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GENERALIZATIONS OF GALOIS AND PATAKI CONNECTIONS TO RELATOR SPACES

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Abstract. In this paper, by using the elementwise composition and inversion, and a unary operation for relators (families of binary relations), we introduce some natural generalizations of Galois and Pataki connections from ordered sets to relator spaces.

More concretely, if $\mathcal{R}$ is a relator on X to Y, $\mathcal{S}$ is a relator on Z to W, $\mathcal{F}$ is a relator on X to Z, $\mathcal{G}$ is a relator on W to Y, and $\square$ is a unary operation for relators such that

$$\left(\mathcal{S}^{\square}\circ\mathcal{F}^{\square}\right)^{\square}=\left(\left(\mathcal{G}^{\square}\right)^{-1}\circ\mathcal{R}^{\square}\right)^{\square},$$

then we say that the relator $\mathcal{F}$ is $\square$–$\mathcal{G}$–normal with respect to the relators $\mathcal{R}$ and $\mathcal{S}$, or that the relators $\mathcal{F}$ and $\mathcal{G}$ form a Galois connection between the relator spaces $(X, Y)(\mathcal{R})$ and $(Z, W)(\mathcal{S})$ with respect to the operation $\square$.

By using appropriate definitions, we show that if in particular $Z=W$, and $\square$ is increasing, inversion and composition compatible, then under the notation $\Phi=\mathcal{G}\circ\mathcal{F}$ we have

$$\left(\left(\mathcal{F}^{\square}\right)^{-1}\circ\mathcal{S}^{\square}\circ\mathcal{F}^{\square}\right)^{\square}=\left(\left(\Phi^{\square}\right)^{-1}\circ\mathcal{R}^{\square}\right)^{\square}.$$

This can be expressed by saying that the relator $\mathcal{F}$ is $\square$–Φ–regular with respect to the relators $\mathcal{R}$ and $\mathcal{S}$, or that the relators $\mathcal{F}$ and Φ form a Pataki connection between the relator spaces $(X, Y)(\mathcal{R})$ and $Z(\mathcal{S})$ with respect to the operation $\square$.

Actually, instead of the above equalities, we shall rather investigate the corresponding inclusions since they are closely connected with the upper and lower semicontinuities, and mild continuities of relators on one relator space to another.

1. A few basic fats on relations

A subset F of a product set $X{\times}Y$ is called a *relation on X to Y*. If in particular $F\subset X^2$, with $X^2=X\times X$, then we may simply say that F is a *relation on X*. In particular, $\Delta_X=\{(x, x):\ x\in X\}$ is called the *identity relation on X*.

If F is a relation on X to Y, then for any $x\in X$ and $A\subset X$ the sets $F(x)=\{y\in Y:\ (x, y)\in F\}$ and $F[A]=\bigcup_{a\in A}F(a)$ are called the *images of x and A under F*, respectively. If $(x, y)\in F$, then we may also write $x\,F\,y$.

Moreover, the sets $D_F=\{x\in X:\ F(x)\neq\emptyset\}$ and $R_F=F[X]$ are called the *domain and range of F*, respectively. If in particular $D_F=X$, then we say that F is a *relation of X to Y*, or that *F is a non-partial relation on X to Y*.

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If F is a relation on X to Y, then $F = \bigcup_{x \in X} \{x\} \times F(x)$. Therefore, the values $F(x)$, where $x \in X$, uniquely determine F. Thus, a relation F on X to Y can be naturally defined by specifying $F(x)$ for all $x \in X$.

For instance, the *complement relation* F^c *of* F can be naturally defined such that $F^c(x) = F(x)^c = Y \setminus F(x)$ for all $x \in X$. And, the *inverse relation* F^{-1} *of* F can be naturally defined such that $F^{-1}(y) = \{x \in X : y \in F(x)\}$ for all $y \in Y$.

Moreover, if in addition G is a relation on Y to Z, then the *composition relation* $G \circ F$ *of* G *and* F can be naturally defined such that $(G \circ F)(x) = G[F(x)]$ for all $x \in X$. Thus, we also have $(G \circ F)[A] = G[F[A]]$ for all $A \subset X$.

Now, a relation F on X may, for instance, be called *reflexive* if $\Delta_X \subset F$, and *transitive* if $F \circ F \subset F$. Moreover, F may be called *symmetric* if $F^{-1} \subset F$, and *antisymmetric* if $F \cap F^{-1} \subset \Delta_X$.

Thus, a reflexive and transitive (symmetric) transitive relation may be called a *preorder (tolerance) relation*. And, a symmetric (antisymmetric) preorder relation may be called an *equivalence (partial order) relation*.

For instance, for any $A \subset X$, the *Pervin relation* $R_A = A^2 \cup A^c \times X$ is a preorder relation on X. While, for any pseudo-metric d on X and $r > 0$, the *surrounding* $B_r^d = \big\{(x, y) \in X^2 : \ d(x, y) < r\big\}$ is a tolerance relation on X.

Note that if F is a symmetric relation on X, then $F^{-1} = F$, but F need not be *involutive* in the sense that $F \circ F = \Delta_X$. While, if F is a preorder relation on X, then F is already *idempotent* in the sense that $F \circ F = F$.

For any relation F on X, we define $F^0 = \Delta_X$, and $F^n = F \circ F^{n-1}$ if $n \in \mathbb{N}$. Moreover, we also define $F^\infty = \bigcup_{n=0}^{\infty} F^n$. Thus, it can be shown that F^∞ is just the smallest preorder relation on X containing F.

In particular, a relation f on X to Y is called a *function* if for each $x \in D_f$ there exists $y \in Y$ such that $f(x) = \{y\}$. In this case, by identifying singletons with their elements, we may simply write $f(x) = y$ in place of $f(x) = \{y\}$.

Moreover, a function $\star$ of X to itself is called a *unary operation on* X. While, a function $*$ of X^2 to X is called a *binary operation on* X. And, for any $x, y \in X$, we usually write $x^\star$ and $x * y$ instead of $\star(x)$ and $*((x, y))$, respectively.

If F is a relation on X to Y, then a function f of D_F to Y is called a *selection of* F if $f \subset F$, i.e., $f(x) \in F(x)$ for all $x \in D_F$. Thus, the *Axiom of Choice* can be briefly expressed by saying that every relation has a selection.

For any relation F on X to Y, we may naturally define two *set-valued functions*, $F^\diamond$ on X to $\mathcal{P}(Y)$ and $F^{\Diamond}$ on $\mathcal{P}(X)$ to $\mathcal{P}(Y)$, such that $F^\diamond(x) = F(x)$ for all $x \in X$ and $F^{\Diamond}(A) = F[A]$ for all $A \subset X$.

Functions on X to $\mathcal{P}(Y)$ can be identified with relations on X to Y. While, functions on $\mathcal{P}(X)$ to $\mathcal{P}(Y)$ are more powerful tools than relations on X to Y [**?**]. However, it seems now to be more convenient to work with relations.

2. A few basic fats on relators

A family $\mathcal{R}$ of relations on one set X to another Y is called a *relator on* X *to* Y. And, the ordered pair $(X, Y)(\mathcal{R}) = \big((X, Y), \ \mathcal{R}\big)$ is called a *relator space*. (For the origins, see [**?**] and the references therein.)

If in particular $\mathcal{R}$ is a relator on X to itself, then we may simply say that $\mathcal{R}$ is a *relator on* X. And, by identifying singletons with their elements, we may naturally write $X(\mathcal{R})$ in place of $(X, X)(\mathcal{R})$. Namely, $(X, X) = \{\{X\}\}$.

Relator spaces of this simpler type are already substantial generalizations of the various *ordered sets* and *uniform spaces* [?]. However, they are insufficient for several important purposes. (See, for instance, [?] and [?].)

A relator $\mathcal{R}$ on X to Y, or a relator space $(X, Y)(\mathcal{R})$ is called *simple* if there exists a relation R on X to Y such that $\mathcal{R} = \{\mathcal{R}\}$. In this case, by identifying singletons with their elements, we may write $(X, Y)(R)$ in place of $(X, Y)(\{R\})$.

Following Birkhoff [?, p. 1], a simple relator space $X(\leq)$ may be called a *goset* (generalized ordered set). Moreover, by Ganter and Wille [?, p. 17], a simple relator space $(X, Y)(R)$ may be called a *formal context*.

A relator $\mathcal{R}$ on X, or a relator space $X(\mathcal{R})$, may, for instance, be naturally called *reflexive* if each member of $\mathcal{R}$ is reflexive. Thus, we may also naturally speak of *preorder, tolerance*, and *equivalence relators*.

For instance, for any family $\mathcal{A}$ of subsets of X, the family $\mathcal{R}_{\mathcal{A}} = \{ R_A : \ A \in \mathcal{A}\}$ is a preorder relator on X. While, for any family $\mathcal{D}$ of pseudo-metrics on X, the family $\mathcal{R}_{\mathcal{D}} = \{ B^d_r : \ r > 0 , \ d \in \mathcal{D}\}$ is a tolerance relator on X.

A function $\square$ of the class of all relator spaces is called a *direct (indirect) unary operation for relators* if, for any relator $\mathcal{R}$ on X to Y, the value $\square\big((X, Y)(\mathcal{R})\big)$ is a relator on X to Y (on Y to X). In this case, trusting to the reader's good sense to avoid confusions, we shall simply write $\mathcal{R}^{\square}$ instead of $\mathcal{R}^{\square_{XY}} = \square\big((X, Y)(\mathcal{R})\big)$.

An unary operation $\square$ for relators is called *increasing* if for any two relators $\mathcal{R}$ and S on X to Y, with $\mathcal{R} \subset \mathcal{S}$, we have $\mathcal{R}^{\square} \subset \mathcal{S}^{\square}$. Moreover, the operation $\square$ is called *extensive, involutive*, and *idempotent* if, for any relator $\mathcal{R}$ on X to Y, we have $\mathcal{R} \subset \mathcal{R}^{\square}$, $\mathcal{R}^{\square\square} = \mathcal{R}$, and $\mathcal{R}^{\square\square} = \mathcal{R}^{\square}$, respectively,

For instance, the functions c and -1, defined by

$$\mathcal{R}^{c} = \big\{ R^{c} : \ \ R \in \mathcal{R} \big\} \qquad \text{and} \qquad \mathcal{R}^{-1} = \big\{ R^{-1} : \ \ R \in \mathcal{R} \big\}$$

for any relator $\mathcal{R}$ on X to Y, are increasing, involutive operations for relators such that $\big(\mathcal{R}^{c}\big)^{-1} = \big(\mathcal{R}^{-1}\big)^{c}$. Thus, the operation c is inversion compatible.

And, the functions ∞ and ∂, defined by

$$\mathcal{R}^{\infty} = \big\{ R^{\infty} : \ \ R \in \mathcal{R} \big\} \qquad \text{and} \qquad \mathcal{R}^{\partial} = \big\{ S \subset X^{2} : \ \ S^{\infty} \in \mathcal{R} \big\}$$

for any relator $\mathcal{R}$ on X, are increasing, idempotent operations for relators such that, for any relator $\mathcal{S}$ on X, we have $\mathcal{R}^{\infty} \subset \mathcal{S}$ if and only if $\mathcal{R} \subset \mathcal{S}^{\partial}$. Thus, the operations ∞ and ∂ set up a Galois connection [?, p. 155].

While, the functions $*$, $\#$, $\wedge$, and $\vartriangle$, defined by

$$\begin{aligned}
\mathcal{R}^{*} &= \big\{ S \subset X \times Y : \quad \exists \ R \in \mathcal{R} : \quad R \subset S \big\}, \\
\mathcal{R}^{\#} &= \big\{ S \subset X \times Y : \quad \forall \ A \subset X : \quad \exists \ R \in \mathcal{R} : \quad R[A] \subset S[A] \big\}, \\
\mathcal{R}^{\wedge} &= \big\{ S \subset X \times Y : \quad \forall \ x \in X : \quad \exists \ R \in \mathcal{R} : \quad R(x) \subset S(x) \big\},
\end{aligned}$$

and

$$\mathcal{R}^{\vartriangle} = \big\{ S \subset X \times Y : \quad \forall \ x \in X : \quad \exists \ \ u \in X : \quad \exists \ R \in \mathcal{R} : \quad R(u) \subset S(x) \big\}$$

for any relator $\mathcal{R}$ on X to Y, are increasing, idempotent operations for relators such that $\mathcal{R} \subset \mathcal{R}^* \subset \mathcal{R}^\# \subset \mathcal{R}^\wedge \subset \mathcal{R}^\triangle$. If in particular $X = Y$, then we can also state that $\mathcal{R}^\infty \subset \mathcal{R}^{*\infty} \subset \mathcal{R}^{\infty *} \subset \mathcal{R}^*$.

Beside the above unary operations, we may also naturally define several natural *binary operations for relators.* For instance, for suitable relators $\mathcal{R}$ and $\mathcal{S}$, we may naturally define

$$\mathcal{R} \curlywedge \mathcal{S} = \big\{ R \cap S : \ R \in \mathcal{R}, \ S \in \mathcal{S} \big\} \quad \text{and} \quad \mathcal{S} \circ \mathcal{R} = \big\{ S \circ R : \ R \in \mathcal{R}, \ S \in \mathcal{S} \big\}.$$

Thus, we evidently have $\big(\mathcal{S} \circ \mathcal{R} \big)^{-1} = \mathcal{R}^{-1} \circ \mathcal{S}^{-1}$ for any relators $\mathcal{R}$ on X to Y and S on Y to Z. However, concerning the operation c we can only prove that $\mathcal{S}^c \circ \mathcal{R} \subset \big(\big(\mathcal{S} \circ \mathcal{R} \big)^c \big)^*$ if $D_R = X$ for all $R \in \mathcal{R}$, and $\mathcal{S} \circ \mathcal{R}^c \subset \big(\big(\mathcal{S} \circ \mathcal{R} \big)^c \big)^*$ if $R_S = Z$ for all $S \in \mathcal{S}$.

The latter inclusions, and the formula $\big(\mathcal{R} \curlywedge \mathcal{R}^{-1} \big)^c \subset \big(\{ \Delta_Y \}^c \big)^*$ put forward by [**?**, Remark 9.2], strongly suggest that, in addition to an operation $\square$ for relators, we have also to consider a *dual operation* $\circledcirc = c \,\square\, c$.

3. Some further facts on relators

The above important unary operations for relators can be most naturally obtained from the various structures for relators [**?**, **?**] with the help of Pataki and Galois connections between power sets [**?**, **?**].

A *structure* $\mathfrak{F}$ *for relators* is a function of the class of all relator spaces such that, for any relator $\mathcal{R}$ on X to Y, the value $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_{\mathcal{R}}^{XY} = \mathfrak{F}\big((X, Y)(\mathcal{R}) \big)$ belongs to a power set depending only on X and Y.

For instance, for any relator $\mathcal{R}$ on X to Y, we may naturally define two relations $\operatorname{Int}_{\mathcal{R}}$ and $\operatorname{Lb}_{\mathcal{R}}$ on $\mathcal{P}(Y)$ to $\mathcal{P}(X)$ such that

$$\operatorname{Int}_{\mathcal{R}}(B) = \big\{ A \subset X : \quad \exists\ R \in \mathcal{R} : \quad R[A] \subset B \big\}$$

and

$$\operatorname{Lb}_{\mathcal{R}}(B) = \big\{ A \subset X : \quad \exists\ R \in \mathcal{R} : \quad A \times B \subset R \big\}$$

for all $B \subset Y$. Thus, $\operatorname{Int}_{\mathcal{R}}$ and $\operatorname{Lb}_{\mathcal{R}}$ are elements of $\mathcal{P}\big(\mathcal{P}(Y) \times \mathcal{P}(X) \big)$.

