal{R}$ *and* $\mathcal{S}$ *,*

(2) $(\Delta_X\,,\ G)$ *is properly mildly continuous with respect to* $\mathcal{R}^{\vartriangle}$ *and* $\mathcal{S}$ *.*

Proof. To check this, note that now, by Remark 6.3, the assertion (2) of Theorem 20.4 can be written in the shorter form that $G^{-1} \circ \mathcal{S} \circ \Delta_X = G^{-1} \circ \mathcal{S} \subseteq \mathcal{R}^{\vartriangle}$.

However, it now more important to note that, by using Theorems 20.3 and 17.8, we can also easily prove the following improvement of [43, Theorems 6.5] .

Theorem 20.6. *If* $F \in \mathcal{F}$ *and* $G \in \mathcal{G}$ *such that* F *is total and* G *is denseness preserving, then the pair* $(F\,,G)$ *is properly, uniformly, and proximally mildly continuous with respect to the relators* $\mathcal{R}^{\vartriangle}$ *and* $\mathcal{S}$ *.*

Proof. If $x \in X$ and $S \in \mathcal{S}$, then since $F(x) \neq \emptyset$ we have $S\,[\,F(x)\,] \in \mathcal{E}_{\mathcal{S}}$. Hence, by using Theorem 20.3, we can infer that $G^{-1}\big[\,S\,[\,F(x)\,]\,\big] \in \mathcal{E}_{\mathcal{R}}$. Therefore, by Theorem 17.8, the required assertion is also true.

From this theorem, by using Theorem 17.10, we can immediately derive

Corollary 20.7. *If* f *is a function of* X *onto* Z *and* $G \in \mathcal{G}$ *such that* G *is denseness preserving, and* $\mathcal{S} \neq \emptyset$ *, then the pair* $(f\,,G)$ *is properly, uniformly, and proximally mildly continuous with respect to the relators* $\mathcal{R}^{\vartriangle}$ *and* $\mathcal{S}^{\vartriangle}$ *.*

Moreover, by using Theorems 20.3 and 17.8, we can also easily prove the following improvement of [43, Theorems 6.4] .

Theorem 20.8. *If* $G \in \mathcal{G}$ *such that there exists a function* f *on* X *onto* Z *such that the pair* $(f\,,G)$ *is properly, uniformly, or proximally mildly continuous with respect to the relators* $\mathcal{R}^{\vartriangle}$ *and* $\mathcal{S}$ *, then* G *is denseness preserving.*

Proof. By Theorem 20.3, it suffices to show that G^{-1} is fatness preserving. For this, note that if $E \in \mathcal{E}_{\mathcal{S}}$, then by Remark 20.2 there exist $z \in Z$ and $S \in \mathcal{S}$ such that $S(z) \subseteq E$. Moreover, since f is a function on X onto Z , there exists $x \in X$ such that $z = f(x)$. Therefore, we also have $S\big(\,f(x)\big) \subseteq E$, and hence $G^{-1}\,\big[\,S\big(\,f(x)\big)\,\big] \subseteq G^{-1}[\,E\,]$. Moreover, by Theorem 17.8, we have $G^{-1}\,\big[\,S\big(\,f(x)\big)\,\big] \in \mathcal{E}_{\mathcal{R}}$. Therefore, $G^{-1}[\,E\,] \in \mathcal{E}_{\mathcal{R}}$ also holds.

Now, as an immediate consequence of Theorems 20.6 and 20.8, we can also state

Corollary 20.9. *For for any function* f *of* X *onto* Y *and* $G \in \mathcal{G}$ *, the following assertions are equivalent :*

(1) G *is denseness preserving ,*

(2) $(f\,,G)$ *is properly, uniformly, or proximally mildly continuous with respect to* $\mathcal{R}^{\vartriangle}$ *and* $\mathcal{S}$ *.*

Moreover, in addition Theorem 20.8, we can also prove the following improvement of [27, Theorem 9.17] .

Theorem 20.10. *If* $G \in \mathcal{G}$ *such that there exists* $F \in \mathcal{F}$ *such that the pair* $(F\,,G)$ *is properly, uniformly, or proximally mildly continuous with respect to the relators* $\mathcal{R}^{\vartriangle}$ *and* $\mathcal{S}^{\vartriangle}$ *, and moreover either* $F \neq \emptyset$ *, or* $X \neq \emptyset$ *and* $\mathcal{R}$ *is total, then* G *is denseness preserving.*

Proof. By Theorem 20.3, it suffices to show that G^{-1} is fatness preserving. For this, note that if $E \in \mathcal{E}_{\mathcal{S}}$, then by Remark 6.3 the relation $V = X \times E$ is in $\mathcal{S}^{\triangle}$. Therefore, if the pair (F, G) is properly mildly continuous with respect to the relators $\mathcal{R}^{\triangle}$ and $\mathcal{S}^{\triangle}$, then the relation $U = G^{-1} \circ V \circ F$ is in $\mathcal{R}^{\triangle}$. Thus, by Remark 6.3, for any $x \in X$, we have $U(x) \in \mathcal{E}_{\mathcal{R}}$.

