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INCREASING FUNCTIONS AND CLOSURE OPERATIONS ON GENERALIZED ORDERED SETS

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Abstract. In this paper, having in mind Galois connections, we establish several consequences of the following definitions.

An ordered pair $X(\leq)=(X, \leq)$ consisting of a set X and a relation $\leq$ on X is called a goset (generalized ordered set).

A function f of one goset X to another Y is called increasing if $u \leq v$ implies $f(u) \leq f(v)$ for all $u, v \in X$.

In particular, an increasing function φ of X to itself is called a closure operation on X if $x \leq \varphi(x)$ and $\varphi\big(\varphi(x)\big) \leq \varphi(x)$ for all $x \in X$.

The results obtained extend and supplement some former results on increasing functions and closure operations, and can be generalized to relator spaces.

Introduction

Ordered sets and Galois connections occur almost everywhere in mathematics [?]. They allow of transposing problems and results from one world of our imagination to another one.

In [?], having in mind a terminology of Birkhoff [?, p. 1], an ordered pair $X(\leq)=(X, \leq)$ consisting of a set X and a relation $\leq$ on X is called a *goset* (generalized ordered set).

In particular, a goset $X(\leq)$ is called a *proset* (preordered set) if the relation $\leq$ is reflexive and transitive. And, a proset is $X(\leq)$ called a *poset* (partially ordered set) if the relation $\leq$ is in addition antisymmetric.

In [?], according to [?, Definition 7.23], an ordered pair (f, g) of functions f of one goset X to another Y and g of Y to X is called a *Galois connection* if for any $x \in X$ and $y \in Y$ we have $f(x) \leq y$ if and only if $x \leq g(y)$.

In this case, by taking $\varphi = g \circ f$, we can at once see that $f(u) \leq f(v) \iff u \leq g\big(f(v)\big) \iff u \leq (g \circ f)(v) \iff u \leq \varphi(v)$ for all $u, v \in X$. Therefore, the ordered pair (f, φ) is a *Pataki connection* by a terminology of [?].

A function f of one goset X to another Y is called *increasing* if $u \leq v$ implies $f(u) \leq f(v)$ for all $u, v \in X$. And, an increasing function φ of X to itself is called a *closure operation* if $x \leq \varphi(x)$ and $\varphi\big(\varphi(x)\big) \leq \varphi(x)$ for all $x \in X$.

In [?], we have proved that if (f, φ) is a Pataki connection between the prosets X and Y, then f is increasing and φ is a closure operation such that $f \leq f \circ \varphi$ and $f \circ \varphi \leq f$. Thus, $f = f \circ \varphi$ if in particular Y is a poset.

Moreover, we have also proved that a function φ of a proset X to itself is a closure operation if and only if (φ, φ) is a Pataki connection, or equivalently (f, φ) is a Pataki connection for some function f of X to another proset Y.

Thus, increasing functions are, in a certain sense, natural generalizations not only closure operations, but also Pataki and Galois connections. Therefore, it seems plausible to extend some results on these connections to increasing functions.

For instance, having in mind a supremum property of Galois connections [?], we shall show that a function f of one goset X to another Y is increasing if it preserves upper bounds in the sense that $f[\operatorname{ub}(A)] \subseteq \operatorname{ub}\big(f[A]\big)$ for all $A \subseteq X$.

If X is reflexive in the sense that the inequality relation in it is reflexive, then we may write max instead of ub. While, if X and Y are sup–complete and antisymmetric, then we can also state that $\sup\big(f[A]\big) \leq f\big(\sup(A)\big)$.

In particular, we shall also show that if φ is a closure operation on a sup–complete, transitive and antisymmetric goset X, then $\varphi\big(\sup(A)\big) = \varphi\big(\sup\big(\varphi[A]\big)\big)$ for all $A \subseteq X$. Moreover, if $Y = \varphi[X]$ and $A \subseteq Y$, then $\sup_Y(A) = \varphi\big(\sup_X(A)\big)$.

Finally, by writing R and S in place of the inequalities in the gosets X and

Moreover, the sets $D_F = \{ x \in X : \; F(x) \neq \emptyset \}$ and $R_F = F[X]$ are called the *domain* and *range* of F, respectively. If in particular $D_F = X$, then we say that F is a relation of X to Y, or that F is a *non-partial relation* on X to Y.

In particular, a relation f on X to Y is called a *function* if for each $x \in D_f$ there exists $y \in Y$ such that $f(x) = \{y\}$. In this case, by identifying singletons with their elements, we may simply write $f(x) = y$ in place of $f(x) = \{y\}$.

Moreover, a function $\star$ of X to itself is called a *unary operation* on X. While, a function $*$ of X^2 to X is called a *binary operation* on X. And, for any $x, y \in X$, we usually write $x^\star$ and $x * y$ instead of $\star(x)$ and $*\big((x, y)\big)$.

If F is a relation on X to Y, then a function f of D_F to Y is called a *selection* of F if $f \subseteq F$, i.e., $f(x) \in F(x)$ for all $x \in D_F$. Thus, the *Axiom of Choice* can be briefly expressed by saying that every relation has a selection.

For any relation F on X to Y, we may naturally define two *set-valued functions*, $F^\diamond$ of X to $\mathcal{P}(Y)$ and $F^\Diamond$ of $\mathcal{P}(X)$ to $\mathcal{P}(Y)$, such that $F^\diamond(x) = F(x)$ for all $x \in X$ and $F^\Diamond(A) = F[A]$ for all $A \subset X$.

Functions of X to $\mathcal{P}(Y)$ can be identified with relations on X to Y. While, functions of $\mathcal{P}(X)$ to $\mathcal{P}(Y)$ are in a sense more general objects than relations on X to Y [**?**]. However, they are frequently less convenient than relations.

If F is a relation on X to Y, then $F = \bigcup_{x \in X} \{x\} \times F(x)$. Therefore, the values $F(x)$, where $x \in X$, uniquely determine F. Thus, a relation F on X to Y can be naturally defined by specifying $F(x)$ for all $x \in X$.

For instance, the *complement relation* F^c can be naturally defined such that $F^c(x) = F(x)^c = Y \setminus F(x)$ for all $x \in X$. The latter notation will not cause confusions, since thus we also have $F^c = X \times Y \setminus F$.

Quite similarly, the *inverse relation* F^{-1} can be naturally defined such that $F^{-1}(y) = \{x \in X : \; y \in F(x)\}$ for all $y \in Y$. Thus, the operations c and -1 are compatible in the sense $\big(F^c\big)^{-1} = \big(F^{-1}\big)^c$.

Moreover, if in addition G is a relation on Y to Z, then the *composition relation* $G \circ F$ can be naturally defined such that $(G \circ F)(x) = G[F(x)]$ for all $x \in X$. Thus, we also have $(G \circ F)[A] = G\big[F[A]\big]$ for all $A \subseteq X$.

While, if G is a relation on Z to W, then the *box product relation* $F \boxtimes G$ can be naturally defined such that $(F \boxtimes G)(x, z) = F(x) \times G(z)$ for all $x \in X$ and $z \in Z$. Thus, we have $(F \boxtimes G)[A] = G \circ A \circ F^{-1}$ for all $A \subseteq X \times Z$ [**?**].

Hence, by taking $A = \{(x, z)\}$, and $A = \Delta_Y$ if $Y = Z$, one can see that the box and composition products are actually equivalent tools. However, the box product can be immediately defined for arbitrary families of relations.

Now, a relation R on X may be briefly called *reflexive* if $\Delta_X \subseteq R$, and *transitive* if $R \circ R \subseteq R$. Moreover, R may be briefly called *symmetric* if $R^{-1} \subseteq R$, and *antisymmetric* if $R \cap R^{-1} \subseteq \Delta_X$.