By the the corresponding definitions, it is clear that

$$\begin{aligned} A \times B \subset R \iff \forall\, a \in A : \ B \subset R(a) \iff \forall\, a \in A : \ R(a)^c \subset B^c \\ \iff \forall\, a \in A : \ R^c(a) \subset B^c \iff R^c[A] \subset B^c. \end{aligned}$$

Therefore, we have

$$A \in \operatorname{Lb}_{\mathcal{R}}(B) \iff A \in \operatorname{Int}_{\mathcal{R}^c}(B^c) \iff A \in \big(\operatorname{Int}_{\mathcal{R}^c} \circ\, \mathcal{C} \big)(B),$$

with the *complement function* $\mathcal{C}$, defined by $\mathcal{C}(B) = B^c$ for all $B \subset Y$.

Hence, we can already see that

$$\operatorname{Lb}_{\mathcal{R}} = \operatorname{Int}_{\mathcal{R}^c} \circ\, \mathcal{C}, \qquad \text{and thus also} \qquad \operatorname{Int}_{\mathcal{R}} = \operatorname{Lb}_{\mathcal{R}^c} \circ\, \mathcal{C}.$$

Therefore, in contrast to a common belief, the topological and order theoretic structures are just as closely related to each other by the above equalities as the exponential and the trigonometric functions are by the Euler formulas [**?**, p. 227].

Now, for any relator $\mathcal{R}$ on X to Y, we may also naturally define two relations $\text{int}_{\mathcal{R}}$ and $\text{lb}_{\mathcal{R}}$ on $\mathcal{P}(Y)$ to X such that, by identifying singletons with their elements, we have

$$\text{int}_{\mathcal{R}}(B) = \text{Int}_{\mathcal{R}}(B) \cap X \qquad \text{and} \qquad \text{lb}_{\mathcal{R}}(B) = \text{Lb}_{\mathcal{R}}(B) \cap X$$

for all $B \subset Y$. Thus, we evidently have $\text{lb}_{\mathcal{R}} = \text{int}_{\mathcal{R}^c} \circ \mathcal{C}$ and $\text{int}_{\mathcal{R}} = \text{lb}_{\mathcal{R}^c} \circ \mathcal{C}$. Moreover, for instance, it can be shown that $\text{Int}_{\mathcal{R}^\wedge}(B) = \mathcal{P}\big(\text{int}_{\mathcal{R}}(B)\big)$.

If in particular $\mathcal{R}$ is a relator on X, then we may also naturally define

$$\tau_{\mathcal{R}} = \big\{ A \subset X : \quad A \in \text{Int}_{\mathcal{R}}(A) \big\} \qquad \text{and} \qquad \ell_{\mathcal{R}} = \big\{ A \subset X : \quad A \in \text{Lb}_{\mathcal{R}}(A) \big\},$$

and quite similarly

$$\mathcal{T}_{\mathcal{R}} = \big\{ A \subset X : \quad A \subset \text{int}_{\mathcal{R}}(A) \big\} \qquad \text{and} \qquad \mathcal{L}_{\mathcal{R}} = \big\{ A \subset X : \quad A \subset \text{lb}_{\mathcal{R}}(A) \big\}.$$

However, these families are, in general, much weaker tools than the above relations. Moreover, they cannot already be expressed in terms of each others. However, for instance, we have $\tau_{\mathcal{R}^\wedge} = \mathcal{T}_{\mathcal{R}}$.

For any relator $\mathcal{R}$ on X to Y, we may also naturally define

$$\mathcal{E}_{\mathcal{R}} = \big\{ B \subset Y : \quad \text{int}_{\mathcal{R}}(B) \neq \emptyset \big\} \qquad \text{and} \qquad \mathfrak{E}_{\mathcal{R}} = \big\{ B \subset Y : \quad \text{lb}_{\mathcal{R}}(B) \neq \emptyset \big\}.$$

Thus, we already have $\mathfrak{E}_{\mathcal{R}} = \big(\mathcal{E}_{\mathcal{R}^c}\big)^c$ and $\mathcal{E}_{\mathcal{R}} = \big(\mathfrak{E}_{\mathcal{R}^c}\big)^c$.

The family $\mathcal{E}_{\mathcal{R}}$ of all *fat sets* is frequently a more important tool than the families $\tau_{\mathcal{R}}$ and $\mathcal{T}_{\mathcal{R}}$ of all *proximally and topologically open sets*. Moreover, for instance, we have $\tau_{\mathcal{R}^\triangle} = \mathcal{E}_{\mathcal{R}} \cup \{\emptyset\}$ for any relator $\mathcal{R}$ on X.

By using Pataki connections, it can be shown that, for any relator $\mathcal{R}$ on X to Y, $\mathcal{S} = \mathcal{R}^{\#}$ is the largest relator on X to Y such that $\text{Int}_{\mathcal{S}} = \text{Int}_{\mathcal{R}}$. While, for any relator $\mathcal{R}$ on X, $\mathcal{S} = \mathcal{R}^{\#\partial}$ is the largest relator on X such that $\tau_{\mathcal{S}} = \tau_{\mathcal{R}}$.

Unfortunately, for the family $\mathcal{T}_{\mathcal{R}}$ of all topologically open subsets of the relator space $X(\mathcal{R})$, there is no largest relator $\mathcal{S}$ on X such that $\mathcal{T}_{\mathcal{S}} = \mathcal{T}_{\mathcal{R}}$. However, $\mathcal{S} = \mathcal{R}^{\wedge\infty}$ is the largest preorder relator on X such that $\mathcal{T}_{\mathcal{S}} = \mathcal{T}_{\mathcal{R}}$. (See [?].)

Instead of the *proximal interior relation* $\text{Int}_{\mathcal{R}}$ suggested by Smirnov [?] one can much better work with the *coherence ralation* $\text{Lim}_{\mathcal{R}}$ suggested by Efremović and Švarc [?]. This is already an almost as strong tool as the relator $\mathcal{R}$ itself.

If $\mathcal{R}$ is a relator on X to Y, then for any preordered set $\Gamma(\leq)$, and nets $x \in X^{\Gamma}$ and $y \in Y^{\Gamma}$ we write $x \in \text{Lim}_{\mathcal{R}}(y)$ if the net (x, y) is eventually in each $R \in \mathcal{R}$ in the sense that $(x, y)^{-1}[R] \in \mathcal{E}_{\leq}$.

Thus, in contrast to $\text{Int}_{\mathcal{R}} = \bigcup_{R \in \mathcal{R}} \text{Int}_{\mathcal{R}}$, we now have $\text{Lim}_{\mathcal{R}} = \bigcap_{R \in \mathcal{R}} \text{Lim}_{\mathcal{R}}$. However, by using Pataki connections, we can also show that, for any relator $\mathcal{R}$ on X to Y, $\mathcal{S} = \mathcal{R}^*$ is the largest relator on X to Y such that $\text{Lim}_{\mathcal{S}} = \text{Lim}_{\mathcal{R}}$.

Now, following the ideas of Császár [?], one may also naturally consider the hyperrelators $\mathfrak{H}_{\mathcal{R}} = \big\{\text{Int}_R : \quad R \in \mathcal{R}\big\}$ and $\mathfrak{K}_{\mathcal{R}} = \big\{\text{Lim}_R : \quad R \in \mathcal{R}\big\}$, which are much stronger tools than the relations $\text{Int}_{\mathcal{R}}$ and $\text{Lim}_{\mathcal{R}}$ themselves.

In the light of the several disadvantages of the family $\mathcal{T}_{\mathcal{R}}$, it is rather curious that most of the works in topology and analysis have been based on open sets suggested by Tietze [?] and standardized by Bourbaki [?] and Kelley [?].

Moreover, it also a striking fact that, despite the results of Pervin [?], Fletcher and Lindgren [?], and the present author [?], generalized topologies and minimal structures are still intensively investigated by a great number of mathematicians.

4. Closure operations and regular structures for relators

Definition 4.1. An extensive, increasing operation is called a *preclosure operation.* And, an idempotent preclosure operation is called a *closure operation.*

Moreover, an extensive, idempotent operation is called a *semiclosure operation.* And, an increasing, idempotent operation is called a *modification operation.*

Remark 4.2. Note that if $\Box$ is an extensive operation for relators, then $\Box$ is already lower semi-idempotent in the sense that $\mathcal{R}^{\Box} \subset \mathcal{R}^{\Box\Box}$ for any relator $\mathcal{R}$.

Definition 4.3. Let $\mathfrak{F}$ be a structure for relators. Then, we say that:

(1) $\mathfrak{F}$ is *quasi-increasing* if $\mathfrak{F}_R \subset \mathfrak{F}_{\mathcal{R}}$ for any relator $\mathcal{R}$ on X to Y and $R \in \mathcal{R}$,

(2) $\mathfrak{F}$ is *increasing* if $\mathfrak{F}_{\mathcal{R}} \subset \mathfrak{F}_{\mathcal{S}}$ for any two relators $\mathcal{R}$ and $\mathcal{S}$ on X to Y with $\mathcal{R} \subset \mathcal{S}$,

(3) $\mathfrak{F}$ is *union-preserving* if $\mathfrak{F}_{\bigcup_{i\in I} \mathcal{R}_i} = \bigcup_{i\in I} \mathfrak{F}_{\mathcal{R}_i}$ for any family $\big(\mathcal{R}_i\big)_{i\in I}$ of relators on X to Y.

Remark 4.4. Note that thus "union-preserving" implies "increasing" implies "quasi-increasing".

Theorem 4.5. *For any structure $\mathfrak{F}$ for relators, the following assertions are equivalent:*

(1) $\mathfrak{F}$ is union-preserving,

(2) $\mathfrak{F}_{\mathcal{R}} = \bigcup_{R\in\mathcal{R}} \mathfrak{F}_R$ for any relator $\mathcal{R}$ on X to Y,

(3) $\mathfrak{F}$ is quasi-increasing and $\mathfrak{F}_{\mathcal{R}} \subset \bigcup_{R\in\mathcal{R}} \mathfrak{F}_R$ for any relator $\mathcal{R}$ on X to Y.

Proof. To prove the implication (2) $\Longrightarrow$ (1), note that if (2) holds, then $\mathfrak{F}$ is increasing. Therefore, for any family $\big(\mathcal{R}_i\big)_{i\in I}$ of relators on X to Y, we have $\bigcup_{i\in I} \mathfrak{F}_{\mathcal{R}_i} \subset \mathfrak{F}_{\bigcup_{i\in I} \mathcal{R}_i}$. Thus, we need only prove the converse inclusion.

For this, note that now for the relator $\mathcal{R} = \bigcup_{i\in I} \mathcal{R}_i$ we have (2). Therefore, if $\Omega \in \mathfrak{F}_{\mathcal{R}}$, then there exists $R \in \mathcal{R}$ such that $\Omega \in \mathfrak{F}_R$. Moreover, there exists $i_o \in I$ such that $R \in \mathcal{R}_{i_o}$. Hence, we can see that $\Omega \in \mathfrak{F}_R \subset \mathfrak{F}_{\mathcal{R}_{i_o}} \subset \bigcup_{i\in I} \mathfrak{F}_{\mathcal{R}_i}$.

Remark 4.6. By using the corresponding definitions and Theorem 4.5, it can be easily seen that the structures Int, Lb, int, lb, τ, ℓ, and $\mathcal{E}$ are union-preserving. However, the increasing structures $\mathcal{T}$ and $\mathcal{L}$ are not union-preserving.

Definition 4.7. Let $\mathfrak{F}$ be a structure and $\Box$ be an operation for relators. Then, we say that:

(1) $\mathfrak{F}$ is *upper $\Box$–semiregular* if $\mathfrak{F}_{\mathcal{R}} \subset \mathfrak{F}_{\mathcal{S}}$ implies $\mathcal{R} \subset \mathcal{S}^{\Box}$ for any two relators $\mathcal{R}$ and $\mathcal{S}$ on X to Y,

(2) $\mathfrak{F}$ is *lower $\Box$–semiregular* if $\mathcal{R} \subset \mathcal{S}^{\Box}$ implies $\mathfrak{F}_{\mathcal{R}} \subset \mathfrak{F}_{\mathcal{S}}$ for any two relators $\mathcal{R}$ and $\mathcal{S}$ on X to Y.

Remark 4.8. If $\mathfrak{F}$ is an upper $\Box$–semiregular structure for relators, then because of the fundamental work of Pataki [?] we may also say that the pair $(\mathfrak{F}, \Box)$ is an *upper Pataki connection.*

Now, the structure $\mathfrak{F}$ may be naturally called *$\Box$–regular* if it is both upper and lower $\Box$–semiregular. Moreover, for instance, $\mathfrak{F}$ may be naturally called regular if it is $\Box$–regular for some operation $\Box$.

Remark 4.9. In the theory of relators, Pataki connections can also be most naturally obtained from the Galois ones [**?**].

Namely, if F is a function of $\mathcal{P}(X)$ to $\mathcal{P}(Y)$ and G is a function of $\mathcal{P}(Y)$ to $\mathcal{P}(X)$ such that, for any $A \subset X$ and $B \subset Y$, we have $F(A) \subset B$ if and only if $A \subset G(B)$, then for any $U, V \subset X$ we have

$$F(U) \subset F(V) \iff U \subset G\big(F(V)\big) \iff U \subset (G \circ F)(V).$$

Thus, in particular concerning the modification operations ∞ and ∂ considered in Section 2, we can note that ∞, as a structure for relators, is $\infty\partial$–regular. By the forthcoming Theorems 6.3 and 6.1, this will imply that $\infty\partial$ is a closure operation for relators such that $\infty = \infty\partial\infty$.

5. Operations derived from structures for relators

Definition 5.1. For any structure $\mathfrak{F}$ for relators, we define an operation $\Box_{\mathfrak{F}}$ for relators such that

$$\mathcal{R}^{\Box_{\mathfrak{F}}} = \big\{ S \subset X \times Y : \quad \mathfrak{F}_S \subset \mathfrak{F}_{\mathcal{R}} \big\}$$

for any relator $\mathcal{R}$ on X to Y.

Remark 5.2. Thus, for instance, if $S \in \mathcal{R}^{\Box_{\mathrm{Int}}}$, then $\mathrm{Int}_S \subset \mathrm{Int}_{\mathcal{R}}$. Hence, since $A \in \mathrm{Int}_S\big(S[A]\big)$ for all $A \subset X$, we can infer that $A \in \mathrm{Int}_{\mathcal{R}}\big(S[A]\big)$ for all $A \subset X$. Thus, for each $A \subset X$, there exists $R \in \mathcal{R}$ such that $R[A] \subset S[A]$. Therefore, $S \in \mathcal{R}^{\#}$.

This shows that $\mathcal{R}^{\Box_{\mathrm{Int}}} \subset \mathcal{R}^{\#}$. Moreover, by using the corresponding definitions, we can even more easily see that the converse inclusion is also true. Therefore, we actually have $\mathcal{R}^{\#} = \mathcal{R}^{\Box_{\mathrm{Int}}}$ for any relator $\mathcal{R}$, and thus $\# = \Box_{\mathrm{Int}}$.

The appropriateness of Definition 5.1, is already apparent from the following extensions and supplements of the corresponding results of Pataki [**?**].