Moreover, we can note that $U(x) = G^{-1}\big[\,V\,[\,F(x)\,]\,\big]$, and thus

$$U(x) = G^{-1}\,[\,E\,] \quad \text{if} \quad x \in D_F \qquad \text{and} \qquad U(x) = \emptyset \quad \text{if} \quad x \in D_F^{\,c}.$$

Hence, if $F \neq \emptyset$, and thus $D_F \neq \emptyset$, by choosing $x \in D_F$, we can see that $G^{-1}[\,E\,] = U(x) \in \mathcal{E}_{\mathcal{R}}$. While, if $\mathcal{R}$ is total, then by Remark 6.23 we have $\emptyset \notin \mathcal{E}_R$. Hence, we can see that $D_F = X$, and thus F is also total. Therefore, $D_F \neq \emptyset$, and thus $F \neq \emptyset$, whenever $X \neq \emptyset$ also holds. Thus, by the former case, $G^{-1}[\,E\,] \in \mathcal{E}_{\mathcal{R}}$ again holds.

Now, because of Corollary 20.7 and Theorem 20.10, we can also state

Corollary 20.11. *If $X \neq \emptyset$ and $\mathcal{S} \neq \emptyset$, then for any function f of X onto Y and $G \in \mathcal{G}$ the following assertions are equivalent:*

(1) G is denseness preserving,

(2) (f, G) is properly, uniformly, or proximally mildly continuous with respect to the relators $\mathcal{R}^{\triangle}$ and $\mathcal{S}^{\triangle}$.

Moreover, by using Theorem 20.10 and Corollary 17.5, we can also easily prove

Theorem 20.12. *If $G \in \mathcal{G}$ such that there exists $F \in \mathcal{F}$ such that the pair (F, G) is paratopologically mildly continuous, and moreover $X \neq \emptyset$ and $\mathcal{R}$ is total, then the relation $G \circ \psi$ is denseness preserving for all $\psi \in Y^{Y}$.*

Proof. Now, by Corollary 17.5, the pair $\big(F \circ \varphi,\ G \circ \psi\big)$ is properly mildly continuous with respect to the relators $\mathcal{R}^{\triangle}$ and $\mathcal{S}^{\triangle}$ for all $\varphi \in X^{X}$ and $\psi \in Y^{Y}$. Hence, by Theorem 20.10, we can already see that the required assertion is also true.

Remark 20.13. To see the usefulness of denseness preserving relations, we can also note that if $\mathcal{R}$ is total, then $Y \in \mathcal{D}_{\mathcal{R}}$. Therefore, if $G \in \mathcal{G}$ such that G is denseness preserving, then $G\,[\,Y\,] \in \mathcal{E}_{\mathcal{S}}$.

Hence, we can infer that $W \in \mathcal{D}_{\mathcal{S}}$, and thus $\mathcal{S}$ is also total. Moreover, if in particular $Z = W$ and $G\,[\,Y\,] \in \mathcal{F}_{\mathcal{S}}$, then $W = Z = \operatorname{cl}_{\mathcal{S}}\big(G\,[\,Y\,]\big) \subseteq G\,[\,Y\,] \subseteq W$, and thus $G\,[\,Y\,] = W$ also holds.

In [43], having in mind the definition of contra continuous functions [6, 24], the first author also introduced following

Definition 20.14. For any $G \in \mathcal{G}$, we say that the relation

(1) G is *fatness reversing*, with respect to the relators $\mathcal{R}$ and $\mathcal{S}$, if $E \in \mathcal{E}_{\mathcal{R}}$ implies $G\,[\,E\,] \in \mathcal{D}_{\mathcal{S}}$,

(2) G is *denseness reversing*, with respect to the relators $\mathcal{R}$ and $\mathcal{S}$, if $D \in \mathcal{D}_{\mathcal{R}}$ implies $G\,[\,D\,] \in \mathcal{E}_{\mathcal{S}}$.

Now, by using some similar arguments as above, we can also easily prove the following two theorems of [43].

Theorem 20.15. *For any $G \in \mathcal{G}$, the following assertions are equivalent:*

(1) *G is fatness (denseness) reversing,*

(2) *G^{-1} is fatness (denseness) reversing.*

Remark 20.16. Note that the implication (2) $\Longrightarrow$ (1) can now be derived from the converse implication by using the fact that $G = \big(G^{-1}\big)^{-1}$.

Moreover, it is also worth noticing that "the fatness reversing part" of the above theorem can be more easily proved with the help of the following theorem.