Thus, a reflexive and transitive (symmetric) transitive relation may be called a *preorder (tolerance) relation.* And, a symmetric (antisymmetric) preorder relation may be called an *equivalence (partial order) relation.*

For instance, for $A \subseteq X$, the *Pervin relation* $P_A = A^2 \cup A^c \times X$ is a preorder relation on X [**?**]. While, for a *pseudo-metric* d on X and $r > 0$, the *surrounding* $B_r^d = \big\{(x, y) \in X^2 : \; d(x, y) < r\big\}$ is a tolerance relation on X.

Moreover, we may recall that if $\mathcal{A}$ is a *partition* of X, i.e., a family of pairwise disjoint, nonvoid subsets of X such that $X = \bigcup \mathcal{A}$, then $E_{\mathcal{A}} = \bigcup_{A \in \mathcal{A}} A^2$ is an equivalence relation on X, which can, to some extent, be identified with $\mathcal{A}$.

Finally, we note that, for any relation R on X, we define $R^0 = \Delta_X$, and $R^n = R \circ R^{n-1}$ if $n \in \mathbb{N}$. Moreover, we also define $R^\infty = \bigcup_{n=0}^{\infty} R^n$. Thus, R^∞ is just the smallest preorder relation on X containing R [?].

2. A few basic facts on generalized ordered sets

According to [?], an ordered pair $X(\leq) = \big(X, \leq\big)$, consisting of a set X and a relation $\leq$ on X, will be called *generalized ordered set*, or an ordered set without axioms. And, we shall usually write X in place of $X(\leq)$.

In the sequel, a generalized ordered set $X(\leq)$ will, for instance, be called *reflexive* if the relation $\leq$ is reflexive. Moreover, the generalized ordered set $X'(\leq') = X(\leq^{-1})$ will be called the *dual* of $X(\leq)$.

Having in mind the terminology of Birkhoff [?, p. 1], a generalized ordered set may be briefly called a *goset*. Moreover, a preordered (partially ordered) set may be call a *proset (poset)*.

Thus, every set X is a poset with the identity relation Δ_X. Moreover, X is a proset with the universal relation X^2. And every subfamily of the power set $\mathcal{P}(X)$ of X is a poset with the ordinary set inclusion $\subseteq$.

The usual definitions on posets can be naturally extended to gosets [?]. (And, even to arbitrary *relator spaces* [?] which include *ordered sets* [?], *context spaces* [?], and *uniform spaces* [?] as the most important particular cases.)

For instance, for any subset A of a goset X, we may naturally define

$$\mathrm{lb}(A) = \big\{ x \in X : \quad \forall\ a \in A : \quad x \leq a \big\},$$

$$\mathrm{ub}(A) = \big\{ x \in X : \quad \forall\ a \in A : \quad a \leq x \big\},$$

$$\min(A) = A \cap \mathrm{lb}(A), \qquad \max(A) = A \cap \mathrm{ub}(A),$$

$$\inf(A) = \max\big(\mathrm{lb}(A)\big), \qquad \sup(A) = \min\big(\mathrm{ub}(A)\big).$$

In the sequel, by identifying singletons with their elements, we shall, for instance, write $\mathrm{ub}(x)$ in place of $\mathrm{ub}(\{x\})$ for all $x \in X$. Thus, we have

$$\mathrm{ub}(x) = \leq(x) = [x, +\infty[= \big\{ y \in X : \quad x \leq y \big\}$$

for all $x \in X$.

Now, as an immediate of the corresponding definitions, we can state

Theorem 2.1. *For any subset A of a goset X, we have*

(1) $\mathrm{lb}(A) = \bigcap_{a \in A} \mathrm{lb}(a)$, *(2)* $\mathrm{ub}(A) = \bigcap_{a \in A} \mathrm{ub}(a)$.

Hence, it is clear that in particular we also have

Corollary 2.2. *If X is a goset, then*

(1) $\mathrm{lb}(\emptyset) = X$ *and* $\mathrm{ub}(\emptyset) = X$,

(2) $\mathrm{lb}(B) \subseteq \mathrm{lb}(A)$ *and* $\mathrm{ub}(B) \subseteq \mathrm{ub}(A)$ *for all* $A \subseteq B \subseteq X$.

By using Theorem 2.1, we can also easily prove the following

Theorem 2.3. *If* Φ *is a unary operation on* $\mathcal{P}(X)$, *for some set* X, *such that* $\Phi(A) = \bigcap_{a \in A} \Phi(a)$ *for all* $A \subseteq X$, *then there exists a relation* $\leq$ *on* X *such that* $\Phi = \mathrm{lb}_{\leq}$ $(\Phi = \mathrm{ub}_{\leq})$.

However, it is now more important to note that we also have the following

Theorem 2.4. *For any two subsets* A *and* B *of a goset* X, *we have*

$$A \subseteq \mathrm{lb}(B) \iff B \subseteq \mathrm{ub}(A).$$

Proof. By the corresponding definitions, each of the above inclusions is equivalent to the property that $a \leq b$ for all $a \in A$ and $b \in B$, which can be briefly expressed by writing that $A \times B \subseteq \leq$. (That is, $A \in \mathrm{Lb}_{\leq}(B)$ or $B \in \mathrm{Ub}_{\leq}(A)$ by [**?**].)

Remark 2.5. The above theorem shows that

$$\mathrm{lb}(A) \subseteq' B \iff A \subseteq \mathrm{ub}(B)$$

for all $A, B \subset X$.

Therefore, the set-functions lb and ub form a *Galois connection* between the poset $\mathcal{P}(X)$ and its dual in the sense of [**?**, Definition 7.23], suggested by Schmidt's reformulation [**?**, p. 209] of Ore's definition of Galois connexions [**?**].

Remark 2.6. Hence, by taking $\Phi = \mathrm{ub} \circ \mathrm{lb}$, we can easily see that

$$\mathrm{lb}(A) \subseteq' \mathrm{lb}(B) \iff A \subseteq \Phi(B)$$

for all $A, B \subset X$.

Therefore, the set-functions lb and Φ form a *Pataki connection* between the poset $\mathcal{P}(X)$ and its dual in the sense of [**?**, Remark 3.8] suggested by a fundamental unifying work of Pataki [**?**] on the basic refinements of relators studied each separately by the present author in [**?**].

By [**?**], the letter fact implies that $\mathrm{lb} = \mathrm{lb} \circ \Phi$, and the function Φ is a *closure operation* on the poset $\mathcal{P}(X)$ in the sense of [**?**, p. 111]. By an observation, attributed to Dedekind by Erné [**?**, p. 50], this is equivalent to the requirement that the function Φ with itself form a Pataki connection between the poset $\mathcal{P}(X)$ and itself.

3. Some further results on gosets

Concerning minima and maxima, and infima and suprema, one can easily prove the following theorems.

Theorem 3.1. *For any subset* A *of a goset* X, *we have*

(1) $\min(A) = \{x \in A : A \subseteq \mathrm{ub}(x)\}$, *(2)* $\max(A) = \{x \in A : A \subseteq \mathrm{lb}(x)\}$.

Remark 3.2. By this theorem, for instance, we may also naturally define

$$\mathrm{ub}^*(A) = \{x \in X : A \cap \mathrm{ub}(x) \subseteq \mathrm{lb}(x)\}.$$

Thus, $\max^*(A) = A \cap \mathrm{ub}^*(A)$ is just the family of all *maximal elements of* A.

Theorem 3.3. *For any subset* A *of a goset* X, *we have*

(1) $\inf(A) = \mathrm{lb}(A) \cap \mathrm{ub}\big(\mathrm{lb}(A)\big)$, *(2)* $\sup(A) = \mathrm{ub}(A) \cap \mathrm{lb}\big(\mathrm{ub}(A)\big)$.

Theorem 3.4. *For any subset A of a goset X, we have*

(1) $\inf(A) = \sup\big(\operatorname{lb}(A)\big)$, *(2)* $\sup(A) = \inf\big(\operatorname{ub}(A)\big)$,

(3) $\min(A) = A \cap \inf(A)$, *(4)* $\max(A) = A \cap \sup(A)$.