Theorem 5.3. *If $\mathfrak{F}$ is a structure and $\Box$ is an operation for relators such that $\mathfrak{F}$ is $\Box$–regular, then $\Box = \Box_{\mathfrak{F}}$.*

Proof. By the corresponding definitions,

$$S \in \mathcal{R}^{\Box} \iff \{S\} \subset \mathcal{R}^{\Box} \iff \mathfrak{F}_{\{S\}} \subset \mathfrak{F}_{\mathcal{R}} \iff \mathfrak{F}_S \subset \mathfrak{F}_{\mathcal{R}} \iff S \in \mathcal{R}^{\Box_{\mathfrak{F}}}$$

for any relator $\mathcal{R}$ and relation S on X to Y.

Now, as some immediate consequences of this theorem, we can also state

Corollary 5.4. *If $\mathfrak{F}$ is a structure for relators, then there exists at most one operation $\Box$ for relators such that $\mathfrak{F}$ is $\Box$–regular.*

Corollary 5.5. *If $\mathfrak{F}$ is a regular structure for relators, then $\Box = \Box_{\mathfrak{F}}$ is the unique operation for relators such that $\mathfrak{F}$ is $\Box$–regular.*

Corollary 5.6. *A structure $\mathfrak{F}$ for relators is regular if and only if it is $\Box_{\mathfrak{F}}$–regular.*

Theorem 5.7. *If $\mathfrak{F}$ is a quasi-increasing structure for relators, then*

(1) $\mathfrak{F}$ is upper $\Box_{\mathfrak{F}}$–semiregular, *(2) $\Box_{\mathfrak{F}}$ is extensive.*

Proof. If $\mathcal{R}$ and $\mathcal{S}$ are relators on X to Y such that $\mathfrak{F}_{\mathcal{R}} \subset \mathfrak{F}_{\mathcal{S}}$, then by the quasi-increasingness of $\mathfrak{F}$, for any $R \in \mathcal{R}$, we also have $\mathfrak{F}_R \subset \mathfrak{F}_{\mathcal{S}}$. Hence, by Definition 5.1, we can already see that $R \in \mathcal{S}^{\square_{\mathfrak{F}}}$. Therefore, $\mathcal{R} \subset \mathcal{S}^{\square_{\mathfrak{F}}}$. Thus, by Definition 4.7, assertion (1) is true.

Now, from the inclusion $\mathfrak{F}_{\mathcal{R}} \subset \mathfrak{F}_{\mathcal{R}}$, by using (1), we can infer that $\mathcal{R} \subset \mathcal{R}^{\square_{\mathfrak{F}}}$. Therefore, (2) is also true.

Remark 5.8. Note that if in particular the structure $\mathfrak{F}$ is increasing, then by Definition 5.1, the operation $\square_{\mathfrak{F}}$ is also increasing. Therefore, by assertion (2), it is already a preclosure operation for relators.

From the above theorem, by Corollary 5.6, it is clear that in particular we also have

Corollary 5.9. *A quasi-increasing structure $\mathfrak{F}$ for relators is regular if and only if it is lower $\square_{\mathfrak{F}}$–semiregular.*

Remark 5.10. By considering the equivalence relator $\mathcal{R} = \{ X^2 \}$ on a set X with $\operatorname{card}(X) > 2$, it can be shown that, the preclosure operation $\square_{\mathcal{T}} = \wedge\partial$ is not idempotent despite that the structure $\mathcal{T}$ is increasing [**?**, Example 7.2].

Moreover, it is also noteworthy that now we also have $\mathcal{R}^{\#\partial} \neq \mathcal{R}$. Thus, in contrast to $\#$ and $\#\infty$, the operation $\#\partial$ is already not stable. Therefore, in our former papers, we have mostly used $\#\infty$ instead of $\#\partial$.

Theorem 5.11. *If $\mathfrak{F}$ is an union-preserving structure for relators, then*

(1) $\mathfrak{F}$ is $\square_{\mathfrak{F}}$–regular, (2) $\square_{\mathfrak{F}}$ is a closure operation.

Proof. Suppose that $\mathcal{R}$ and $\mathcal{S}$ are relators on X to Y such that $\mathcal{R} \subset \mathcal{S}^{\square_{\mathfrak{F}}}$, and $\Omega \in \mathfrak{F}_{\mathcal{R}}$. Then, by Theorem 4.5, there exists $R \in \mathcal{R}$ such that $\Omega \in \mathfrak{F}_R$. Now, since $\mathcal{R} \subset \mathcal{S}^{\square_{\mathfrak{F}}}$, we also have $R \in \mathcal{S}^{\square_{\mathfrak{F}}}$. Hence, by Definition 5.1, we can infer that $\mathfrak{F}_R \subset \mathfrak{F}_{\mathcal{S}}$. Therefore, we also have $\Omega \in \mathfrak{F}_{\mathcal{S}}$. Consequently, $\mathfrak{F}_{\mathcal{R}} \subset \mathfrak{F}_{\mathcal{S}}$. This shows that $\mathfrak{F}$ is lower $\square_{\mathfrak{F}}$–semiregular. Hence, by Theorem 5.7, we can see that (1) is true. Assertion (2) is a consequence of (1) by the forthcoming Theorem 6.3.

Remark 5.12. By Remark 4.6, the structure Int is a union-preserving. Therefore, by Theorem 5.11, Int is $\square_{\operatorname{Int}}$–regular, and $\square_{\operatorname{Int}}$ is a closure operation. Moreover, by Remark 5.2, we have $\# = \square_{\operatorname{Int}}$.

Therefore, the structure Int is actually $\#$–regular, and $\#$ is a closure operation. Thus, in particular, for any two relators $\mathcal{R}$ and $\mathcal{S}$ on X to Y, we have

$$\operatorname{Int}_{\mathcal{R}} \subset \operatorname{Int}_{\mathcal{S}} \iff \mathcal{R} \subset \mathcal{S}^{\#}.$$

Remark 5.13. Hence, by using the equality $\operatorname{Lb}_{\mathcal{R}} = \operatorname{Int}_{\mathcal{R}^c} \circ \mathcal{C}$ and the corresponding definitions, we can easily that

$$\operatorname{Lb}_{\mathcal{R}} \subset \operatorname{Lb}_{\mathcal{S}} \iff \operatorname{Int}_{\mathcal{R}^c} \circ \mathcal{C} \subset \operatorname{Int}_{\mathcal{S}^c} \circ \mathcal{C} \iff \operatorname{Int}_{\mathcal{R}^c} \subset \operatorname{Int}_{\mathcal{S}^c}$$
$$\iff \mathcal{R}^c \subset \mathcal{S}^{c\#} \iff \mathcal{R} \subset \mathcal{S}^{c\#c} \iff \mathcal{R} \subset \mathcal{S}^{\circledast}$$

for any two relators $\mathcal{R}$ and $\mathcal{S}$ on X to Y.

Therefore, the structure Lb is $\circledast$–regular, and thus by Theorem 5.3 we necessarily have $\square_{\operatorname{Lb}} = \circledast$. Moreover, by Remark 4.6, the structure Lb is also union-preserving. Hence, by Theorem 5.11, we can see that $\circledast$ is also a closure operation for relators.

6. Some further results on regular structures

Analogously to the results of [**?**, **?**], we can also easily prove the following theorems.

Theorem 6.1. *If $\mathfrak{F}$ is a $\Box$–regular structure for relators, then*

(1) $\Box$ is extensive, *(2) $\mathfrak{F}$ is increasing,*

(3) $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_{\mathcal{R}^{\Box}}$ for any relator $\mathcal{R}$ on X to Y.

Proof. If $\mathcal{R}$ is a relator on X to Y, then from the inclusion $\mathcal{R}^{\Box} \subset \mathcal{R}^{\Box}$, by using the lower $\Box$–semiregularity of $\mathfrak{F}$, we can infer that $\mathfrak{F}_{\mathcal{R}^{\Box}} \subset \mathfrak{F}_{\mathcal{R}}$.

On the other hand, from the inclusion $\mathfrak{F}_{\mathcal{R}} \subset \mathfrak{F}_{\mathcal{R}}$, by using the upper $\Box$–semiregularity of $\mathfrak{F}$, we can infer that $\mathcal{R} \subset \mathcal{R}^{\Box}$. Therefore, (1) is true.

Now, if $\mathcal{S}$ is a relator on X to Y such that $\mathcal{R} \subset \mathcal{S}$, then by using (1) we can see that $\mathcal{R} \subset \mathcal{S}^{\Box}$ also holds. Hence, by using the lower $\Box$–semiregularity of $\mathfrak{F}$, we can infer that $\mathfrak{F}_{\mathcal{R}} \subset \mathfrak{F}_{\mathcal{S}}$. Therefore, assertion (2) is also true.

Now, from the inclusion $\mathcal{R} \subset \mathcal{R}^{\Box}$, by using (2), we can infer that $\mathfrak{F}_{\mathcal{R}} \subset \mathfrak{F}_{\mathcal{R}^{\Box}}$. Therefore, assertion (3) is also true.

From this theorem, by Theorem 5.11, it is clear that in particular we also have

Corollary 6.2. *If $\mathfrak{F}$ is a union-preserving structure for relators, then $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_{\mathcal{R}^{\Box_{\mathfrak{F}}}}$ for any relator $\mathcal{R}$ on X to Y.*

Theorem 6.3. *For an operation $\Box$ for relators, the following assertions are equivalent:*

(1) $\Box$ is self-regular, *(2) $\Box$ is a closure operation,*

(3) there exists at least one $\Box$–regular structure $\mathfrak{F}$ for relators.

Proof. If (2) holds, and $\mathcal{R}$ is a relator on X to Y, then by the extensivity of $\Box$ we have $\mathcal{R} \subset \mathcal{R}^{\Box}$. Therefore, if $\mathcal{S}$ is a relator on X to Y such that $\mathcal{R}^{\Box} \subset \mathcal{S}^{\Box}$, then we also have $\mathcal{R} \subset \mathcal{S}^{\Box}$. Thus, $\Box$ is upper $\Box$–semiregular.

On the other hand, if $\mathcal{R} \subset \mathcal{S}^{\Box}$, then by the increasingness of $\Box$ we also have $\mathcal{R}^{\Box} \subset \mathcal{S}^{\Box\Box}$. Hence, by the idempotency of $\Box$, it follows that $\mathcal{R}^{\Box} \subset \mathcal{S}^{\Box}$. Therefore, $\Box$ is also lower $\Box$–semiregular. Thus, $\Box$ is $\Box$-regular, and so (1) also holds.

Now, since (3) can be trivially derived from (1) by taking $\mathfrak{F} = \Box$, we need only show that (3) also implies (2).

For this, note that if (3) holds, then by Theorem 6.1 the operation $\Box$ is extensive. Moreover, for any relator $\mathcal{R}$ on X to Y, we have $\mathfrak{F}_{\mathcal{R}^{\Box}} = \mathfrak{F}_{\mathcal{R}}$. Hence, by taking $\mathcal{R}^{\Box}$ in place of $\mathcal{R}$, we can see that $\mathfrak{F}_{\mathcal{R}^{\Box\Box}} = \mathfrak{F}_{\mathcal{R}^{\Box}}$, and thus $\mathfrak{F}_{\mathcal{R}^{\Box\Box}} = \mathfrak{F}_{\mathcal{R}}$ also holds. Hence, by using the upper $\Box$–semiregularity of $\mathfrak{F}$, we can already infer that $\Box$ is upper semiidempotent in the sense that $\mathcal{R}^{\Box\Box} \subset \mathcal{R}^{\Box}$. Now, by Remark 4.2, we can see that $\Box$ is actually idempotent.

Thus, to obtain (2), it remains only to show that $\Box$ is also increasing. For this, note that if $\mathcal{R}$ and $\mathcal{S}$ are relators on X to Y such that $\mathcal{R} \subset \mathcal{S}$, then by Theorem 6.1 we also have $\mathfrak{F}_{\mathcal{R}} \subset \mathfrak{F}_{\mathcal{S}}$. Moreover, we have $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_{\mathcal{R}^{\Box}}$, and thus also $\mathfrak{F}_{\mathcal{R}^{\Box}} \subset \mathfrak{F}_{\mathcal{S}}$. Hence, by using the upper $\Box$–semiregularity of $\mathfrak{F}$, we can already infer that $\mathcal{R}^{\Box} \subset \mathcal{S}^{\Box}$.

From this theorem, by Theorem 5.3, it is clear that in particular we also have

Corollary 6.4. *If* $\Diamond$ *is a closure operation for relators, then* $\Diamond = \Box_{\Diamond}$.

Moreover, from Theorem 6.3, by using Definition 4.7, we can immediately derive

Corollary 6.5. *For a structure* $\mathfrak{F}$ *and an operation* $\Box$ *for relators, the following assertions are equivalent;*

(1) $\mathfrak{F}$ *is* $\Box$*–regular,*

(2) $\Box$ *is a closure operation, and for any two relators* $\mathcal{R}$ *and* $\mathcal{S}$ *on* X *to* Y*, we have* $\mathfrak{F}_{\mathcal{R}} \subset \mathfrak{F}_{\mathcal{S}}$ *if and only if* $\mathcal{R}^{\Box} \subset \mathcal{S}^{\Box}$.

7. A further characterization of regular structures

Theorem 7.1. *For a structure* $\mathfrak{F}$ *and an operation* $\Box$ *for relators, the following assertions are equivalent:*

(1) $\mathfrak{F}$ *is* $\Box$*–regular,*

(2) $\mathfrak{F}$ *is increasing, and for every relator* $\mathcal{R}$ *on* X *to* Y*,* $\mathcal{S} = \mathcal{R}^{\Box}$ *is the largest relator on* X *to* Y *such that* $\mathfrak{F}_{\mathcal{S}} \subset \mathfrak{F}_{\mathcal{R}}$.

Proof. If (1) holds, then by Theorem 6.1 the structure $\mathfrak{F}$ is increasing, and for any relator $\mathcal{R}$ on X to Y we have $\mathfrak{F}_{\mathcal{R}^{\Box}} = \mathfrak{F}_{\mathcal{R}}$. Moreover, if $\mathcal{S}$ is a relator on X to Y such that $\mathfrak{F}_{\mathcal{S}} \subset \mathfrak{F}_{\mathcal{R}}$, then by using the upper $\Box$–semiregulaty of $\mathfrak{F}$ we can see that $\mathcal{S} \subset \mathcal{R}^{\Box}$. Thus, in particular, (2) also holds.

On the other hand, if (2) holds, and $\mathcal{R}$ and $\mathcal{S}$ are relators on X to Y such that $\mathfrak{F}_{\mathcal{S}} \subset \mathfrak{F}_{\mathcal{R}}$, then from the assumed maximality property of $\mathcal{R}^{\Box}$ we can see that $\mathcal{S} \subset \mathcal{R}^{\Box}$. Therefore, $\mathfrak{F}$ is upper $\Box$–semiregular.

Conversely, if $\mathcal{R}$ and $\mathcal{S}$ are relators on X to Y such that $\mathcal{S} \subset \mathcal{R}^{\Box}$, then by using the assumed increasingness of $\mathfrak{F}$ we can see that $\mathfrak{F}_{\mathcal{S}} \subset \mathfrak{F}_{\mathcal{R}^{\Box}}$. Hence, by the assumed inclusion $\mathfrak{F}_{\mathcal{R}^{\Box}} \subset \mathfrak{F}_{\mathcal{R}}$, it follows that $\mathfrak{F}_{\mathcal{S}} \subset \mathfrak{F}_{\mathcal{R}}$. Therefore, $\mathfrak{F}$ is also lower $\Box$–semiregular, and thus (1) also holds.