Theorem 20.17. *Under the notation $F = X \times Z$, for any $G \in \mathcal{G}$, the following assertions are equivalent:*

(1) *G is fatness reversing,* *(2)* $\mathcal{S}^{-1} \circ \{G\} \circ \mathcal{R} \subseteq \{F\}$,

(3) $Z = S^{-1}\big[\,G\big[\,R(x)\,\big]\,\big]$ *for all* $x \in X$, $R \in \mathcal{R}$, *and* $S \in \mathcal{S}$.

Proof. If $x \in X$ and $R \in \mathcal{R}$, then by Remark 20.2 we have $R(x) \in \mathcal{E}_{\mathcal{R}}$. Hence, if assertion (1) holds, we can infer that $G\,[\,R(x)\,] \in \mathcal{D}_{\mathcal{S}}$. Therefore, by Remark 20.2, we have $Z = S^{-1}\big[\,G\big[\,R(x)\,\big]\,\big]$ for all $S \in \mathcal{S}$, and thus assertion (3) also holds.

While, if $E \in \mathcal{E}_{\mathcal{R}}$, then by Remark 20.2, there exist $x \in X$ and $R \in \mathcal{R}$ such that $R(x) \subseteq E$, and thus $S^{-1}\big[\,G\,[\,R(x)\,]\,\big] \subseteq S^{-1}\big[\,G\,[\,E\,]\,\big]$ for all $S \in \mathcal{S}$. Hence, if assertion (3) holds, we can infer that $Z = S^{-1}\big[\,G\,[\,E\,]\,\big]$ for all $S \in \mathcal{S}$. Therefore, by Remark 20.2, we also have $G\,[\,E\,] \in \mathcal{D}_{\mathcal{S}}$, and thus assertion (1) also holds.

The equivalence of the assertions (2) and (3) is immediate from the corresponding definitions.

Remark 20.18. Note that if in particular the relators $\mathcal{R}$ and $\mathcal{S}$ are nonvoid, then instead of (2) we may also write that $\{F\} = \mathcal{S}^{-1} \circ \{G\} \circ \mathcal{R}$.

Moreover, it is also worth noticing that

$$\mathrm{cl}_S\big(G\big[\,R(x)\,\big]\big) = S^{-1}\big[\,G\big[\,R(x)\,\big]\,\big] = \big(S^{-1} \circ G \circ R\big)(x) = \mathrm{cl}_{R \boxtimes S}\big(\{G\}\big)(x)$$

for all $x \in X$, $R \in \mathcal{R}$ and $S \in \mathcal{S}$.

Now, as a detailed reformulation of Theorem 20.17, we can also state

Corollary 20.19. *For any $G \in \mathcal{G}$, the following assertions are equivalent:*

(1) *G is fatness reversing,*

(2) $S(z) \cap G\,[\,R(x)\,] \neq \emptyset$ *for all* $x \in X$, $z \in Z$, $R \in \mathcal{R}$ *and* $S \in \mathcal{S}$,

(3) for any $x \in X$, $z \in Z$, $R \in \mathcal{R}$, *and* $S \in \mathcal{S}$, *there exist* $y \in R(x)$ *and* $w \in S(z)$ *such that* $w \in G(y)$.

However, it is now more important to note that, as an immediate consequence of Theorem 10.11 and 13.6, and the fact that, under the notation $F = X \times Z$, we have $\{F\} = \{F\}^{\Diamond}$ for any stable unary operation $\Diamond$ for relators, we can also state the following

Theorem 20.20. *If in particular $\Box$ is a stable, inversion and composition compatible closure operation, then under the notation $F = X \times Z$, for any $G \in \mathcal{G}$, the following assertions are equivalent:*

(1) *G is fatness reversing with respect to $\mathcal{R}$ and $\mathcal{S}$,*

(2) *$(\mathcal{R}, \mathcal{S})$ is mildly $\Box$–continuous with respect to $\{F\}$ and $\{G\}$.*

Remark 20.21. Note that now, under the notation $F = X \times Z$, for any $R \in \mathcal{R}$ and $S \in \mathcal{S}$, we have to consider the diagram:

$$\begin{array}{ccc} X & \xrightarrow{R} & Y \\ {\scriptstyle F}\big\downarrow & & \big\downarrow{\scriptstyle G} \\ Z & \xrightarrow{S} & W \end{array}$$

Remark 20.22. In addition to the fatness and denseness preserving and reversing relations, the *proximal and topological openness and closedness preserving and reversing relations* should also be investigated.

However, by the results of [52], these relations are rather connected with the proximally and topologically lower and upper continuous relations than with the proximally and topologically mildly continuous ones.

Surprisingly enough, in the context of topological spaces and their obvious generalizations, functions and relations with topological openness and closedness reversing inverses, under the name "*contra-continuous functions and upper and lower contra-continuous multifunctions*", have also been intensively investigated by a great number of mathematicians. (See, for instance, [6, 24, 9, 13, 1, 19].)

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