Theorem 3.5. *Under the notation $\Phi = \min$, $\max$, $\inf$, or $\sup$, for any subset A of an antisymmetric goset X, we have* $\operatorname{card}\big(\Phi(A)\big) \leq 1$.

Remark 3.6. Conversely, one can also easily see that if X is a reflexive goset such that $\operatorname{card}\big(\Phi(A)\big) \leq 1$ for all $A \subseteq X$, with $\operatorname{card}(A) = 2$, then X is antisymmetric.

In [**?**], by using the notation $\mathcal{L} = \{A \subseteq X : \ A \subseteq \operatorname{lb}(A)\}$, we have first proved that a reflexive goset X is antisymmetric if and only if $\operatorname{card}(A) \leq 1$ for all $A \in \mathcal{L}$.

Definition 3.7. A goset X is called *inf–complete* (*sup–complete*) if $\inf(A) \neq \emptyset$ ($\sup(A) \neq \emptyset$) for all $A \subseteq X$.

Remark 3.8. Quite similarly, a goset X may, for instance, be also naturally called *min–complete* if $\min(A) \neq \emptyset$ for all nonvoid subset A of X.

Thus, the set $\mathbb{N}$ of all natural numbers is min–, but not inf–complete. While, the extended real line $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ is inf–, but not min–complete.

Now, as an immediate consequence of Theorem 3.4, we can state the following straightforward extension of [**?**, Theorem 3, p. 112].

Theorem 3.9. *For a goset X, the following assertions are equivalent:*

(1) X is inf–complete, *(2) X is sup–complete.*

Remark 3.10. Similar equivalences of several modified inf– and sup–completeness properties of gosets have been established in [**?**] and [**?**].

Definition 3.11. A goset X is called *linear* if for any $u, v \in X$, with $u \neq v$, we have either $u \leq v$ or $v \leq u$.

Remark 3.12. If X is a goset, then for any $u, v \in X$ we write $u < v$ if both $u \leq v$ and $u \neq v$.

Therefore, if the goset X is linear, then for any $u, v \in X$, with $u \neq v$, we actually have either $u < v$ or $v < u$.

Moreover, as a consequence of the corresponding definitions, we can also state

Theorem 3.13. *For a goset X, the following assertions are equivalent:*

(1) X is is reflexive and linear,

(2) for any $u, v \in X$, we have either $u \leq v$ or $v \leq u$,

(3) $\min(A) \neq \emptyset$ ($\max(A) \neq \emptyset$) *for all $A \subseteq X$ with $1 \leq \operatorname{card}(A) \leq 2$.*

Hence, it is clear that in particular we also have

Corollary 3.14. *If X is a min-complete (max-complete) goset, then X is reflexive and linear.*

The importance of reflexive, linear, and antisymmetric gosets is apparent from the following

Theorem 3.15. *(1) If X is an antisymmetric goset, then $u < v$ implies $v \not\leq u$ for all $u, v \in X$,*

(2) While, if X is a reflexive and linear goset, then $u \not\leq v$ implies $v < u$ for all $u, v \in X$.

Proof. To check (2), note that if $u \not\leq v$, then by Theorem 3.13 we have $v \leq u$. Moreover, by the reflexivity of X, we also have $v \neq u$. Therefore, we also have $v < u$.

Now, as an immediate consequence of this theorem, we can also state the following very particular Galois-type connection.

Corollary 3.16. *If X is a reflexive, linear and antisymmetric goset, then for any $u, v \in X$ we have $u < v$ if and only if $v \not\leq u$.*

Remark 3.17. If $X(\leq_X)$ is a goset and $Y \subseteq X$, then by taking $\leq_Y = \leq_X \cap Y^2$ we can at once see that $Y(\leq_Y)$ is also a goset which inherits several basic properties of the original goset $X(\leq_X)$.

Moreover, concerning subgosets, we can also easily prove the following

Theorem 3.18. *If X is a goset and $Y \subseteq X$, then for any $A \subseteq Y$ we have*

(1) $\min_Y(A) = \min_X(A)$, *(2)* $\min_Y(A) = \min_X(A)$,

(3) $\mathrm{lb}_Y(A) = \mathrm{lb}_X(A) \cap Y$, *(4)* $\mathrm{ub}_Y(A) = \mathrm{ub}_X(A) \cap Y$,

(5) $\inf_X(A) \cap Y \subseteq \inf_Y(A)$, *(6)* $\sup_X(A) \cap Y \subseteq \sup_Y(A)$.

Proof. To check (5), note that if $\alpha \in \inf_X(A)$, then by Theorem 3.3 we have $\alpha \in \mathrm{lb}_X(A)$ and $\alpha \in \mathrm{ub}_X\big(\mathrm{lb}_X(A)\big)$. Hence, if $\alpha \in Y$ also holds, by using (3) we can see that $\alpha \in \mathrm{lb}_Y(A)$.

Moreover, if $v \in \mathrm{lb}_Y(A)$, then by using (3) we can see that $v \in Y$ and $v \in \mathrm{lb}_X(A)$. Hence, since $\alpha \in \mathrm{ub}_X\big(\mathrm{lb}_X(A)\big)$, we can infer that $v \leq \alpha$. This shows that $\alpha \in \mathrm{ub}_X\big(\mathrm{lb}_Y(A)\big)$. Hence, since $\alpha \in Y$, by using (3) we can already infer that $\alpha \in \mathrm{ub}_Y\big(\mathrm{lb}_Y(A)\big)$. Thus, by Theorem 3.3, $\alpha \in \sup_Y(A)$ also holds.

Remark 3.19. In connection with (5), Tamás Glavosits, my PhD student, showed that the corresponding equality need not be true even if X is finite poset.

For this, he took $X = \{a, b, c, d\}$, $Y = X \setminus \{b\}$, $A = Y \setminus \{a\}$, and considered the preorder $\leq$ on X generated by the relation $R = \{(a, b), (b, c), (b, d)\}$.

Thus, he could at once see that $\inf_Y(A) = \max\big(\mathrm{lb}_Y(A)\big) = \max\big(\{a\}\big) = \{a\}$, but $\inf_X(A) = \max\big(\mathrm{lb}_X(A)\big) = \max\big(\{a, b\}\big) = \{b\}$, and thus $\inf_X(A) \cap Y = \emptyset$.

4. Increasing functions of one goset to another

Increasing functions are usually called isotone, monotone, or order-preserving in algebra. Moreover, in [**?**, p. 186] even the extensive maps are called increasing. However, we prefer to use the following terminology of analysis [**?**, p. 128].

Definition 4.1. If f is a function of one goset X to another Y, then we say that:

(1) f is *increasing* if $u \leq v$ implies $f(u) \leq f(v)$ for all $u, v \in X$,

(2) f is *strictly increasing* if $u < v$ implies $f(u) < f(v)$ for all $u, v \in X$.

Remark 4.2. Quite similarly, the function f may, for instance, be called *decreasing* if $u \leq v$ implies $f(v) \leq f(u)$ for all $u,\, v \in X$.

Thus, we can note that f is a decreasing function of X to Y if and only if it is an increasing function of X to the dual Y' of Y.

Therefore, the study of decreasing functions can be traced back to that of the increasing ones. The following two obvious theorems show that almost the same is true in connection with the strictly increasing ones.

Theorem 4.3. *If f is an injective, increasing function of one goset X to another Y, then f is strictly increasing.*

Remark 4.4. Conversely, we can at once see that if f is a strictly increasing function of an arbitrary goset X to a reflexive one Y, then f is increasing.

Moreover, we can also easily prove the following

Theorem 4.5. *If f is a strictly increasing function of a linear goset X to an arbitrary one Y, then f is injective.*

Proof. If $u,\, v \in X$ such that $u \neq v$, then by Remark 3.12 we have either $u < v$ or $v < u$. Hence, by using the strict increasingness of f, we can already infer that either $f(u) < f(v)$ or $f(v) < f(u)$, and thus $f(u) \neq f(v)$.