From this theorem, by Theorem 6.1, it is clear that in particular we also have

Corollary 7.2. *If* $\mathfrak{F}$ *is a* $\Box$*–regular structure for relators, then for any relator* $\mathcal{R}$ *on* X *to* Y*,* $\mathcal{S} = \mathcal{R}^{\Box}$ *is the largest relator on* X *to* Y *such that* $\mathfrak{F}_{\mathcal{S}} = \mathfrak{F}_{\mathcal{R}}$.

From the above results, by Theorem 5.3, it is clear that we also have

Corollary 7.3. *If* $\mathfrak{F}$ *is a regular structure for relators, then for any relator* $\mathcal{R}$ *on* X *to* Y*,* $\mathcal{S} = \mathcal{R}^{\Box_{\mathfrak{F}}}$ *is the largest relator on* X *to* Y *such that* $\mathfrak{F}_{\mathcal{S}} \subset \mathfrak{F}_{\mathcal{R}}$ $\big(\mathfrak{F}_{\mathcal{S}} = \mathfrak{F}_{\mathcal{R}} \big)$.

Hence, by Theorem 5.13, it is clear that in particular we also have

Corollary 7.4. *If* $\mathfrak{F}$ *is a union-preserving structure for relators, then for any relator* $\mathcal{R}$ *on* X *to* Y*,* $S = \mathcal{R}^{\Box_{\mathfrak{F}}}$ *is the largest relator on* X *to* Y *such that* $\mathfrak{F}_{\mathcal{S}} \subset \mathfrak{F}_{\mathcal{R}}$ $\big(\mathfrak{F}_{\mathcal{S}} = \mathfrak{F}_{\mathcal{R}} \big)$.

Remark 7.5. Hence, by Remark 5.12, we can see that, for any relator $\mathcal{R}$ on X to Y, $\mathcal{S} = \mathcal{R}^{\#}$ is the largest relator on X to Y such that $\operatorname{Int}_{\mathcal{S}} \subset \operatorname{Int}_{\mathcal{R}}$ $\big(\operatorname{Int}_{\mathcal{S}} = \operatorname{Int}_{\mathcal{R}} \big)$.

From Theorem 7.1, by Theorem 6.3, it is clear that in particular we also have

Corollary 7.6. *For an operation* $\Box$ *for relators, the following assertions are equivalent:*

(1) $\Box$ *is a closure operation,*

(2) $\Box$ *is increasing, and for every relator* $\mathcal{R}$ *on* X *to* Y, $\mathcal{S} = \mathcal{R}^{\Box}$ *is the largest relator on* X *to* Y *such that* $\mathcal{S}^{\Box} \subset \mathcal{R}^{\Box}$,

(3) there exists an increasing structure $\mathfrak{F}$ *for relators such that, for every relator* $\mathcal{R}$ *on* X *to* Y, $\mathcal{S} = \mathcal{R}^{\Box}$ *is the largest relator on* X *to* Y *such that* $\mathfrak{F}_{\mathcal{S}} \subset \mathfrak{F}_{\mathcal{R}}$.

However, by using some direct arguments, we can also prove the following

Theorem 7.7. *For an operation* $\Box$ *for relators, the following assertions are equivalent:*

(1) $\Box$ *is a semiclosure operation,*

(2) for every relator $\mathcal{R}$ *on* X *to* Y, $\mathcal{S} = \mathcal{R}^{\Box}$ *is the largest relator on* X *to* Y *such that* $\mathcal{S}^{\Box} \subset \mathcal{R}^{\Box}$ $\big(\mathcal{S}^{\Box} = \mathcal{R}^{\Box}\big)$,

(3) there exists a structure $\mathfrak{F}$ *for relators such that, for every relator* $\mathcal{R}$ *on* X *to* Y, $\mathcal{S} = \mathcal{R}^{\Box}$ *is the largest relator on* X *to* Y *such that* $\mathfrak{F}_{\mathcal{S}} \subset \mathfrak{F}_{\mathcal{R}}$ $\big(\mathfrak{F}_{\mathcal{S}} = \mathfrak{F}_{\mathcal{R}}\big)$.

Proof. If (1) holds and $\mathcal{R}$ is a relator on X to Y, then by the idempotency of $\Box$ we have $\big(\mathcal{R}^{\Box}\big)^{\Box} = \mathcal{R}^{\Box}$. Moreover, if $\mathcal{S}$ is a relator on X to Y such that $\mathcal{S}^{\Box} \subset \mathcal{R}^{\Box}$, then by the expansivity of $\Box$, we also have $\mathcal{S} \subset \mathcal{R}^{\Box}$. Therefore, (2) also holds.

Now, since (3) can be trivially derived form (2) by taking $\mathfrak{F} = \Box$, we need only show that (3) also implies (1). For this, note that if (3) holds and $\mathcal{R}$ is a relator on X to Y, then from the equality $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_{\mathcal{R}}$, by using any of the assumed maximality properties of $\mathcal{R}^{\Box}$, we can infer that $\mathcal{R} \subset \mathcal{R}^{\Box}$. Therefore, $\Box$ is extensive.

On the other hand, by taking $\mathcal{R}^{\Box}$ in place of $\mathcal{R}$ in (3), we can also see that $\mathfrak{F}_{\mathcal{R}^{\Box\Box}} \subset \mathfrak{F}_{\mathcal{R}^{\Box}}$ $\big(\mathfrak{F}_{\mathcal{R}^{\Box\Box}} = \mathfrak{F}_{\mathcal{R}^{\Box}}\big)$. Now, since $\mathfrak{F}_{\mathcal{R}^{\Box}} \subset \mathfrak{F}_{\mathcal{R}}$ $\big(\mathfrak{F}_{\mathcal{R}^{\Box}} = \mathfrak{F}_{\mathcal{R}}\big)$ also holds, we can also see that $\mathfrak{F}_{\mathcal{R}^{\Box\Box}} \subset \mathfrak{F}_{\mathcal{R}}$ $\big(\mathfrak{F}_{\mathcal{R}^{\Box\Box}} = \mathfrak{F}_{\mathcal{R}}\big)$. Hence, by the assumed maximality properties of $\mathcal{R}^{\Box}$, we can already infer that $\mathcal{R}^{\Box\Box} \subset \mathcal{R}^{\Box}$. Now, by Remark 4.2, we can see that the corresponding equality is also true. Therefore, $\Box$ is idempotent, and thus (1) also holds.

8. Some further results on closure operations

By using a direct argument, we can easily prove the following extension of the last statement of Remark 5.13.

Theorem 8.1. *If* $\Box$ *is a closure and* $\diamond$ *is an increasing involution operation for relators, then* $\Diamond = \diamond\Box\diamond$ *is also a closure operation for relators.*

Proof. Since $\diamond$ and $\Box$ are increasing, it is clear that $\diamond\Box$, and thus $(\diamond\,\Box)\diamond$ is also increasing. Therefore, $\Diamond$ is also increasing.

Moreover, if $\mathcal{R}$ is a relator, then by using the extensivity of $\Box$ we can see that $\mathcal{R}^{\diamond} \subset \mathcal{R}^{\diamond\Box}$. Hence, by using the increasingness of $\diamond$, we can infer that $\mathcal{R}^{\diamond\diamond} \subset \mathcal{R}^{\diamond\Box\diamond}$. Now, since $\mathcal{R}^{\diamond\diamond} = \mathcal{R}$, we can see that $\mathcal{R} \subset \mathcal{R}^{\Diamond}$, and thus $\Diamond$ is also extensive.

Finally, by using the involutiveness of $\diamond$ and the idempontency of $\Box$, we can see that

$$\begin{aligned}\Diamond\,\Diamond &= (\diamond\Box\diamond)(\diamond\Box\diamond) = (\diamond\Box)\big((\diamond\diamond)(\Box\diamond)\big)\\ &= (\diamond\Box)\big(\Delta(\Box\diamond)\big) = (\diamond\Box)\big(\Box\diamond\big) = \diamond\big((\Box\Box)\diamond\big) = \diamond(\Box\diamond) = \Diamond,\end{aligned}$$

where Δ is the identity operation for relators. Therefore, $\Diamond$ is also idempotent.

Remark 8.2. Thus, in particular, $\circledast = c * c$ is also a closure operation for relators. Moreover, it is also worth noticing that, for any relator $\mathcal{R}$ on X to Y, we have

$$\mathcal{R}^{\circledast} = \bigcup_{R \in \mathcal{R}} \mathcal{P}(R) = \big\{ S \subset X \times Y : \quad \exists\ R \in \mathcal{R} : \ S \subset R \big\}.$$

Namely, if for instance $S \in \mathcal{R}^{\circledast}$, then $S \in \mathcal{R}^{c*c}$, and thus $S^c \in \mathcal{R}^{c*}$. Therefore, there exists $R \in \mathcal{R}$ such that $R^c \subset S^c$. Hence, it follows that $S \subset R$, and thus $S \in \mathcal{P}(R)$. Therefore, $S \in \bigcup_{R \in \mathcal{R}} \mathcal{P}(R)$ also holds.

Remark 8.3. By using the corresponding definitions and Theorem 4.5, it can be easily seen that the operations c, -1, ∞, ∂, and $*$ are union-preserving.

Moreover, we can also easily see that the composition of two union-preserving operations is also union-preserving. Thus, in particular, $\circledast$ is also union-preserving.

Unfortunately, the important closure operations $\#$, $\wedge$, and $\vartriangle$ are not union-preserving. However, we can easily prove the following

Theorem 8.4. *If $\square$ is a closure operation for relators, then for any family $\big(\mathcal{R}_i\big)_{i \in I}$ of relators on X to Y, we have*

$$\Big(\bigcup_{i \in I} \mathcal{R}_i \Big)^{\square} = \Big(\bigcup_{i \in I} \mathcal{R}_i^{\square} \Big)^{\square}.$$

Proof. If $\mathcal{R} = \bigcup_{i \in I} \mathcal{R}_i$, then for each $i \in I$ we have $\mathcal{R}_i \subset \mathcal{R}$. Hence, by using the increasingness of $\square$, we can infer that $\mathcal{R}_i^{\square} \subset \mathcal{R}^{\square}$. Therefore, we have $\bigcup_{i \in I} \mathcal{R}_i^{\square} \subset \mathcal{R}^{\square}$. Hence, by using the increasingness and the idempotency of $\square$, we can already infer that $\big(\bigcup_{i \in I} \mathcal{R}_i^{\square} \big)^{\square} \subset \mathcal{R}^{\square\square} = \mathcal{R}^{\square}$.

On the other hand, by the extensivity of $\square$, for each $i \in I$ we have $\mathcal{R}_i \subset \mathcal{R}_i^{\square}$, and hence also $\mathcal{R}_i \subset \bigcup_{i \in I} \mathcal{R}_i^{\square}$. Therefore, $\mathcal{R} = \bigcup_{i \in I} \mathcal{R}_i \subset \bigcup_{i \in I} \mathcal{R}_i^{\square}$. Hence, by using the increasingness of $\square$, we can already infer that $\mathcal{R}^{\square} \subset \big(\bigcup_{i \in I} \mathcal{R}_i^{\square} \big)^{\square}$. Therefore, the required equality is also true.

From this theorem, by calling a relator $\mathcal{R}$ to be $\square$–invariant if $\mathcal{R}^{\square} = \mathcal{R}$, we can immediately derive the following

Corollary 8.5. *If $\square$ is a closure operation for relators, then for any family $\big(\mathcal{R}_i\big)_{i \in I}$ of relators on X to Y, we have $\big(\bigcup_{i \in I} \mathcal{R}_i \big)^{\square} = \bigcup_{i \in I} \mathcal{R}_i^{\square}$ if and only the relator $\bigcup_{i \in I} \mathcal{R}_i^{\square}$ is $\square$–invariant.*

Analogously to Theorem 8.4, we can also easily prove the following

Theorem 8.6. *If $\square$ is a closure operation for relators, then for any family $\big(\mathcal{R}_i\big)_{i \in I}$ of relators on X to Y, we have*

$$\bigcap_{i \in I} \mathcal{R}_i^{\square} = \Big(\bigcap_{i \in I} \mathcal{R}_i^{\square} \Big)^{\square}.$$

Proof. If $\mathcal{R} = \bigcap_{i \in I} \mathcal{R}_i$, then for each $\in I$ we have $\mathcal{R} \subset \mathcal{R}_i$, and hence also $\mathcal{R}^{\square} \subset \mathcal{R}_i^{\square}$. Therefore, $\big(\bigcap_{i \in I} \mathcal{R}_i \big)^{\square} = \mathcal{R}^{\square} \subset \bigcap_{i \in I} \mathcal{R}_i^{\square}$.

Hence, by taking $\mathcal{R}_i^{\square}$ in place of $\mathcal{R}_i$, we can already infer that

$$\Big(\bigcap_{i\in I}\mathcal{R}_i^{\square}\Big)^{\square}\subset\bigcap_{i\in I}\mathcal{R}_i^{\square\square}=\bigcap_{i\in I}\mathcal{R}_i^{\square}\subset\Big(\bigcap_{i\in I}\mathcal{R}_i^{\square}\Big)^{\square}.$$

Therefore, the required equality is also true.

From this theorem, we can immediately derive the following

Corollary 8.7. *If $\square$ is a closure operation for relators, then for any family $\left(\mathcal{R}_i\right)_{i\in I}$ of relators on X to Y, the relator $\bigcap_{i\in I}\mathcal{R}_i^{\square}$ is $\square$–invariant.*

Hence, it is clear that in particular we also have

Corollary 8.8. *If $\square$ is a closure operation for relators, then for any family $\left(\mathcal{R}_i\right)_{i\in I}$ of $\square$-invariant relators on X to Y, the relator $\bigcap_{i\in I}\mathcal{R}_i$ is also $\square$–invariant.*

Remark 8.9. Note that the proofs of the above three theorems also yield some useful statements for the corresponding generalizations of closure operations.

Remark 8.10. In our former papers, a closure operation $\square$ for relator has been usually called a *refinement operation*. Therefore, the $\square$–invariant relators have been rather called $\square$*–fine*, than $\square$*–closed*.

Moreover, two relators $\mathcal{R}$ and $\mathcal{S}$ on X to Y have been called $\square$*–equivalent* if $\mathcal{R}^{\square}=\mathcal{S}^{\square}$. And, the relator $\mathcal{R}$ has been called $\square$*–simple* if it is $\square$–equivalent to a simple relator $\{R\}$ on X to Y.

Now, a relator $\mathcal{R}$ on X to Y may, for instance, be naturally called *properly, uniformly, proximally, topologically, and paratopologically simple* if it is $\square$–simple with $\square=\Delta, *, \#, \wedge,$ and $\vartriangle$, respectively.

9. Inversion compatible operations for relators

Definition 9.1. An unary operation $\square$ for relators is called *inversion compatible* if for any relator $\mathcal{R}$ on X to Y we have

$$\left(\mathcal{R}^{\square}\right)^{-1}=\left(\mathcal{R}^{-1}\right)^{\square}.$$

The usefulness of this definition is apparent from the next simple theorems.

Theorem 9.2. *If $\square$ is an inversion compatible operation for relators, then for any relator $\mathcal{R}$ on X to Y the following assertions are equivalent:*

(1) $\mathcal{R}$ is $\square$–invariant, *(2) $\mathcal{R}^{-1}$ is $\square$–invariant.*

Proof. If (1) holds, then we have $\left(\mathcal{R}^{-1}\right)^{\square}=\left(\mathcal{R}^{\square}\right)^{-1}=\mathcal{R}^{-1}$, and thus (2) also holds.