Now, as an immediate consequence of the above results, we can also state

Corollary 4.6. *For a function f of a linear goset X to a reflexive one Y, the following assertions are equivalent:*

(1) f is strictly increasing, *(2) f is injective and increasing.*

In this respect, it also worth proving the following

Theorem 4.7. *If f is a strictly increasing function of a linear goset X onto an antisymmetric one Y, then f^{-1} is a strictly increasing function of Y onto X.*

Proof. From Theorem 4.5, we know that f is injective. Hence, since $f[X] = Y$, we can see that f^{-1} is a function of Y onto Y. Therefore, we need only show that f^{-1} is also strictly increasing.

For this, suppose that $z,\, w \in Y$ such that $z < w$. Define $u = f^{-1}(z)$ and $v = f^{-1}(w)$. Then, $u,\, v \in X$ such that $z = f(u)$ and $w = f(v)$. Hence, since $z \neq w$, we can also see that $u \neq v$. Moreover, by Remark 3.12, we have either $u < v$ or $v < u$. However, if $v < u$, then by the strict increasingness of f we also have $f(v) < f(u)$, and thus $w < z$. Hence, by using the inequality $z < w$ and the antisymmetry of Y, we can already infer that $z = w$. This contradiction proves that $u < v$, and thus $f^{-1}(z) < f^{-1}(w)$.

Hence, by using Theorem 4.3 and Remark 4.4, we can immediately derive

Corollary 4.8. *If f is an injective, increasing function of a reflexive, linear goset X onto an antisymmetric one Y, then f^{-1} is an injective, increasing function of Y onto X.*

Analogously to [**?**], we shall now also use the following

Definition 4.9. If φ is an unary operation on a goset X, then we say that:

(1) φ is *extensive (intensive)* if $\Delta_X \leq \varphi$ $\big(\varphi \leq \Delta_X\big)$,

(2) φ is *upper (lower) semiidempotent* if $\varphi \leq \varphi^2$ $\big(\varphi^2 \leq \varphi\big)$.

Remark 4.10. Moreover, φ may be naturally called *upper (lower) semiinvolutive* if φ^2 is extensive (intensive). That is, $\Delta_X \leq \varphi^2$ $(\varphi^2 \leq \Delta_X)$.

Remark 4.11. In this respect, it is also worth noticing that φ is upper (lower) semiidempotent if and only if its restriction to its range is extensive (intensive). Therefore, if φ is extensive (intensive), then φ is upper (lower) semiidempotent.

The importance of extensive operations is also apparent from the following

Theorem 4.12. *If φ is a strictly increasing operation on a min-complete, antisymmetric goset X, then φ is extensive.*

Proof. If φ is not extensive, then $A = \{ x \in X : \ x \not\leq \varphi(x) \}$ is a nonvoid subset of X. Therefore, by the min–completeness of X, there exists $a \in X$ such that $a \in \min(A)$. Hence, it follows that $a \in A$ and $a \in \mathrm{lb}(A)$. Thus, in particular $a \not\leq \varphi(a)$. Hence, by using Corollary 3.14 and Theorem 3.15, we can infer that $\varphi(a) < a$. Thus, since φ is strictly increasing, we also have $\varphi\big(\varphi(a)\big) < \varphi(a)$. Hence, by using the antisymmetry of X and Theorem 3.15, we can infer that $\varphi(a) \not\leq \varphi\big(\varphi(a)\big)$, and thus $\varphi(a) \in A$. Now, by using that $a \in \mathrm{lb}(A)$, we can see that $a \leq \varphi(a)$. This contradiction proves the theorem.

Remark 4.13. To feel the importance of extensive operations, it is also worth noticing that if φ is an extensive operation on an antisymmetric goset, then each maximal element x of X is already a fixed point of φ in the sense that $\varphi(x) = x$.

This fact has also been strongly emphasized by Brøndsted [**?**]. Moreover, fixed point theorems for extensive maps (which are sometimes called expansive, progressive, increasing, or inflationary) were also proved in [**?**], [**?**, p. 188], and [**?**].

The following theorem shows that, in contrast to the injective, increasing functions the inverse of an injective, extensive operation need not be extensive.

Theorem 4.14. *If φ is an injective and extensive operation on antisymmetric goset X such that $X = \varphi[X]$ and φ^{-1} is also extensive, then $\varphi = \Delta_X$.*

Proof. By the extensivity of φ and φ^{-1}, for every $x \in X$, we have

$$x \leq \varphi(x) \qquad \text{and} \qquad \varphi(x) \leq \varphi^{-1}\big(\varphi(x)\big).$$

Hence, by noticing that $\varphi^{-1}\big(\varphi(x)\big) = x$ and using the antisymmetry of X, we can already infer that $\varphi(x) = x$, and thus $\varphi(x) = \Delta_X(x)$. Therefore, the required equality is also true.

From this theorem, by using Theorems 4.7 and 4.12, we can immediately derive

Corollary 4.15. *If φ is a strictly increasing operation on a min-complete, antisymmetric goset X such that $X = \varphi[X]$, then $\varphi = \Delta_X$.*

Proof. Now, by Corollary 3.14 and Theorem 4.7, φ^{-1} is also strictly increasing. Thus, by Theorem 4.12, both φ and φ^{-1} are extensive. Therefore, by Theorem 4.14, the required equality is also true.

In general, the idempotent operations are quite different from the both upper and lower semiidempotent ones. However, we may still naturally have the following

Definition 4.16. An increasing, extensive (intensive) operation is called a *preclosure (preinterior) operation.* And, a lower semiidempotent (upper semiidempotent) preclosure (preinterior) operation is called a *closure (interior) operation.*

Moreover, an extensive (intensive) lower semiidempotent (upper semidempotent) operation is called a *semiclosure (semiinterior) operation.* While, an increasing and upper (lower) semiidempotent operation is called an *upper (lower) semimodification operation.*

Remark 4.17. Thus, φ is, for instance, an interior operation on a goset X if and only if it is a closure operation on its dual X'.

5. Some further important properties of increasing functions

Concerning increasing functions, we can also easily prove the following

Theorem 5.1. *For a function f of one goset X to another Y, the following assertions are equivalent:*

(1) f *is increasing,*

(2) $f\,[\,\mathrm{ub}(x)\,] \subseteq \mathrm{ub}\big(f(x)\big)$ *for all* $x \in X$,

(3) $f\,[\,\mathrm{ub}(A)\,] \subseteq \mathrm{ub}\big(f\,[\,A\,]\big)$ *for all* $A \subseteq X$.

Proof. If $A \subseteq X$ and $y \in f\,[\,\mathrm{ub}(A)\,]$, then there exists $x \in \mathrm{ub}(A)$ such that $y = f(x)$. Thus, for any $a \in A$, we have $a \leq x$. Hence, if (1) holds, we can infer that $f(a) \leq f(x)$, and thus $f(a) \leq y$. Therefore, $y \in \mathrm{ub}\big(f\,[\,A\,]\big)$, and thus (3) also holds.

The remaining implications $(3) \Longrightarrow (2) \Longrightarrow (1)$ are even more obvious.

Remark 5.2. Note that f is an increasing function of X to Y if and only if it is an increasing function of X' to Y'.

Therefore, in the above theorem we may write lb in place of ub. However, because of Theorem 3.3 and Corollary 2.2, we cannot write $\sup$ instead of ub.

Despite this, by using Theorem 5.1, we can easily prove the following

Theorem 5.3. *For a function f of a reflexive goset X to an arbitrary one Y, the following assertions are equivalent:*

(1) f *is increasing,*

(2) $f\,[\,\max(A)\,] \subseteq \mathrm{ub}\big(f\,[\,A\,]\big)$ *for all* $A \subseteq X$,

(3) $f\,[\,\max(A)\,] \subseteq \max\big(f\,[\,A\,]\big)$ *for all* $A \subseteq X$,

(4) $f\,[\,\max(A)\,] \subseteq \mathrm{ub}\big(f\,[\,A\,]\big)$ *for all* $A \subseteq X$ *with* $\mathrm{card}(A) \leq 2$.

Proof. If (1) holds, then by Theorem 5.1, for any $A \subseteq X$, we have

$$\begin{aligned} f\,[\,\max(A)\,] = f\,[\,A \cap \mathrm{ub}(A)\,] &\subseteq f\,[\,A\,] \cap f\,[\,\mathrm{ub}(A)\,] \\ &\subseteq f\,[\,A\,] \cap \mathrm{ub}\big(f\,[\,A\,]\big) = \max\big(f\,[\,A\,]\big). \end{aligned}$$

Therefore, (3) also holds even if X is not assumed to be reflexive.

Thus, since the implication $(3) \Longrightarrow (2) \Longrightarrow (4)$ trivially hold, we need only show that (4) also implies (1). For this, note that if $u, v \in X$ such that $u \leq v$, then by taking $A = \{u, v\}$ and using the reflexivity of X we can see that $v \in \operatorname{ub}(A)$, and thus

$$v \in A \cap \operatorname{ub}(A) = \max(A).$$

Hence, if (4) holds, we can infer that

$$f(v) \in f[\max(A)] \subseteq \operatorname{ub}\big(f[A]\big) = \operatorname{ub}\big(\{f(u), f(v)\}\big).$$

Thus, in particular $f(u) \leq f(v)$, and thus (1) also holds.

Now, as a useful consequence of this theorem, we can also easily prove

Corollary 5.4. *If f is a function on a reflexive goset X to an arbitrary one Y such that*

$$f[\sup(A)] \subseteq \sup\big(f[A]\big)$$

for all $A \subseteq X$ with $\operatorname{card}(A) \leq 2$, then f is already increasing.

Proof. If A is as above, then by Theorems 3.4 and 3.3 we have

$$f[\max(A)] \subseteq f[\sup(A)] \subseteq \sup\big(f[A]\big) \subseteq \operatorname{ub}\big(f[A]\big).$$

Therefore, by Theorem 5.3, f is increasing.

By Theorem 3.3 and Corollary 2.2, a converse of this corollary is certainly not true. However, by using Theorem 5.1, we can also easily prove the following two theorems.

Theorem 5.5. *If f is an increasing function of one goset X to another Y, then for any $A \subseteq X$ we have*

$$\operatorname{lb}\big(\operatorname{ub}\big(f[A]\big)\big) \subseteq \operatorname{lb}(f[\operatorname{ub}(A)]).$$

Proof. Now, by Theorem 5.1, we have $f[\operatorname{ub}(A)] \subseteq \operatorname{ub}\big(f[A]\big)$. Hence, by using Corollary 2.2, we can immediately get the required inclusion.

Theorem 5.6. *If f is an increasing function of one sup-complete, antisymmetric goset X to another Y, then for any $A \subseteq X$ we have*

$$\sup\big(f[A]\big) \leq f\big(\sup(A)\big).$$

Proof. If $\alpha = \sup(A)$, then by Theorems 3.5 and 3.3, and the usual identification of singletons with their elements, we also have $\alpha \in \operatorname{ub}(A)$, and thus $f(\alpha) \in f[\operatorname{ub}(A)]$. Hence, by using Theorem 5.5, we can already infer that $f(\alpha) \in \operatorname{ub}\big(f[A]\big)$.

While, if $\beta = \sup\big(f[A]\big)$, then by Theorems 3.5 and 3.3, and the usual identification of singletons with their elements, we also have $\beta \in \operatorname{lb}\big(\operatorname{ub}\big(f[A]\big)\big)$. Hence, by using that $f(\alpha) \in \operatorname{ub}\big(f[A]\big)$, we can already infer that $\beta \leq f(\alpha)$, and thus the required equality is also true.

By using a dual of Theorem 5.1 mentioned in Remark 5.2, we can quite similarly prove the following theorem which can also be easily derived from Theorem 5.6 by dualization.

Theorem 5.7. *If f is an increasing function of one inf-complete, antisymmetric goset X to another Y, then for any $A \subseteq X$ we have*

$$f\big(\inf(A)\big) \leq \inf\big(f[A]\big).$$

Remark 5.8. Note that, by Theorem 3.9, in the latter theorem we may also write sup–complete instead of inf-complete.

Therefore, as an immediate consequence of Theorems 5.6 and 5.7, we can state

Corollary 5.9. *If f is an increasing function of a sup-complete, antisymmetric goset X to a sup-complete, transitive and antisymmetric goset Y, and A is a nonvoid subset of X such that $f\big(\inf(A)\big)=f\big(\sup(A)\big)$, then*

$$f\big(\inf(A)\big)=\inf\big(f\,[A]\big)=\sup\big(f\,[A]\big)=f\big(\sup(A)\big)\,.$$

6. Infimum and supremum properties of closure operations

Theorem 6.1. *If φ is a closure operation on an inf-complete, antisymmetric goset X, then for any $A\subseteq X$ we have*

$$\inf\big(\varphi\,[A]\big)=\varphi\,\big(\inf\big(\varphi\,[A]\big)\big)\,.$$

Proof. Now, by Theorem 5.7, we have $\varphi\big(\inf(A)\big)\le\inf\big(\varphi\,[A]\big)$. Hence, by writing $\varphi\,[A]$ in place of A, we can see that

$$\varphi\,\big(\inf\big(\varphi\,[A]\big)\big)\le\inf\,(\varphi\,[\varphi\,[A]])\,.$$

Moreover, by using the antisymmetry of X, we can see that φ is now idempotent. Therefore, $\varphi\,[\varphi\,[A]]=(\varphi\circ\varphi)\,[A]=\varphi^2\,[A]=\varphi\,[A]$. Thus, we actually have

$$\varphi\,\big(\inf\big(\varphi\,[A]\big)\big)\le\inf\,(\varphi\,[A])\,.$$

Moreover, by extensivity of φ, the converse inequality is also true. Hence, by using the antisymmetry of X, we can see that the required equality is also true.

Remark 6.2. Note that an operation φ on a set X is idempotent if and only if $\varphi\,[X]$ is the family of all fixed points of φ.

Namely, $\varphi^2=\varphi$ if and only if $\varphi^2(x)=\varphi(x)$, i.e., $\varphi\big(\varphi(x)\big)=\varphi(x)$ for all $x\in X$. That is, $\varphi(x)\in\mathrm{Fix}(\varphi)$ for all $x\in X$, or equivalently $\varphi\,[X]\subset\mathrm{Fix}(\varphi)$.

Therefore, by using Theorem 6.1 and Remark 6.2, we can also prove

Corollary 6.3. *Under the conditions of Theorem 6.1, for any $A\subseteq\varphi\,[X]$, we have*

$$\inf\big(A\big)=\varphi\big(\inf(A)\big)\,.$$

Proof. Now, by the antisymmetry of X, φ is idempotent. Thus, by Remark 6.2, we have $\varphi(y)=y$ for all $y\in\varphi\,[X]$. Hence, by the assumtion $A\subseteq\varphi\,[X]$, we can see that $\varphi\,[A]=A$. Thus, Theorem 6.1 gives the required equality.

Remark 6.4. Note that if in particular φ is an extensive, idempotent operation on a reflexive goset X, then $\varphi\,[X]$ is also the family of all elements x of X which are φ–closed in the sense that $\varphi(x)\le x$.

Therefore, if in addition to the conditions of Theorem 6.1, X is reflexive, then the assertion of Corollary 6.3 can also be expressed by stating that the infimum of any family of φ–closed elements of X is also φ–closed.