Hence, by writing $\mathcal{R}^{-1}$ in place $\mathcal{R}$, we can see that the converse implication is also true.

Definition 9.3. If $\square$ is an unary operation for relators, then a relator $\mathcal{R}$ on X is called $\square$*–symmetric* if

$$\left(\mathcal{R}^{\square}\right)^{-1}=\mathcal{R}^{\square}.$$

Remark 9.4. Now, the relator $\mathcal{R}$ may, for instance, be naturally called *properly, uniformly, proximally, topologically, and paratopologically symmetric* if it is $\square$–symmetric with $\square=\Delta, *, \#, \wedge,$ and $\vartriangle$, respectively.

Theorem 9.5. *If $\mathcal{R}$ is a properly symmetric relator on X, then $\mathcal{R}$ is $\square$–symmetric for any inversion compatible operation $\square$ for relators.*

Proof. By the corresponding definitions, we have $\mathcal{R}^{\square} = \big(\mathcal{R}^{-1}\big)^{\square} = \big(\mathcal{R}^{\square}\big)^{-1}$.

Theorem 9.6. *If $\square$ is an inversion compatible operation for relators, then for any relator $\mathcal{R}$ on X the following assertions are equivalent:*

(1) *$\mathcal{R}$ is $\square$–symmetric,*

(2) *$\mathcal{R}^{-1}$ is $\square$–symmetric,*

(3) *$\mathcal{R}$ and $\mathcal{R}^{-1}$ are $\square$–equivalent.*

Proof. If (1) holds, then we have $\mathcal{R}^{\square} = \big(\mathcal{R}^{\square}\big)^{-1} = \big(\mathcal{R}^{-1}\big)^{\square}$. Therefore, (3) also holds.

While, if (3) holds, then $\Big(\big(\mathcal{R}^{-1}\big)^{\square}\Big)^{-1} = \big(\mathcal{R}^{\square}\big)^{-1} = \big(\mathcal{R}^{-1}\big)^{\square}$. Therefore, (2) also holds.

Now, from the implication (1) $\Longrightarrow$ (2), by writing $\mathcal{R}^{-1}$ in place of $\mathcal{R}$, we can see that the converse implication is also true.

Remark 9.7. In this respect, it is also worth noticing that if $\square$ is an unary operation for relators and $\mathcal{R}$ is a $\square$–symmetric relator on X to Y such that $\mathcal{R}$ and $\mathcal{R}^{-1}$ are $\square$–equivalent, then $\big(\mathcal{R}^{\square}\big)^{-1} = \mathcal{R}^{\square} = \big(\mathcal{R}^{-1}\big)^{\square}$.

However, it is now more important to note that, in addition to Theorem 9.6, we can also prove the following

Theorem 9.8. *If $\square$ is an inversion compatible closure operation for relators, then for any relator $\mathcal{R}$ on X the following assertions are equivalent:*

(1) *$\mathcal{R}$ is $\square$–symmetric,*

(2) *$\mathcal{R}^{-1} \subset \mathcal{R}^{\square}$;* (3) *$\mathcal{R} \subset \big(\mathcal{R}^{-1}\big)^{\square}$,*

(4) *$\mathcal{R}$ is $\square$-equivalent to a properly symmetric relator $\mathcal{S}$ on X.*

Proof. If (1) holds, then by the extensivity of $\square$, it is clear that $\mathcal{R}^{-1} \subset \big(\mathcal{R}^{\square}\big)^{-1} = \mathcal{R}^{\square}$. Therefore, (2) also holds.

Moreover, if (2) holds, then we can see that $\mathcal{R} \subset \big(\mathcal{R}^{\square}\big)^{-1} = \big(\mathcal{R}^{-1}\big)^{\square}$. Therefore, (3) also holds.

While, if (3) holds, then we can quite similarly see that (2) also holds. From (2) and (3), by using Theorem 6.3, we can infer that $\big(\mathcal{R}^{-1}\big)^{\square} \subset \mathcal{R}^{\square} \subset \big(\mathcal{R}^{-1}\big)^{\square}$, and thus $\mathcal{R}^{\square} = \big(\mathcal{R}^{-1}\big)^{\square}$. Therefore, by Theorem 9.6, (1) also holds.

On the other hand, if (1) holds, then $\mathcal{R}^{\square}$ is properly symmetric. Hence, since $\mathcal{R}^{\square} = \big(\mathcal{R}^{\square}\big)^{\square}$, we can already see that (4) holds with $\mathcal{S} = \mathcal{R}^{\square}$.

Conversely, if (4) holds, then it is clear that $\big(\mathcal{R}^{\square}\big)^{-1} = \big(\mathcal{S}^{\square}\big)^{-1} = \big(\mathcal{S}^{-1}\big)^{\square} = \mathcal{S}^{\square} = \mathcal{R}^{\square}$. Therefore, (1) also holds.

From this theorem, by using Theorem 6.3 and Definition 4.5, we can derive

Corollary 9.9. *If $\square$ is an inversion compatible operation and $\mathfrak{F}$ is a $\square$–regular structure for relators, then for any relator $\mathcal{R}$ on X the following assertions are equivalent:*

(1) *$\mathcal{R}$ is $\square$–symmetric,* (2) *$\mathfrak{F}_{\mathcal{R}^{-1}} \subset \mathfrak{F}_{\mathcal{R}}$;* (3) *$\mathfrak{F}_{\mathcal{R}} \subset \mathfrak{F}_{\mathcal{R}^{-1}}$.*

Hence, by Theorem 5.3, it is clear that in particular we also have

Corollary 9.10. *If $\mathfrak{F}$ is a regular structure for relators such that the induced operation $\square_{\mathfrak{F}}$ is inversion compatible, then for any relator $\mathcal{R}$ on X the following assertions are equivalent:*

(1) $\mathcal{R}$ *is* $\square_{\mathfrak{F}}$*–symmetric,* *(2)* $\mathfrak{F}_{\mathcal{R}^{-1}} \subset \mathfrak{F}_{\mathcal{R}}$*;* *(3)* $\mathfrak{F}_{\mathcal{R}} \subset \mathfrak{F}_{\mathcal{R}^{-1}}$*.*

Remark 9.11. To apply these inclusions to the structure Int, we can note that if $\mathcal{R}$ is a relator on X to Y, then for any $A \subset X$ and $B \subset Y$ we have

$$B \in \operatorname{Int}_{\mathcal{R}^{-1}}(A) \iff \exists\ R \in \mathcal{R}: \quad R^{-1}[B] \subset A.$$

Moreover, we also have

$$R^{-1}[B] \subset A \iff A^c \cap R^{-1}[B] = \emptyset \iff R[A^c] \cap B = \emptyset \iff R[A^c] \subset B^c.$$

Therefore,

$$\begin{aligned} B \in \operatorname{Int}_{\mathcal{R}^{-1}}(A) &\iff A^c \in \operatorname{Int}_{\mathcal{R}}(B^c) \iff B^c \in \operatorname{Int}_{\mathcal{R}}^{-1}(A^c) \\ &\iff \mathcal{C}(B) \in \operatorname{Int}_{\mathcal{R}}^{-1}\big(\mathcal{C}(A)\big) \iff B \in \mathcal{C}^{-1}\big[\operatorname{Int}_{\mathcal{R}}^{-1}\big(\mathcal{C}(A)\big] \\ &\iff B \in \mathcal{C}\big[\operatorname{Int}_{\mathcal{R}}^{-1}\big(\mathcal{C}(A)\big] \iff B \in \big(\mathcal{C} \circ \operatorname{Int}_{\mathcal{R}}^{-1} \circ \mathcal{C}\big)(A). \end{aligned}$$

Hence, we can already infer that

$$\operatorname{Int}_{\mathcal{R}^{-1}} = \mathcal{C} \circ \operatorname{Int}_{\mathcal{R}}^{-1} \circ \mathcal{C}, \qquad \text{and thus also} \qquad \operatorname{Int}_{\mathcal{R}}^{-1} = \mathcal{C} \circ \operatorname{Int}_{\mathcal{R}^{-1}} \circ \mathcal{C}.$$

Remark 9.12. Finally, we note that the theorems proved in this section can be generalized by using an arbitrary increasing involution operation $\diamond$ for relators instead of the inversion -1.

10. The inversion compatibility of some basic operations

Theorem 10.1. *If $\square$ is an union-preserving operation for relators, then the following assertions are equivalent:*

(1) $\square$ *is inversion compatible,*

(2) $\big(\{R\}^{\square}\big)^{-1} = \big\{R^{-1}\big\}^{\square}$ *for any relation R on X to Y.*

Proof. To prove that (2) also implies (1), note that by Remark 8.3 the operation -1 is union-preserving. Therefore, for any relator $\mathcal{R}$ on X to Y, we have

$$\begin{aligned} \big(\mathcal{R}^{\square}\big)^{-1} = \Big(\bigcup_{R \in \mathcal{R}} \{R\}^{\square}\Big)^{-1} &= \bigcup_{R \in \mathcal{R}} \big(\{R\}^{\square}\big)^{-1} \\ &= \bigcup_{R \in \mathcal{R}} \big\{R^{-1}\big\}^{\square} = \Big(\bigcup_{R \in \mathcal{R}} \big\{R^{-1}\big\}\Big)^{\square} = \big(\mathcal{R}^{-1}\big)^{\square}. \end{aligned}$$

Remark 10.2. By using some obvious analogues of our former definitions, we can easily see that, for an operation $\square$ for relations, the following assertions are equivalent:

(1) $\square$ is inversion compatible,

(2) $\big(R^{\square}\big)^{-1} \subset \big(R^{-1}\big)^{\square}$ for any relation R on X to Y,

(3) $\big(R^{-1}\big)^{\square} \subset \big(R^{\square}\big)^{-1}$ for any relation R on X to Y.

Therefore, as an immediate consequence of Theorem 10.1, we can also state

Corollary 10.3. *If* $\square$ *is an union-preserving operation for relators, then the following assertions are equivalent:*

(1) $\square$ *is inversion compatible,*

(2) $\left(\{R\}^{\square}\right)^{-1} \subset \left\{R^{-1}\right\}^{\square}$ *for any relation* R *on* X *to* Y*,*

(3) $\left\{R^{-1}\right\}^{\square} \subset \left(\{R\}^{\square}\right)^{-1}$ *for any relation* R *on* X *to* Y*.*

However, the latter observation cannot actually be used to simplify the proof of the following

Theorem 10.4. *The operations* c, ∞, ∂, *and* $*$ *are inversion compatible.*

Proof. From Remark 8.3, we know that these operations are union-preserving.

Moreover, for instance, for any two relations R and S on X we have

$$\left(R^{\infty}\right)^{-1}=\left(\bigcup_{n=0}^{\infty} R^{n}\right)^{-1}=\bigcup_{n=0}^{\infty}\left(R^{n}\right)^{-1}=\bigcup_{n=0}^{\infty}\left(R^{-1}\right)^{n}=\left(R^{-1}\right)^{\infty}$$

and

$$S \in\left\{R^{-1}\right\}^{\partial} \iff S^{\infty} \in\left\{R^{-1}\right\} \iff S^{\infty}=R^{-1} \iff \left(S^{\infty}\right)^{-1}=R$$
$$\iff \left(S^{-1}\right)^{\infty}=R \iff \left(S^{-1}\right)^{\infty} \in\{R\} \iff S^{-1} \in\{R\}^{\partial} \iff S \in\left(\{R\}^{\partial}\right)^{-1}.$$

Therefore,

$$\left(\{R\}^{\infty}\right)^{-1}=\left(\{R\}^{-1}\right)^{\infty} \qquad \text{and} \qquad \left(\{R\}^{\partial}\right)^{-1}=\left(\{R\}^{-1}\right)^{\partial}.$$

Hence, by Theorem 10.1, we can see that ∞ and ∂ are inversion compatible.

Analogously to Remark 10.2, we can also easily prove the following

Theorem 10.5. *For an operation* $\square$ *on relators, the following assertions are equivalent:*

(1) $\square$ *is inversion compatible,*

(2) $\left(\mathcal{R}^{\square}\right)^{-1} \subset \left(\mathcal{R}^{-1}\right)^{\square}$ *for any relator* $\mathcal{R}$ *on* X *to* Y*,*

(3) $\left(\mathcal{R}^{-1}\right)^{\square} \subset \left(\mathcal{R}^{\square}\right)^{-1}$ *for any relator* $\mathcal{R}$ *on* X *to* Y*.*

Proof. To prove that (2) implies (3), note that if (2) holds, then for any relator $\mathcal{R}$ on X to Y we also have $\left(\left(\mathcal{R}^{-1}\right)^{\square}\right)^{-1} \subset \mathcal{R}^{\square}$, and hence also $\left(\mathcal{R}^{-1}\right)^{\square} \subset \left(\mathcal{R}^{\square}\right)^{-1}$.

Theorem 10.6. *The operation* $\#$ *is also inversion compatible.*

Proof. If $\mathcal{R}$ is a relator on X to Y, then by Remark 9.11 we have

$$\operatorname{Int}_{\mathcal{R}^{-1}}=\mathcal{C} \circ \operatorname{Int}_{\mathcal{R}}^{-1} \circ \mathcal{C}.$$

Moreover, by using Remark 5.12 and Theorem 6.1, we can see that $\operatorname{Int}_{\mathcal{R}^{\#}}=\operatorname{Int}_{\mathcal{R}}$. Therefore, we also have

$$\operatorname{Int}_{(\mathcal{R}^{\#})^{-1}}=\mathcal{C} \circ \operatorname{Int}_{\mathcal{R}^{\#}}^{-1} \circ \mathcal{C}=\mathcal{C} \circ \operatorname{Int}_{\mathcal{R}}^{-1} \circ \mathcal{C}=\operatorname{Int}_{\mathcal{R}^{-1}}.$$

Hence, by using Remark 5.12, we can already infer that $\left(\mathcal{R}^{\#}\right)^{-1} \subset \left(\mathcal{R}^{-1}\right)^{\#}$. Therefore, by Theorem 10.5, the required assertion is also true.

Now, as an immediate consequence Remark 5.12, Theorems 10.6 and 9.6, Corollary 9.10 and Remarks 9.11, we can also state

Theorem 10.7. *For a relator $\mathcal{R}$ on X, the following assertions are equivalent:*

(1) $\mathcal{R}$ *is proximally symmetric,*

(2) $\operatorname{Int}_{\mathcal{R}^{-1}} = \operatorname{Int}_{\mathcal{R}}$, *(3)* $\operatorname{Int}_{\mathcal{R}}^{-1} = \mathcal{C} \circ \operatorname{Int}_{\mathcal{R}} \circ \mathcal{C}$.

Remark 10.8. Unfortunately, the important closure operations $\wedge$ and $\vartriangle$ are not inversion compatible.

Therefore, for any relator $\mathcal{R}$ on X to Y, we must also define

$$\mathcal{R}^{\vee} = \left(\mathcal{R}^{\wedge}\right)^{-1} \qquad \text{and} \qquad \mathcal{R}^{\triangledown} = \left(\mathcal{R}^{\vartriangle}\right)^{-1}.$$

However, these operations have very curious properties [?, ?].

For instance, the operations $\vee\vee$ and $\triangledown\triangledown$ coincide with the extremal closure operations defined by

$$\mathcal{R}^{\bullet} = \left\{ \delta_{\mathcal{R}} \right\}^{*}, \qquad \text{where} \qquad \delta_{\mathcal{R}} = \bigcap \mathcal{R},$$

and

$$\mathcal{R}^{\blacklozenge} = \mathcal{R} \quad \text{if} \quad \mathcal{R} = \left\{ X \times Y \right\} \qquad \text{and} \qquad \mathcal{R}^{\blacklozenge} = \mathcal{P}(X \times Y) \quad \text{if} \quad \mathcal{R} \neq \left\{ X \times Y \right\}.$$

Remark 10.9. Note that the compositions of inversion compatible operations are also inversion compatible.

Therefore, by Theorems 10.4 and 10.6, the operations $\#\infty$, $\#\partial$, and $\circledast = c\#c$ are also inversion compatible.

11. Composition compatible operations for relators

Composition compatibility properties of operations for relators have been first considered in [?] in somewhat different forms.

Definition 11.1. For an operation $\square$ for relators, we say that:

(1) $\square$ is left composition semicompatible if $\left(\mathcal{S} \circ \mathcal{R}\right)^{\square} = \left(\mathcal{S} \circ \mathcal{R}^{\square}\right)^{\square}$ for any two relators $\mathcal{R}$ on X to Y and $\mathcal{S}$ on Y to Z,

(2) $\square$ is right composition semicompatible if $\left(\mathcal{S} \circ \mathcal{R}\right)^{\square} = \left(\mathcal{S}^{\square} \circ \mathcal{R}\right)^{\square}$ for any two relators $\mathcal{R}$ on X to Y and $\mathcal{S}$ on Y to Z.

Remark 11.2. Now, the operation $\square$ may be naturally called composition compatible if it is both left and right composition semicompatible.

Note that, actually, this is also very weak composition compatibility property. However, by the following theorem, it will be sufficient for our subsequent purposes.

Theorem 11.3. *If $\square$ is a composition compatible operation for relators, then for any two relators $\mathcal{R}$ on X to Y and $\mathcal{S}$ on Y to Z we have*

$$\left(\mathcal{S} \circ \mathcal{R}\right)^{\square} = \left(\mathcal{S}^{\square} \circ \mathcal{R}^{\square}\right)^{\square}.$$

Proof. By Definition 11.1, we have $\left(\mathcal{S} \circ \mathcal{R}\right)^{\square} = \left(\mathcal{S} \circ \mathcal{R}^{\square}\right)^{\square} = \left(\mathcal{S}^{\square} \circ \mathcal{R}^{\square}\right)^{\square}$.

Remark 11.4. In this case, by Definition 11.1, we also have

$$\left(\mathcal{S}^{\square} \circ \mathcal{R}\right)^{\square} = \left(\mathcal{S} \circ \mathcal{R}^{\square}\right)^{\square}.$$

From Theorem 11.3, by using the associativity of composition, we can derive

Corollary 11.5. *If $\square$ is a composition compatible operation for relators, then for any three relators $\mathcal{R}$ on X to Y, $\mathcal{S}$ on Y to Z, and $\mathcal{T}$ on Z to W we have*

$$\big(\mathcal{T}\circ\mathcal{S}\circ\mathcal{R}\big)^{\square}=\big(\mathcal{T}^{\square}\circ\mathcal{S}^{\square}\circ\mathcal{R}^{\square}\big)^{\square}.$$

Proof. By using Theorem 11.3, we can see that

$$\big(\mathcal{T}\circ\big(\mathcal{S}\circ\mathcal{R}\big)\big)^{\square}=\Big(\mathcal{T}^{\square}\circ\big(\mathcal{S}\circ\mathcal{R}\big)^{\square}\Big)^{\square}=\Big(\mathcal{T}^{\square}\circ\big(\mathcal{S}^{\square}\circ\mathcal{R}^{\square}\big)\Big)^{\square}.$$

Remark 11.6. In this case, by using Definition 11.1, we can also prove that

$$\big(\mathcal{T}\circ\mathcal{S}\circ\mathcal{R}\big)^{\square}=\big(\mathcal{T}^{\square}\circ\mathcal{S}\circ\mathcal{R}\big)^{\square}=\big(\mathcal{T}\circ\mathcal{S}^{\square}\circ\mathcal{R}\big)^{\square}=\big(\mathcal{T}\circ\mathcal{S}\circ\mathcal{R}^{\square}\big)^{\square}.$$

However, it is now more interesting, that by using the corresponding definitions, we can also easily prove the following

Theorem 11.7. *If $\square$ is a preclosure operation for relators, then for any two relators $\mathcal{R}$ on X to Y and $\mathcal{S}$ on Y to Z we have*

(1) $\big(\mathcal{S}\circ\mathcal{R}\big)^{\square}\subset\big(\mathcal{S}\circ\mathcal{R}^{\square}\big)^{\square}\subset\big(\mathcal{S}^{\square}\circ\mathcal{R}^{\square}\big)^{\square}$,

(2) $\big(\mathcal{S}\circ\mathcal{R}\big)^{\square}\subset\big(\mathcal{S}^{\square}\circ\mathcal{R}\big)^{\square}\subset\big(\mathcal{S}^{\square}\circ\mathcal{R}^{\square}\big)^{\square}$.

Proof. By the extensivity $\square$, we have $\mathcal{R}\subset\mathcal{R}^{\square}$. Hence, by the elementwise definition of composition of relators, we can see that $\mathcal{S}\circ\mathcal{R}\subset\mathcal{S}\circ\mathcal{R}^{\square}$. Thus, by the increasingness of $\square$, we also have $\big(\mathcal{S}\circ\mathcal{R}\big)^{\square}\subset\big(\mathcal{S}\circ\mathcal{R}^{\square}\big)^{\square}$. Hence, by writing $\mathcal{S}^{\square}$ in place of $\mathcal{S}$, we can see that $\big(\mathcal{S}^{\square}\circ\mathcal{R}\big)^{\square}\subset\big(\mathcal{S}^{\square}\circ\mathcal{R}^{\square}\big)^{\square}$. Therefore, the first part of (1) and the second part of (2) are true.

From this theorem, by using Definition 11.1, we can immediately derive

Corollary 11.8. *If $\square$ is a preclosure operation for relators, then*

(1) $\square$ is left composition semicompatible if and only if $\big(\mathcal{S}\circ\mathcal{R}^{\square}\big)^{\square}\subset\big(\mathcal{S}\circ\mathcal{R}\big)^{\square}$ for any two relators $\mathcal{R}$ on X to Y and $\mathcal{S}$ on Y to Z,

(2) $\square$ is right composition semicompatible if and only if $\big(\mathcal{S}^{\square}\circ\mathcal{R}\big)^{\square}\subset\big(\mathcal{S}\circ\mathcal{R}\big)^{\square}$ for any two relators R on X to Y and $\mathcal{S}$ on Y to Z.

Hence, by Theorem 6.3, it is clear that in particular we also have

Corollary 11.9. *If $\square$ is a closure operation for relators, then*

(1) $\square$ is left composition semicompatible if and only if $\mathcal{S}\circ\mathcal{R}^{\square}\subset\big(\mathcal{S}\circ\mathcal{R}\big)^{\square}$ for any two relators $\mathcal{R}$ on X to Y and $\mathcal{S}$ on Y to Z,

(2) $\square$ is right composition semicompatible if and only if $\mathcal{S}^{\square}\circ\mathcal{R}\subset\big(\mathcal{S}\circ\mathcal{R}\big)^{\square}$ for any two relators $\mathcal{R}$ on X to Y and $\mathcal{S}$ on Y to Z.

Remark 11.10. In addition to the above results, it is also worth noticing that an involution operation $\square$ for relators is left composition semicompatible if and only if $\mathcal{S}\circ\mathcal{R}=\mathcal{S}\circ\mathcal{R}^{\square}$ for any two relators $\mathcal{R}$ on X to Y and $\mathcal{S}$ on Y to Z.

Moreover, since $\mathcal{S}\circ\mathcal{R}=\bigcup_{S\in\mathcal{S}}S\circ\mathcal{R}$ holds, we can also at once state that an involution operation $\square$ for relators is left composition semicompatible if and only if $S\circ\mathcal{R}=S\circ\mathcal{R}^{\square}$ for any relator $\mathcal{R}$ on X to Y and relation S on Y to Z.

12. Composition compatibilities of the basic closure operations

Now, by using Corollary 11.9 and Theorem 8.4, we can also prove the following

Theorem 12.1. *If $\square$ is a closure operation for relators, then*

(1) $\square$ is left composition semicompatible if and only if $S \circ \mathcal{R}^{\square} \subset \big(S \circ \mathcal{R} \big)^{\square}$ for any relator $\mathcal{R}$ on X to Y and relation S on Y to Z,

(2) $\square$ is right composition semicompatible if and only if $\mathcal{S}^{\square} \circ R \subset \big(\mathcal{S} \circ R \big)^{\square}$ for any relation R on X to Y and relator $\mathcal{S}$ on Y to Z.

Proof. If $\square$ is left composition semicompatible, then by Corollary 11.9, for any relator $\mathcal{R}$ and relation S on Y to Z, we have $\{S\} \circ \mathcal{R}^{\square} \subset \big(\{S\} \circ \mathcal{R} \big)^{\square}$, and thus $S \circ \mathcal{R}^{\square} \subset \big(S \circ \mathcal{R} \big)^{\square}$. Therefore, the "only if part" of (1) is true.

Conversely, if $\mathcal{R}$ is a relator on X to Y and $\mathcal{S}$ is a relator on Y to Z, and the inclusion $S \circ \mathcal{R}^{\square} \subset \big(S \circ \mathcal{R} \big)^{\square}$ holds for any relation S on Y to Z, then by using the corresponding definitions and Theorem 8.4 we can see that

$$\mathcal{S} \circ \mathcal{R}^{\square} = \bigcup_{S \in \mathcal{S}} S \circ \mathcal{R}^{\square} \subset \bigcup_{S \in \mathcal{S}} \big(S \circ \mathcal{R} \big)^{\square}$$

$$\subset \Big(\bigcup_{S \in \mathcal{S}} \big(S \circ \mathcal{R} \big)^{\square} \Big)^{\square} = \Big(\bigcup_{S \in \mathcal{S}} S \circ \mathcal{R} \Big)^{\square} = \big(\mathcal{S} \circ \mathcal{R} \big)^{\square}.$$

Therefore, by Corollary 11.9, the "if part" of (1) is also true.

By using this theorem, we can somewhat more easily establish the composition compatibility properties of the basic closure operations considered in Section 2.

Theorem 12.2. *The operations $*$ and $\#$ are composition compatible.*

Proof. To prove right composition semicompatibility of $\#$, by Theorem 12.1, it is enough to prove only that, for any relation R on X to Y and relator $\mathcal{S}$ on Y to Z, we have $\mathcal{S}^{\#} \circ R \subset \big(\mathcal{S} \circ R \big)^{\#}$.

For this, suppose that $W \in \mathcal{S}^{\#} \circ R$ and $A \subset X$. Then, there exists $V \in \mathcal{S}^{\#}$ such that $W = V \circ R$. Moreover, there exists $S \in \mathcal{S}$ such that $S \big[R[A] \big] \subset V \big[R[A] \big]$, and thus $(S \circ R)[A] \subset (V \circ R)[A] = W[A]$. Hence, by taking $U = S \circ R$, we can see that $U \in \mathcal{S} \circ R$ such that $U[A] \subset W[A]$. Therefore, $W \in \big(\mathcal{S} \circ R \big)^{\#}$ also holds.

Theorem 12.3. *The operations $\wedge$ and $\vartriangle$ are left composition semicompatible.*

Proof. To prove left composition semicompatibility of $\vartriangle$, by Theorem 12.1, it is enough to prove only that, for any relator $\mathcal{R}$ on X to Y and relation S on Y to Z, we have $S \circ \mathcal{R}^{\vartriangle} \subset \big(S \circ \mathcal{R} \big)^{\vartriangle}$.

For this, suppose that $W \in S \circ \mathcal{R}^{\vartriangle}$ and $x \in X$. Then, there exists $V \in \mathcal{R}^{\vartriangle}$ such that $W = S \circ V$. Moreover, there exist $u \in X$ and $R \in \mathcal{R}$ such that $R(u) \subset V(x)$. Hence, we can infer that

$$(S \circ R)(u) = S \big[R(u) \big] \subset S \big[V(x) \big] = (S \circ V)(x) = W(x).$$

Now, by taking $U = S \circ R$, we can see that $U \in S \circ \mathcal{R}$ such that $U(u) \subset W(x)$. Therefore, $W \in \big(S \circ R \big)^{\vartriangle}$ also holds.

Instead of the right composition compatibility of the operations $\wedge$ and $\vartriangle$, we can only prove the following

Theorem 12.4. *For any two relators* $\mathcal{R}$ *on* X *to* Y *and* $\mathcal{S}$ *on* Y *to* Z, *we have*

(1) $\big(\mathcal{S}\circ\mathcal{R}\big)^{\wedge}=\big(\mathcal{S}^{\#}\circ\mathcal{R}\big)^{\wedge}$, *(2)* $\big(\mathcal{S}\circ\mathcal{R}\big)^{\vartriangle}=\big(\mathcal{S}^{\#}\circ\mathcal{R}\big)^{\vartriangle}$.

Proof. By the extensivity $\#$, we have $\mathcal{S}\subset\mathcal{S}^{\#}$. Hence, by the elementwise definition of composition of relators, we can see that $\mathcal{S}\circ\mathcal{R}\subset\mathcal{S}^{\#}\circ\mathcal{R}$. Thus, by the increasingness of $\wedge$, we also have $\big(\mathcal{S}\circ\mathcal{R}\big)^{\wedge}\subset\big(\mathcal{S}^{\#}\circ\mathcal{R}\big)^{\wedge}$.

To get the converse inclusion, by Theorem 6.3, it is now enough to prove only that $S^{\#}\circ\mathcal{R}\subset\big(\mathcal{S}\circ\mathcal{R}\big)^{\wedge}$. For this, suppose that $W\in S^{\#}\circ\mathcal{R}$ and $x\in X$. Then, there exists $V\in\mathcal{S}^{\#}$ and $R\in\mathcal{R}$ such that $W=V\circ R$. Moreover, there exists $S\in\mathcal{S}$, such that $S\big[R(x)\big]\subset V\big[R(x)\big]$, and thus $(S\circ R)(x)\subset(V\circ R)(x)=W(x)$. Hence, by taking $U=S\circ R$, we can see that $U\in\mathcal{S}\circ\mathcal{R}$ such that $U(x)\subset W(x)$. Therefore, $W\in\big(\mathcal{S}\circ\mathcal{R}\big)^{\wedge}$ also holds.

Thus, we have proved (1). Assertion (2) can now be immediately derived from (1) by using that the operation $\vartriangle$ is $\wedge$–absorbing in the sense that $\mathcal{U}^{\wedge\vartriangle}=\mathcal{U}^{\vartriangle}$ for any relator $\mathcal{U}$ on X to Z.

From this theorem, by using Theorem 12.3, we can immediately derive

Corollary 12.5. *For any two relators* $\mathcal{R}$ *on* X *to* Y *and* $\mathcal{S}$ *on* Y *to* Z, *we have*

(1) $\big(\mathcal{S}\circ\mathcal{R}\big)^{\wedge}=\big(\mathcal{S}^{\#}\circ\mathcal{R}^{\wedge}\big)^{\wedge}$, *(2)* $\big(\mathcal{S}\circ\mathcal{R}\big)^{\vartriangle}=\big(\mathcal{S}^{\#}\circ\mathcal{R}^{\vartriangle}\big)^{\vartriangle}$.