Now, instead of an analogue of Theorem 6.1 for suprema, we can only prove

Theorem 6.5. *If φ is a closure operation on a sup-complete, transitive, and antisymmetric goset X, then for any $A \subseteq X$ we have*

$$\varphi\big(\sup(A)\big) = \varphi\left(\sup\big(\varphi[A]\big)\right).$$

Proof. Define $\alpha = \sup(A)$ and $\beta = \sup\big(\varphi[A]\big)$. Then, by Theorem 5.6, we have $\beta \leq \varphi(\alpha)$. Hence, since φ is increasing, we can infer that $\varphi(\beta) \leq \varphi\big(\varphi(\alpha)\big)$. Moreover, since φ is now idempotent, we also have $\varphi\big(\varphi(\alpha)\big) = \varphi(\alpha)$. Therefore, $\varphi(\beta) \leq \varphi(\alpha)$.

On the other hand, since φ is extensive, for any $x \in A$, we have $x \leq \varphi(x)$. Moreover, since $\beta \in \mathrm{ub}\big(\varphi[A]\big)$, we also have $\varphi(x) \leq \beta$. Hance, by using the transitivity of X, we can infer that $x \leq \beta$ for all $x \in A$, and thus $\beta \in \mathrm{ub}(A)$. Now, by using that $\alpha \in \mathrm{lb}\big(\mathrm{ub}(A)\big)$, we can see that $\alpha \leq \beta$. Hence, by the increasingness of φ, it is clear that $\varphi(\alpha) \leq \varphi(\beta)$ is also true. Therefore, by the antisymmetry of X, we actually have $\varphi(\alpha) = \varphi(\beta)$, and thus the required equality is also true.

From this theorem, we only get the following partial analogue of Theorem 6.1.

Corollary 6.6. *Under the conditions of Theorem 6.5, for any $A \subseteq X$, we have*

$$\sup\big(\varphi[A]\big) = \varphi\left(\sup\big(\varphi[A]\big)\right) \quad \textit{if and only if} \quad \varphi\big(\sup(A)\big) = \sup\big(\varphi[A]\big).$$

Now, in addition to Theorem 3.18, we can also easily prove the following

Theorem 6.7. *If φ is a closure operation on an inf-complete, antisymmetric goset X and $Y = \varphi[X]$, then for any $A \subseteq Y$ we have*

$$\inf_Y(A) = \inf_X(A).$$

Proof. If $\alpha = \inf_X(A)$, then by Theorem 3.5 and Corollary 6.3, we have $\alpha = \varphi(\alpha)$, and hence $\alpha \in Y$. Therefore, $\alpha = \inf_X(A) \cap Y$ also hold.

On the other hand, by Theorem 3.18, we always have $\inf_X(A) \cap Y \subseteq \inf_Y(A)$. Therefore, $\alpha \in \inf_Y(A)$ also holds. Hence, by using Theorem 3.5, we can already see that $\alpha = \inf_Y(A)$ is also true.

From this theorem, it is clear that in particular we also have

Corollary 6.8. *Under the conditions of Theorem 6.7, Y is also inf–complete.*

Remark 6.9. Hence, by Theorem 3.9, we can see that Y is also sup–complete.

7. A further supremum property of closure operations

Instead of establishing an analogue of Theorem 6.6 for $\sup_Y(A)$, it is convenient to prove first some more general theorems.

Theorem 7.1. *If φ is an idempotent operation on a goset X and $Y = \varphi[X]$, then for any $A \subseteq Y$ we have*

$$\mathrm{ub}_Y(A) \subseteq \varphi[\mathrm{ub}_X(A)].$$

Proof. If $\beta \in \mathrm{ub}_Y(A)$, then by Theorem 3.16 we have $\beta \in Y$ and $\beta \in \mathrm{ub}_X(A)$. Hence, by Remark 6.2, we can see that $\beta = \varphi(\beta)$, and thus $\beta \in \varphi[\mathrm{ub}_X(A)]$. Therefore, the required inclusion is also true.

Remark 7.2. By dualization, it is clear that in the above theorem we may also write lb in place of ub.

However, it is now more important to note that we also have the following

Theorem 7.3. *If φ is an extensive operation on a transitive goset X and $Y = \varphi[X]$, then for any $A \subseteq Y$ we have*

$$\varphi[\operatorname{ub}_X(A)] \subseteq \operatorname{ub}_Y(A).$$

Proof. If $\beta \in \operatorname{ub}_X(A)$, then because of $\beta \leq \varphi(\beta)$ and the transitivity of X, we also have $\varphi(\beta) \in \operatorname{ub}_X(A)$. Hence, since $\varphi(\beta) \in Y$, we can already see that $\varphi(\beta) \in \operatorname{ub}_X(A) \cap Y = \operatorname{ub}_Y(A)$, and thus the required inclusion is also true.

Now, as an immediate consequence of the above two theorems, we can also state

Corollary 7.4. *If φ is a semiclosure operation on a transitive, antisymmetric goset X and $Y = \varphi[X]$, then for any $A \subset Y$ we have*

$$\operatorname{ub}_Y(A) = \varphi[\operatorname{ub}_X(A)].$$

However, it is now more important to note that, in addition to Theorem 7.3, we also have the following

Theorem 7.5. *If φ is an increasing, lower semiidempotent operation on a transitive goset X and $Y = \varphi[X]$, then for any $A \subseteq Y$ we have*

$$\varphi\big[\operatorname{lb}_X\big(\operatorname{ub}_X(A)\big)\big] \subseteq \operatorname{lb}_Y\big(\operatorname{ub}_Y(A)\big).$$

Proof. Suppose that $\beta \in \operatorname{lb}_X\big(\operatorname{ub}_X(A)\big)$. If $v \in \operatorname{ub}_Y(A)$, then by Theorem 3.18 $v \in Y$ and $v \in \operatorname{ub}_X(A)$. Hence, by using the assumed property of β, we can infer that $\beta \leq v$. Now, since φ is increasing, we can also state that $\varphi(\beta) \leq \varphi(v)$.

Moreover, since $v \in Y$, we can see that there exists $u \in X$ such that $v = \varphi(u)$. Hence, by using that φ is lower semiidempotent, we can infer that

$$\varphi(v) = \varphi\big(\varphi(u)\big) = \varphi^2(u) \leq \varphi(u) = v.$$

Now, by using the transitivity of X, we can also see that $\varphi(\beta) \leq v$. Therefore, $\varphi(\beta) \in \operatorname{lb}_X\big(\operatorname{ub}_Y(A)\big)$. Hence, by noticing that $\varphi(\beta) \in Y$, we can already infer that $\varphi(\beta) \in \operatorname{lb}_Y\big(\operatorname{ub}_Y(A)\big)$. Therefore, the required inclusion is also true.

Now, as an immediate consequence of Theorems 7.3 and 7.5, we can also state

Theorem 7.6. *If φ is a closure operation on a transitive goset X and $Y = \varphi[A]$, then for any $A \subseteq Y$ we have*

$$\varphi[\sup_X(A)] \subseteq \sup_Y(A).$$

Proof. By Theorems 2.9, 7.3 and 7.5, we have

$$\begin{aligned}&\varphi[\sup_X(A)] = \varphi\big[\operatorname{ub}_X(A) \cap \operatorname{lb}_X\big(\operatorname{ub}_X(A)\big)\big]\\ &\subseteq \varphi[\operatorname{ub}_X(A)] \cap \varphi\big[\operatorname{lb}_X\big(\operatorname{ub}_X(A)\big)\big] \subseteq \operatorname{ub}_Y(A) \cap \operatorname{lb}_Y\big(\operatorname{ub}_Y(A)\big) = \sup_Y(A).\end{aligned}$$

From this theorem, it is clear that in particular we also have

Corollary 7.7. *Under the conditions of Theorem 7.6, the sup–completeness of X implies that of Y.*

From Theorem 7.6, by using Theorem 3.5 we can also immediately derive the following counterpart of Theorem 6.7.