Remark 12.6. By using Theorem 12.1, we can also somewhat more easily prove that the operation $\circledast$, defined in Remark 8.2, is also composition compatible.

13. Seminormal functions

In [?], slightly extending the ideas of Ore [?], Schmidt [?, p. 209], Blyth and Janowitz [?, p. 11], and the present author [?] on Galois connections, residuated mappings, and normal functions, we have introduced the following definition in a somewhat different form.

Definition 13.1. Let X and Y be gosets. Moreover, let f be a function of X to Y and g be a function of Y to X.

Then, we say that:

(1) f is upper g–seminormal if $f(x)\leq y$ implies $x\leq g(y)$ for all $x\in X$ and $y\in Y$,

(2) f is lower g–seminormal if $x\leq g(y)$ implies $f(x)\leq y$ for all $x\in X$ and $y\in Y$.

Now, by writing R and S in place of the inequality relations in X and Y, respectively, we can see that f is upper g–normal if and only if $f(x)\,R\,y$ implies $x\,S\,g(y)$ for all $x\in X$ and $y\in Y$.

This shows an obvious way to a straightforward generalization of Definition 13.1 to relator spaces.

Definition 13.2. Let $(X,Y)(\mathcal{R})$ and $(Z,W)(\mathcal{S})$ be relator spaces. Moreover, let f be a function of X to Z and g be a function of W to Y.

Then, we say that:

(1) f is upper g–seminormal if for each $S \in \mathcal{S}$ there exists $R \in \mathcal{R}$ such that $f(x)\,S\,w$ implies $x\,R\,g(w)$ for all $x \in X$ and $w \in W$,

(2) f is lower g–seminormal if for each $R \in \mathcal{R}$ there exists $S \in \mathcal{S}$ such that $x\,R\,g(w)$ implies $f(x)\,S\,w$ for all $x \in X$ and $w \in W$.

Hence, by noticing the pairwise equivalence of the implications

$$f(x)\,S\,w \implies x\,R\,g(w)\,,$$
$$w \in S\big(f(x)\big) \implies g(w) \in R(x)\,,$$
$$w \in S\big(f(x)\big) \implies w \in g^{-1}\,[\,R(x)\,]\,,$$
$$w \in \big(S \circ f\big)(x) \implies w \in \big(g^{-1} \circ R\big)(x)\,,$$

and the inclusion $\big(S \circ f\big)(x) \subset \big(g^{-1} \circ R\big)(x)$, we can easily get following concise reformulation of Definition 13.2.

Theorem 13.3. *Under the notations of Definition 13.2,*

(1) f is upper g–seminormal if and only if for each $S \in \mathcal{S}$ there exists $R \in \mathcal{R}$ such that $S \circ f \subset g^{-1} \circ R$,

(2) f is lower g–seminormal if and only if for each $R \in \mathcal{R}$ there exists $S \in \mathcal{S}$ such that $g^{-1} \circ R \subset S \circ f$.

Hence, by using the composition of relators, and the operation $\circledast$, defined in Remark 8.2 such that $\circledast = c * c$, and thus

$$\mathcal{R}^{\circledast} = \big\{\, S \subset X \times Y : \quad \exists\ R \in \mathcal{R} : \quad S \subset R \,\big\}$$

for any relator $\mathcal{R}$ on X to Y, we can easily get the following concise reformulation of Theorem 13.3.

Theorem 13.4. *Under the notations of Definition 14.2,*

(1) f is upper g–seminormal if and only if $\mathcal{S} \circ f \subset \big(g^{-1} \circ \mathcal{R}\big)^{\circledast}$,

(2) f is lower g–seminormal if and only if $g^{-1} \circ \mathcal{R} \subset \big(\mathcal{S} \circ f\,\big)^{\circledast}$.

Hence, by using that $\circledast$ is also a composition compatible closure operation for relators, we can immediately derive the following theorem which already shows the $\circledast$–invariance of Definition 13.2.

Theorem 13.5. *Under the notations of Definition 13.2,*

(1) f is upper g–seminormal if and only if $\big(\mathcal{S}^{\circledast} \circ f\,\big)^{\circledast} \subset \big(g^{-1} \circ \mathcal{R}^{\circledast}\big)^{\circledast}$,

(2) f is lower g–seminormal if and only if $\big(g^{-1} \circ \mathcal{R}^{\circledast}\big)^{\circledast} \subset \big(\mathcal{S}^{\circledast} \circ f\,\big)^{\circledast}$.

Remark 13.6. Hence, by using [**?**, Definition 4.1], we can already see that

(1) the function f is upper g–seminormal if and only if the relation pair $\big(f\,,\, g^{-1}\big)$ is upper $\circledast$–semicontinuous,

(2) the function f is lower g–seminormal if and only if the relation pair $\big(f^{-1},\, g\big)$ is lower $\circledast$–semicontinuous.

This shows that, in accordance with [**?**], upper and lower Galois connections (i. e., upper and lower seminormal functions) are very particular cases of upper and lower semicontinuous pairs of relations, respectively.

14. Semiregular functions

In [?], slightly extending the ideas of Pataki [?] and the present author [?], we have also introduced the following definition in a somewhat different form.

Definition 14.1. Let X and Y be gosets. Moreover, let f be a function of X to Y, and φ be a function of X to itself.

Then, we say that:

(1) f is upper φ–semiregular if $f(u) \leq f(v)$ implies $u \leq \varphi(v)$ for all $u, v \in X$,

(2) f is lower φ–semiregular if $u \leq \varphi(v)$ implies $f(u) \leq f(v)$ for all $u, v \in X$.

Now, by writing R and S in place of the inequality relations in X and Y, respectively, we can see that f is upper φ–semiregular if and only if $f(u)\,S\,f(v)$ implies $u\,R\,\varphi(v)$ for all $u, v \in X$.

This shows an obvious way to a straightforward generalization of Definition 14.1 to relator spaces.

Definition 14.2. Let $(X, Y)(\mathcal{R})$ and $Z(\mathcal{S})$ be relator spaces. Moreover, let f be a function of X to Z, and φ be a function of X to Y.

Then, we say that:

(1) f is upper φ–semiregular if for each $S \in \mathcal{S}$ there exists $R \in \mathcal{R}$ such that $f(u)\,S\,f(v)$ implies $u\,R\,\varphi(v)$ for all $u, v \in X$,

(2) f is lower φ–semiregular if for each $R \in \mathcal{R}$ there exists $S \in \mathcal{S}$ such that $u\,R\,\varphi(v)$ implies $f(u)\,S\,f(v)$ for all $u, v \in X$.

Now, by noticing the pairwise equivalence of the implications

$$\begin{aligned} f(u)\,S\,f(v) &\implies u\,R\,\varphi(v)\,, \\ f(v) \in S\big(f(u)\big) &\implies \varphi(v) \in R(u)\,, \\ v \in f^{-1}\big[\,S\big(f(u)\big)\big] &\implies v \in \varphi^{-1}\big[\,R(u)\,\big]\,, \\ v \in \big(f^{-1}\circ S \circ f\big)(u) &\implies v \in \big(\varphi^{-1}\circ R\big)(u)\,, \end{aligned}$$

and the inclusion $\big(f^{-1}\circ S \circ f\big)(u) \subset \big(\varphi^{-1}\circ R\big)(u)$, we can easily get following concise reformulation of Definition 14.2.

Theorem 14.3. *Under the notations of Definition 14.2,*

(1) f is upper φ–semiregular if and only if for each $S \in \mathcal{S}$ there exists $R \in \mathcal{R}$ such that $f^{-1}\circ S \circ f \subset \varphi^{-1}\circ R$,

(2) f is lower φ–semiregular if and only if for each $R \in \mathcal{R}$ there exists $S \in \mathcal{S}$ such that $\varphi^{-1}\circ R \subset f^{-1}\circ S \circ f$.

Now, by using the composition of relators, and the operation $\circledast$ defined in Remark 8.2, we can easily establish the following concise reformulation of Theorem 14.3.

Theorem 14.4. *Under the notations of Definition 14.2,*

(1) f is upper φ–semiregular if and only if $f^{-1}\circ \mathcal{S} \circ f \subset \big(\varphi^{-1}\circ \mathcal{R}\big)^{\circledast}$,

(2) f is lower φ–semiregular if and only if $\varphi^{-1}\circ \mathcal{R} \subset \big(f^{-1}\circ \mathcal{S} \circ f\big)^{\circledast}$.

Hence, by using that $\circledast$ is also a composition compatible closure operation for relators, we can immediately derive the following theorem which already shows the $\circledast$–invariance of Definition 14.2.

Theorem 14.5. *Under the notations of Definition 14.2,*

(1) f *is upper* φ*–semiregular if and only if* $\big(f^{-1}\circ\mathcal{S}^{\circledast}\circ f\big)^{\circledast}\subset\big(\varphi^{-1}\circ\mathcal{R}^{\circledast}\big)^{\circledast}$,

(2) f *is lower* φ*–semiregular if and only if* $\big(\varphi^{-1}\circ\mathcal{R}^{\circledast}\big)^{\circledast}\subset\big(f^{-1}\circ\mathcal{S}^{\circledast}\circ f\big)^{\circledast}$.

Remark 14.6. Hence, by using [**?**, Definition 10.1], and an plausible supplement of it, we can also see that

(1) f is upper φ–semiregular if and only if it is mildly $\circledast$–continuous with respect to the relators $\varphi^{-1}\circ\mathcal{R}^{\circledast}$ and $\mathcal{S}$,

(2) f is lower φ–semiregular if and only if it is mildly $\circledast$–contracontinuous with respect to the relators $\varphi^{-1}\circ\mathcal{R}^{\circledast}$ and $\mathcal{S}$.

Note that now we have $\big(\varphi^{-1}\circ\mathcal{R}\big)^{\circledast}=\big(\varphi^{-1}\circ\mathcal{R}^{\circledast}\big)^{\circledast}$. Therefore, in the above assertions, we could also write $\varphi^{-1}\circ\mathcal{R}$ in place of $\varphi^{-1}\circ\mathcal{R}^{\circledast}$.

Remark 14.7. The above remark shows that, in accordance with [**?**], upper and lower Pataki connections (i.e., upper and lower semiregular functions) are very particular cases of mildly and contra mildly continuous relations, respectively.

Unfortunately, in [**?**] we have not recognized the importance of the operation $\circledast$. Moreover, in [**?**], we have used a reverse terminology. This also shows that the right definitions can usually be found only in the context of relator spaces.

Remark 14.8. For instance, the most natural definitions of Cauchy, completeness, compactness, well-chainedness, and connectedness properties have been given in [**?**] and [**?**]. (The results of [**?**] have to be also generalized and supplemented.)

It has turned out that "convergent" and "Cauchy" are actually equivalent. Moreover, "compact" and "connected" are particular cases of "precompact" and "well-chained", respectively. And, "well-chainedness" is a particular case of "simplicity".

15. Seminormal relators

Now, by Theorem 13.5, we may naturally introduce the following general definition which actually makes seminormalities to be equivalent to semicontinuities.

Definition 15.1. Let $(X, Y)(\mathcal{R})$ and $(Z, W)(\mathcal{S})$ be relator spaces. Moreover, let $\mathcal{F}$ be a relator on X to Z and $\mathcal{G}$ be a relator on W to Y. Furthermore, assume that $\square$ is a direct unary operation for relators.

Then, we shall say that:

(1) $\mathcal{F}$ is *upper* $\square$*–*$\mathcal{G}$*–seminormal* if $\big(\mathcal{S}^{\square}\circ\mathcal{F}^{\square}\big)^{\square}\subset\Big(\big(\mathcal{G}^{\square}\big)^{-1}\circ\mathcal{R}^{\square}\Big)^{\square}$,

(2) $\mathcal{F}$ is *lower* $\square$*–*$\mathcal{G}$*–seminormal* if $\Big(\big(\mathcal{G}^{\square}\big)^{-1}\circ\mathcal{R}^{\square}\Big)^{\square}\subset\big(\mathcal{S}^{\square}\circ\mathcal{F}^{\square}\big)^{\square}$.

Remark 15.2. Now, $\mathcal{F}$ may be naturally called $\square$*–*$\mathcal{G}$*–normal* if it is both upper and lower $\square$–$\mathcal{G}$–seminormal. Moreover, for instance $\mathcal{F}$ may be naturally called $\square$*–normal* if it is $\square$–$\mathcal{G}$–normal for some relator $\mathcal{G}$ on W to Y.

Thus, in accordance with our former terminology, $\mathcal{F}$ may, for instance, be naturally called *properly, uniformly, proximally, topologically, and paratopologically normal* if it is $\square$–normal with $\square=\Delta, *, \#, \wedge$, and $\vartriangle$, respectively.

From Definition 15.1, by using Theorems 6.3 and 11.3, we can immediately derive the following two theorems.

Theorem 15.3. *In in particular* $\Box$ *is a closure operation, then*

(1) $\mathcal{F}$ *is upper* $\Box$–$\mathcal{G}$*–seminormal* $\iff$ $\mathcal{S}^{\Box} \circ \mathcal{F}^{\Box} \subset \Big(\big(\mathcal{G}^{\Box} \big)^{-1} \circ \mathcal{R}^{\Box} \Big)^{\Box}$,

(2) $\mathcal{F}$ *is lower* $\Box$–$\mathcal{G}$*–seminormal* $\iff$ $\big(\mathcal{G}^{\Box} \big)^{-1} \circ \mathcal{R}^{\Box} \subset \big(\mathcal{S}^{\Box} \circ \mathcal{F}^{\Box} \big)^{\Box}$.

Theorem 15.4. *In in particular* $\Box$ *is an inversion and composition compatible operation, then*

(1) $\mathcal{F}$ *is upper* $\Box$–$\mathcal{G}$*–seminormal* $\iff$ $(\mathcal{S} \circ \mathcal{F})^{\Box} \subset \big(\mathcal{G}^{-1} \circ \mathcal{R} \big)^{\Box}$,

(2) $\mathcal{F}$ *is lower* $\Box$–$\mathcal{G}$*–seminormal* $\iff$ $\big(\mathcal{G}^{-1} \circ \mathcal{R} \big)^{\Box} \subset (\mathcal{S} \circ \mathcal{F})^{\Box}$.

Hence, it is clear that in particular we also have

Corollary 15.5. *In in particular* $\Box$ *is an inversion and composition compatible closure operation, then*

(1) $\mathcal{F}$ *is upper* $\Box$–$\mathcal{G}$*–seminormal* $\iff$ $\mathcal{S} \circ \mathcal{F} \subset \big(\mathcal{G}^{-1} \circ \mathcal{R} \big)^{\Box}$,

(2) $\mathcal{F}$ *is lower* $\Box$–$\mathcal{G}$*–seminormal* $\iff$ $\mathcal{G}^{-1} \circ \mathcal{R} \subset (\mathcal{S} \circ \mathcal{F})^{\Box}$.

Remark 15.6. In addition to the above results, it is also worth noticing that if in particular $\Box$ is an increasing involution operation, then

(1) $\mathcal{F}$ is upper $\Box$–$\mathcal{G}$–seminormal $\iff$ $\mathcal{S}^{\Box} \circ \mathcal{F}^{\Box} \subset \big(\mathcal{G}^{\Box} \big)^{-1} \circ \mathcal{R}^{\Box}$,

(2) $\mathcal{F}$ is lower $\Box$–$\mathcal{G}$–seminormal $\iff$ $\big(\mathcal{G}^{\Box} \big)^{-1} \circ \mathcal{R}^{\Box} \subset \mathcal{S}^{\Box} \circ \mathcal{F}^{\Box}$.