Theorem 7.8. *If φ is a closure operation on a sup-complete, transitive, and antisymmetric goset X and $Y = \varphi[A]$ then for any $A \subseteq Y$ we have*

$$\sup_Y(A) = \varphi\big(\sup_X(A)\big).$$

8. A reformulation of increasingness to simple relator spaces

A family $\mathcal{R}$ of relations on one set X to another Y is called a *relator* on X to Y. And, the ordered pair $(X, Y)(\mathcal{R}) = \big((X, Y), \mathcal{R}\big)$ is called a *relator space.* (For the origins, see [**?**] and the references therein.)

If in particular $\mathcal{R}$ is a relator on X to itself, then we may simply say that $\mathcal{R}$ is a relator on X. And, by identifying singletons with their elements, we may naturally write $X(\mathcal{R})$ in place of $(X, X)(\mathcal{R})$, since $(X, X) = \{\{X\}\}$.

Relator spaces of this simpler type are already substantial generalizations of the various ordered sets [**?**] and uniform spaces [**?**]. However, they are insufficient for several important purposes. (See, for instance, [**?**] and [**?**].)

A relator $\mathcal{R}$ on X to Y, or a relator space $(X, Y)(\mathcal{R})$ is called *simple* if there exists a relation R on X to Y such that $\mathcal{R} = \{\mathcal{R}\}$. In this case, by identifying singletons with their elements, we may write $(X, Y)(R)$ in place of $(X, Y)\big(\{R\}\big)$.

According to our former definition, a simple relator space $X(R)$ may be called a goset. Moreover, by Ganter and Wille [**?**, p. 17], a simple relator space $\big(X, Y\big)(R)$ may be called a *formal context* or *context space.*

A relator $\mathcal{R}$ on X, or a relator space $X(\mathcal{R})$, may, for instance, be naturally called *reflexive* if each member of $\mathcal{R}$ is reflexive. Thus, we may also naturally speak of *preorder, tolerance*, and *equivalence relators.*

For instance, for any family $\mathcal{A}$ of subsets of X, the family $\mathcal{R}_{\mathcal{A}} = \{R_A : \ A \in \mathcal{A}\}$ is a preorder relator on X. While, for any family $\mathcal{D}$ of pseudo-metrics on X, the family $\mathcal{R}_{\mathcal{D}} = \{B_r^d : \ r > 0, \ d \in \mathcal{D}\}$ is a tolerance relator on X.

Now, according to Definition 4.1, a function f of one simple relator space $X(R)$ to another $Y(S)$ may be naturally called increasing if for any $u, v \in X$

$$u\,R\,v \quad \Longrightarrow \quad f(u)\,S\,f(v).$$

Hence, by noticing that

$$u\,R\,v \iff v \in R(u) \iff (u, v) \in R,$$

and

$$f(u)\,S\,f(v) \iff f(v) \in S\big(f(u)\big) \iff \big(f(u), f(v)\big) \in S,$$

that is

$$f(u) S\,f(v) \iff f(v) \in (S \circ f)(u) \iff (f \boxtimes f)(u, v) \in S,$$

we can easily establish the following

Theorem 8.1. *For a function f of one simple relator space $X(R)$ to another $Y(S)$, the following assertions are equivalent:*

(1) f *is increasing,* *(2)* $f \circ R \subseteq S \circ f$,

(3) $(f \boxtimes f)[R] \subseteq S$, *(4)* $f \circ R \circ f^{-1} \subseteq S$,

(5) $R \subseteq (f \boxtimes f)^{-1}[S]$, *(6)* $R \subseteq f^{-1} \circ S \circ f$.

Proof. To prove the equivalence of (1) and (2) note that, by the above argument and the corresponding definitions

$$
\begin{aligned}
(1) \iff & f(v) \in (S \circ f)(u) \quad \text{for all} \quad u \in X \quad \text{and} \quad v \in R(u) \\
\iff & f[R(u)] \subset (S \circ f)(u) \quad \text{for all} \quad u \in X \\
\iff & \big(f \circ R\big)(u) \subset (S \circ f)(u) \quad \text{for all} \quad u \in X \iff (2).
\end{aligned}
$$

From this theorem, by using the closure operation $*$, defined by

$$\mathcal{R}^* = \big\{ S \subset X \times Y : \quad \exists \ R \in \mathcal{R} : \quad R \subset S \big\}$$

for any relator $\mathcal{R}$ on X to Y, we can immediately derive the following

Corollary 8.2. *For a function f of one simple relator space $X(R)$ to another $Y(S)$, the following assertions are equivalent:*

(1) f *is increasing,* *(2)* $S \circ f \in \big\{ f \circ R \big\}^*$,

(3) $S \in \big\{ (f \boxtimes f)[R] \big\}^*$, *(4)* $S \in \big\{ f \circ R \circ f^{-1} \big\}^*$,

(5) $(f \boxtimes f)^{-1}[S] \in \{R\}^*$, *(6)* $f^{-1} \circ S \circ f \in \{R\}^*$.

Remark 8.3. Note that, by using the notations $\mathcal{F} = \{f\}$, $\mathcal{R} = \{R\}$ and $\mathcal{S} = \{S\}$, instead of (2) we may also write the more instructive inclusions

$$\mathcal{S} \circ \mathcal{F} \subseteq \big(\mathcal{F} \circ \mathcal{R}\big)^*, \qquad (\mathcal{S}^* \circ \mathcal{F})^* \subseteq (\mathcal{F} \circ \mathcal{R}^*)^*, \qquad (\mathcal{S}^* \circ \mathcal{F}^*)^* \subseteq (\mathcal{F}^* \circ \mathcal{R}^*)^*.$$

The second one, whenever we think arbitrary relators in place of $\mathcal{R}$ and $\mathcal{S}$, already shows the $*$–invariance of the increasingness of $\mathcal{F}$ with respect to those relators.

9. Generalizations of increasingness to arbitrary relator spaces

From Corollary 8.2, by using the following obvious extensions of the operations -1 and $\circ$ from relations to relators, defined by

$$\mathcal{R}^{-1} = \big\{ R^{-1} : \quad R \in \mathcal{R} \big\} \qquad \text{and} \qquad \mathcal{S} \circ \mathcal{R} = \big\{ S \circ R : \quad R \in \mathcal{R}, \quad S \in \mathcal{S} \big\}$$

for any relator $\mathcal{R}$ on X to Y and $\mathcal{S}$ on Y to Z, we can also easily arrive at the following generalization of [**?**, Definition 4.1], which is also closely related to [**?**, Definition 15].

Definition 9.1. Let $(X, Y)(\mathcal{R})$ and $(Z, W)(\mathcal{S})$ be relator spaces, and suppose that $\square$ is a direct unary operation for relators. Then, for any two relators $\mathcal{F}$ on X to Z and $\mathcal{G}$ on Y to W, we say that

(1) $\big(\mathcal{F}, \mathcal{G}\big)$ is mildly $\square$–increasing if $\Big(\big(\mathcal{G}^{\square}\big)^{-1} \circ \mathcal{S}^{\square} \circ \mathcal{F}^{\square}\Big)^{\square} \subseteq \mathcal{R}^{\square\square}$,

(2) $\big(\mathcal{F}, \mathcal{G}\big)$ is upper $\square$–semiincreasing if $\big(\mathcal{S}^{\square} \circ \mathcal{F}^{\square}\big)^{\square} \subseteq \big(\mathcal{G}^{\square} \circ \mathcal{R}^{\square}\big)^{\square}$,

(3) $\big(\mathcal{F}, \mathcal{G}\big)$ is lower $\square$–semiincreasing if $\Big(\big(\mathcal{G}^{\square}\big)^{-1} \circ \mathcal{S}^{\square}\Big)^{\square} \subseteq \Big(\mathcal{R}^{\square} \circ \big(\mathcal{F}^{\square}\big)^{-1}\Big)^{\square}$.

Remark 9.2. A function $\Box$ of the class of all relator spaces to itself is called a *direct (indirect) unary operation for relators* if, for any relator $\mathcal{R}$ on X to Y, the value $\Box\big((X, Y)(\mathcal{R})\big)$ is a relator on X to Y (on Y to X).

In this case, trusting to the reader's good sense to avoid confusions, we shall simply write $\mathcal{R}^{\Box}$ instead of $\mathcal{R}^{\Box_{XY}} = \Box\big((X, Y)(\mathcal{R})\big)$. Note that thus $*$ is a direct, while -1 is an indirect unary operation for relators.

In addition to the operation $*$, the functions $\#$, $\wedge$, and $\vartriangle$, defined by

$$\mathcal{R}^{\#} = \big\{ S \subset X \times Y : \quad \forall\ A \subset X : \quad \exists\ R \in \mathcal{R} : \quad R[A] \subset S[A] \big\},$$

$$\mathcal{R}^{\wedge} = \big\{ S \subset X \times Y : \quad \forall\ x \in X : \quad \exists\ R \in \mathcal{R} : \quad R(x) \subset S(x) \big\},$$

and

$$\mathcal{R}^{\vartriangle} = \big\{ S \subset X \times Y : \quad \forall\ x \in X : \quad \exists\ \ u \in X : \quad \exists\ R \in \mathcal{R} : \quad R(u) \subset S(x) \big\}$$

for any relator $\mathcal{R}$ on X to Y, are also important closure operations for relators.

Thus, we evidently have $\mathcal{R} \subset \mathcal{R}^{*} \subset \mathcal{R}^{\#} \subset \mathcal{R}^{\wedge} \subset \mathcal{R}^{\vartriangle}$ for any relator $\mathcal{R}$ on X to Y. Moreover, if in particular $X = Y$, then in addition to the above inclusions we can also easily prove that $\mathcal{R}^{\infty} \subset \mathcal{R}^{*\,\infty} \subset \mathcal{R}^{\infty *} \subset \mathcal{R}^{*}$ where

$$\mathcal{R}^{\infty} = \big\{ R^{\infty} : \ \ R \in \mathcal{R} \big\}.$$

In addition to ∞, it is also worth considering the operation ∂, defined by

$$\mathcal{R}^{\partial} = \big\{ S \subset X^{2} : \ \ S^{\infty} \in \mathcal{R} \big\}$$

for any relator $\mathcal{R}$ on X. Namely, the operations ∞ and ∂ also form a Galois connection. And thus, $\infty\,\partial$ is also a closure operation and $\infty = \infty\,\partial\,\infty$.

Moreover, for any relator $\mathcal{R}$ on X to Y, we may also naturally define

$$\mathcal{R}^{c} = \big\{ R^{c} : \quad R \in \mathcal{R} \big\},$$

where $R^{c} = X \times Y \setminus R$. Thus, for instance, we may also naturally consider the operation $\circledast = c * c$ which seems to play as important role in algebra as the operation $*$ does in analysis.

Unfortunately, the operations $\wedge$ and $\vee$ are not inversion compatible, therefore, in addition to the above compound operations we have to consider also the operations $\vee = \wedge\, -1$ and $\triangledown = \vartriangle -1$, which already have very curious properties.

For instance, the operations $\vee\vee$ and $\triangledown\triangledown$ coincide with the extremal closure operations $\bullet$ and $\blacklozenge$, defined by

$$\mathcal{R}^{\bullet} = \big\{ \delta_{\mathcal{R}} \big\}^{*}, \qquad \text{where} \qquad \delta_{\mathcal{R}} = \bigcap \mathcal{R},$$

and

$$\mathcal{R}^{\blacklozenge} = \mathcal{R} \quad \text{if} \quad \mathcal{R} = \big\{ X \times Y \big\} \qquad \text{and} \qquad \mathcal{R}^{\blacklozenge} = \mathcal{P}(X \times Y) \quad \text{if} \quad \mathcal{R} \neq \big\{ X \times Y \big\}.$$

Because of the above important operations for relators, Definition 9.1 offers an abundance of natural increasingness properties for relations. Moreover, if $\mathcal{R}$ is a relator on X to Y, then by taking

$$\mathrm{Lb}_{\mathcal{R}}(B) = \big\{ A \subset X : \quad \exists\ R \in \mathcal{R} : \ \ A \times B \subset R \big\} \quad \text{and} \quad \mathrm{lb}_{\mathcal{R}}(B) = X \cap \mathrm{lb}_{\mathcal{R}}(B)$$

for all $B \subseteq Y$, from the dual of Theorem 5.1 one can also immediately derive some reasonable definitions for the increasingness of relations.

Analogously, to the relations $\mathrm{Lb}_{\mathcal{R}}$ and lb_{R}, we may also naturally define

$$\mathrm{Int}_{\mathcal{R}}(B) = \big\{ A \subset X : \quad \exists\ R \in \mathcal{R} : \ \ R[A] \subset B \big\} \quad \text{and} \quad \mathrm{int}_{\mathcal{R}}(B) = X \cap \mathrm{Int}_{\mathcal{R}}(B)$$

for all $B \subseteq Y$. Thus, in contrast to a common belief, the relations $\mathrm{lb}_{\mathcal{R}}$ and int_R are not independent of each other. Namely, it can be easily shown that

$$\mathrm{Lb}_{\mathcal{R}} = \mathrm{Int}_{\mathcal{R}^c} \circ \mathcal{C}\,, \qquad \text{and thus also} \qquad \mathrm{lb}_{\mathcal{R}} = \mathrm{int}_{\mathcal{R}^c} \circ \mathcal{C}\,,$$

where $\mathcal{C}(B) = Y \setminus B$ for all $B \subseteq Y$. The above formulas closely resemble to the famous Euler formulas on exponential and trigonometric functions [?, p. 227].

To see the importance of the operations $\#$ and $\circledast = c \# c$, by using Pataki connections on power sets, it can be shown that, for any relator $\mathcal{R}$ on X to Y, $\mathcal{S} = \mathcal{R}^{\#}$ $\big(\mathcal{S} = \mathcal{R}^{\circledast}\big)$ is the largest relator on X to Y such that $\mathrm{Int}_{\mathcal{S}} = \mathrm{Int}_{\mathcal{R}}$ $\big(\mathrm{Lb}_{\mathcal{S}} = \mathrm{Lb}_{\mathcal{R}}\big)$.

Concerning the operations $\wedge$ and $\circledwedge = c \wedge c$, we can quite similarly see that $\mathcal{S} = \mathcal{R}^{\wedge}$ $\big(\mathcal{S} = \mathcal{R}^{\circledwedge}\big)$ is the largest relator on X to Y such that $\mathrm{int}_{\mathcal{S}} = \mathrm{int}_{\mathcal{R}}$ $\big(\mathrm{lb}_{\mathcal{S}} = \mathrm{lb}_{\mathcal{R}}\big)$.

However, if in particular $\mathcal{R}$ is a relator on X, then for the families

$$\mathcal{T}_{\mathcal{R}} = \big\{ A \subseteq X : \quad A \subseteq \mathrm{int}_{\mathcal{R}}(A) \big\} \qquad \text{and} \qquad \mathcal{L}_{\mathcal{R}} = \big\{ A \subseteq X : \quad A \subseteq \mathrm{lb}_{\mathcal{R}}(A) \big\}$$

there does not exist a largest relator $\mathcal{S}$ on X such that $\mathcal{T}_{\mathcal{S}} = \mathcal{T}_{\mathcal{R}}$ $\big(\mathcal{L}_{\mathcal{S}} = \mathcal{L}_{\mathcal{R}}\big)$.

Finally, we note that to obtain some similar generalizations of closure operations, one can observe that an unary operation φ on a simple relator space $X(R)$ is extensive (lower semiidempotent) if and only if $\varphi \subset R$ $\big(\varphi \,|\, \varphi[X] \subseteq R^{-1}\big)$.

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