Now, as a straightforward extension of the corresponding results of Section 13, we can also easily prove the following

Theorem 15.7. *Under the notations of Definition 15.1, the following assertions are equivalent:*

(1) $\mathcal{F}$ *is upper* $\circledast$–$\mathcal{G}$*–seminormal*,

(2) *for any* $S \in \mathcal{S}$ *and* $F \in \mathcal{F}$ *there exist* $G \in \mathcal{G}$ *and* $R \in \mathcal{R}$ *such that* $S \circ F \subset G^{-1} \circ R$,

(3) *for any* $S \in \mathcal{S}$ *and* $F \in \mathcal{F}$ *there exist* $G \in \mathcal{G}$ *and* $R \in \mathcal{R}$ *such that* $F(x) \cap S^{-1}(w) \neq \emptyset$ *implies* $G(w) \cap R(x) \neq \emptyset$ *for all* $x \in X$ *and* $w \in W$.

Proof. To prove the implications (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3), not that if (1) holds, then by Remarks 8.2, 10.9 and 12.6, and Corollary 15.5, we have

$$\mathcal{S} \circ \mathcal{F} \subset \big(\mathcal{G}^{-1} \circ \mathcal{R} \big)^{\circledast}.$$

Thus, by the corresponding definitions, for any $S \in \mathcal{S}$ and $F \in \mathcal{F}$ there exist $G \in \mathcal{G}$ and $R \in \mathcal{R}$ such that

$$S \circ F \subset G^{-1} \circ R.$$

Hence, we can infer that

$$\big(S \circ F \big)(x) \subset \big(G^{-1} \circ R \big)(x), \qquad \text{and thus} \qquad S \big[F(x) \big] \subset G^{-1} \big[R(x) \big]$$

for all $x \in X$. Therefore,

$$w \in S \big[F(x) \big] \quad \Longrightarrow \quad w \in G^{-1} \big[R(x) \big],$$

and thus

$$S^{-1}(w) \cap F(x) \neq \emptyset \quad \Longrightarrow \quad G(w) \cap R(x) \neq \emptyset$$

for all $x \in X$ and $w \in W$.

From this theorem, it is clear that in particular we also have

Corollary 15.8. *If in particular each member of the families $\mathcal{F}$ and $\mathcal{G}$ is a function, then the following assertions are equivalent:*

(1) $\mathcal{F}$ is upper $\circledast$–$\mathcal{G}$–seminormal,

(2) for any $S \in \mathcal{S}$ and $f \in \mathcal{F}$ there exist $g \in \mathcal{G}$ and $R \in \mathcal{R}$ such that $f(x) S w$ implies $x\, R\, g(w)$ for all $x \in X$ and $w \in W$.

Now, analogously to Theorem 15.7, we can also prove the following

Theorem 15.9. *Under the notations of Definition 15.1, the following assertions are equivalent:*

(1) $\mathcal{F}$ is lower $\circledast$–$\mathcal{G}$–seminormal,

(2) for any $G \in \mathcal{G}$ and $R \in \mathcal{R}$ there exist $S \in \mathcal{S}$ and $F \in \mathcal{F}$ such that $G^{-1} \circ R \subset S \circ F$,

(3) for any $G \in \mathcal{G}$ and $R \in \mathcal{R}$ there exist $S \in \mathcal{S}$ and $F \in \mathcal{F}$ such that $G(w) \cap R(x) \neq \emptyset$ implies $F(x) \cap S^{-1}(w) \neq \emptyset$ for all $x \in X$ and $w \in W$.

Hence, it is clear that in particular we also have

Corollary 15.10. *If in particular each member of the families $\mathcal{F}$ and $\mathcal{G}$ is a function function, then the following assertions are equivalent:*

(1) $\mathcal{F}$ is lower $\circledast$–$\mathcal{G}$–seminormal,

(2) for any $g \in \mathcal{G}$ and $R \in \mathcal{R}$ there exist $S \in \mathcal{S}$ and $f \in \mathcal{F}$ such that $x\, R\, g(w)$ implies $f(x) S\, w$ for all $x \in X$ and $w \in W$.

16. Semiregular relators

Now, by Theorem 14.5, we may naturally introduce the following general definition which brings semiregulaties quite close to mild continuities.

Definition 16.1. Let $(X, Y)(\mathcal{R})$ and $Z(\mathcal{S})$ be relator spaces. Moreover, let $\mathcal{F}$ be a relator on X to Z and Φ be a relator on X to Y. Furthermore, assume that $\square$ is a direct unary operation for relators.

Then, we shall say that:

(1) $\mathcal{F}$ is *upper $\square$–Φ–semiregular* if $\Big(\big(\mathcal{F}^{\square}\big)^{-1} \circ \mathcal{S}^{\square} \circ \mathcal{F}^{\square} \Big)^{\square} \subset \Big(\big(\Phi^{\square}\big)^{-1} \circ \mathcal{R}^{\square} \Big)^{\square}$,

(2) $\mathcal{F}$ is *lower $\square$–Φ–semiregular* if $\Big(\big(\Phi^{\square}\big)^{-1} \circ \mathcal{R}^{\square} \Big)^{\square} \subset \Big(\big(\mathcal{F}^{\square}\big)^{-1} \circ \mathcal{S}^{\square} \circ \mathcal{F}^{\square} \Big)^{\square}$.

Remark 16.2. Now, $\mathcal{F}$ may be naturally called *$\square$–Φ–regular* if it is both upper and lower $\square$–Φ–semiregular. Moreover, for instance $\mathcal{F}$ may be naturally called *$\square$–regular* if it is $\square$–Φ–regular for some relator Φ on X to Y.

Thus, in accordance with our usual terminology, $\mathcal{F}$ may, for instance, be naturally called *properly, uniformly, proximally, topologically, and paratopologically regular* if it is $\square$–regular with $\square = \Delta, *, \#, \wedge,$ and $\vartriangle$, respectively.

From Definition 16.1, by using Theorems 6.3 and 11.3 and Corollary 11.5, we can immediately derive the following two theorems.

Theorem 16.3. *In in particular $\square$ is a closure operation, then*

(1) $\mathcal{F}$ is upper $\square$–Φ–semiregular $\iff$ $\big(\mathcal{F}^{\square}\big)^{-1} \circ \mathcal{S}^{\square} \circ \mathcal{F}^{\square} \subset \Big(\big(\Phi^{\square}\big)^{-1} \circ \mathcal{R}^{\square} \Big)^{\square}$,

(2) $\mathcal{F}$ *is lower* $\Box$–Φ–*semiregular* $\iff$ $\big(\Phi^{\Box}\big)^{-1}\circ\mathcal{R}^{\Box}\subset\Big(\big(\mathcal{F}^{\Box}\big)^{-1}\circ\mathcal{S}^{\Box}\circ\mathcal{F}^{\Box}\Big)^{\Box}$.

Theorem 16.4. *In in particular* $\Box$ *is an inversion and composition compatible operation, then*

(1) $\mathcal{F}$ *is upper* $\Box$–Φ–*semiregular* $\iff$ $\big(\mathcal{F}^{-1}\circ\mathcal{S}\circ\mathcal{F}\big)^{\Box}\subset\big(\Phi^{-1}\circ\mathcal{R}\big)^{\Box}$,

(2) $\mathcal{F}$ *is lower* $\Box$–Φ–*semiregular* $\iff$ $\big(\Phi^{-1}\circ\mathcal{R}\big)^{\Box}\subset\big(\mathcal{F}^{-1}\circ\mathcal{S}\circ\mathcal{F}\big)^{\Box}$.

Hence, it is clear that in particular we also have

Corollary 16.5. *In in particular* $\Box$ *is an inversion and composition compatible closure operation, then*

(1) $\mathcal{F}$ *is upper* $\Box$–Φ–*semiregular* $\iff$ $\mathcal{F}^{-1}\circ\mathcal{S}\circ\mathcal{F}\subset\big(\Phi^{-1}\circ\mathcal{R}\big)^{\Box}$,

(2) $\mathcal{F}$ *is lower* $\Box$–Φ–*semiregular* $\iff$ $\Phi^{-1}\circ\mathcal{R}\subset\big(\mathcal{F}^{-1}\circ\mathcal{S}\circ\mathcal{F}\big)^{\Box}$.

Remark 16.6. In addition to the above results, it is also worth noticing that if in particular $\Box$ is an increasing involution operation, then

(1) $\mathcal{F}$ is upper $\Box$–Φ–semiregular $\iff$ $\big(\mathcal{F}^{\Box}\big)^{-1}\circ\mathcal{S}^{\Box}\circ\mathcal{F}^{\Box}\subset\big(\Phi^{\Box}\big)^{-1}\circ\mathcal{R}^{\Box}$,

(2) $\mathcal{F}$ is lower $\Box$–Φ–semiregular $\iff$ $\big(\Phi^{\Box}\big)^{-1}\circ\mathcal{R}^{\Box}\subset\big(\mathcal{F}^{\Box}\big)^{-1}\circ\mathcal{S}^{\Box}\circ\mathcal{F}^{\Box}$.

Now, as a straightforward extension of the corresponding results of Section 14, we can also easily prove the following

Theorem 16.7. *Under the notations of Definition 16.1, the following assertions are equivalent:*

(1) $\mathcal{F}$ *is upper* $\circledast$–Φ–*semiregular,*

(2) for any $F_1,\,F_2\in\mathcal{F}$ *and* $S\in\mathcal{S}$ *there exist* $\phi\in\Phi$ *and* $R\in\mathcal{R}$ *such that* $F_2^{-1}\circ S\circ F_1\subset\phi^{-1}\circ R$,

(3) for any $F_1,\,F_2\in\mathcal{F}$ *and* $S\in\mathcal{S}$ *there exist* $\phi\in\Phi$ *and* $R\in\mathcal{R}$ *such that* $F_2(v)\cap S\big[F_1(u)\big]\neq\emptyset$ *implies* $\phi(v)\cap R(u)\neq\emptyset$ *for all* $u,\,v\in X$.

Proof. To prove the implications (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3), not that if (1) holds, then by Remarks 8.2, 10.9 and 12.6, and Corollary 16.5 we have

$$\mathcal{F}^{-1}\circ\mathcal{S}\circ\mathcal{F}\subset\big(\Phi^{-1}\circ\mathcal{R}\big)^{\circledast}$$

Thus, by the corresponding definitions, for any $F_1,\,F_2\in\mathcal{F}$ and $S\in\mathcal{S}$ there exist $\phi\in\Phi$ and $R\in\mathcal{R}$ such that

$$F_2^{-1}\circ S\circ F_1\subset\phi^{-1}\circ R\,.$$

Hence, we can infer that

$$\big(F_2^{-1}\circ S\circ F_1\big)(u)\subset\big(\phi^{-1}\circ R\big)(u)\,,\quad\text{and thus}\quad F_2^{-1}\,\big[\,S\,\big[\,F_1(u)\,\big]\,\big]\subset\phi^{-1}\,\big[\,R(u)\,\big]$$

for all $u\in X$. Therefore,

$$v\in F_2^{-1}\,\big[\,S\,\big[\,F_1(u)\,\big]\,\big]\quad\Longrightarrow\quad v\in\phi^{-1}\,\big[\,R(u)\,\big]$$

and thus

$$F_2(v)\cap S\big[\,F_1(u)\,\big]\neq\emptyset\quad\Longrightarrow\quad\phi(v)\cap R(u)\neq\emptyset$$

for all $u,\,v\in X$.

From this theorem, it is clear that in particular we also have

Corollary 16.8. *If in particular each member of the families* $\mathcal{F}$ *and* Φ *is a function, then the following assertions are equivalent:*

(1) $\mathcal{F}$ *is upper* $\circledast$–Φ*–semiregular,*

(2) for any $f_1,\, f_2 \in \mathcal{F}$ *and* $S \in \mathcal{S}$ *there exist* $\varphi \in \Phi$ *and* $R \in \mathcal{R}$ *such that* $f_1(u)\, S\, f_2(v)$ *implies* $u\, R\, \varphi(v)$ *for all* $u,\, v \in X$.

Now, analogously to Theorem 16.7, we can also prove the following

Theorem 16.9. *Under the notations of Definition 16.1, the following assertions are equivalent:*

(1) $\mathcal{F}$ *is lower* $\circledast$–Φ*–semiregular,*

(2) for any $\phi \in \Phi$ *and* $R \in \mathcal{R}$ *there exist* $F_1, F_2 \in \mathcal{F}$ *and* $S \in \mathcal{S}$ *such that* $\phi^{-1} \circ R \subset F_2^{-1} S \circ F_1$,

(3) for any $\phi \in \Phi$ *and* $R \in \mathcal{R}$ *there exist* $F_1, F_2 \in \mathcal{F}$ *and* $S \in \mathcal{S}$ *such that* $\phi(v) \cap R(x) \neq \emptyset$ *implies* $F_2(v) \cap S\,\big[\, F_1(u)\,\big] \neq \emptyset$ *for all* $u,\, v \in X$.

Hence, it is clear that in particular we also have

Corollary 16.10. *If in particular each member of the families* $\mathcal{F}$ *and* $\mathcal{G}$ *is a function function, then the following assertions are equivalent:*

(1) $\mathcal{F}$ *is lower* $\circledast$–Φ*–semiregular,*

(2) for any $\varphi \in \Phi$ *and* $R \in \mathcal{R}$ *there exist* $f_1, f_2 \in \mathcal{F}$ *and* $S \in \mathcal{S}$ *such that* $x\, R\, \varphi(v)$ *implies* $f_1(v)\, S\, f_2(v)$ *for all* $u,\, v \in X$.

17. $\circledast$–SEMINORMALITIES AND $\circledast$–SEMIREGULARITIES WITH RESPECT TO COMPLEMENT RELATORS

By using Theorem 15.7, we can also easily prove the following

Theorem 17.1. *Under the notations of Definition 15.1, the following assertions are equivalent:*

(1) $\mathcal{F}$ *is upper* $\circledast$–$\mathcal{G}$*–seminormal with respect to the relators* $\mathcal{R}^c$ *and* $\mathcal{S}^c$,

(2) for any $S \in \mathcal{S}$ *and* $F \in \mathcal{F}$ *there exist* $G \in \mathcal{G}$ *and* $R \in \mathcal{R}$ *such that* $G(w) \subset R(x)$ *implies* $F(x) \subset S^{-1}(w)$ *for all* $x \in X$ *and* $w \in W$.

Proof. To prove the implication (1) $\Longrightarrow$ (2), note that if (1) holds, then by Theorem 15.7 for any $S \in \mathcal{S}$ and $F \in \mathcal{F}$ there exist $G \in \mathcal{G}$ and $R \in \mathcal{R}$ such that, for any $x \in X$ and $w \in W$,

$$F(x) \cap \big(\, S^{\,c}\,\big)^{-1}(w) \neq \emptyset \quad\Longrightarrow\quad G(w) \cap R^c(x) \neq \emptyset\,,$$

and thus

$$G(w) \cap R^c(x) = \emptyset \quad\Longrightarrow\quad F(x) \cap \big(\, S^{\,c}\,\big)^{-1}(w) = \emptyset\,.$$

Hence, by using that $R^c(x) = R(x)^c$ and $\big(\, S^{\,c}\,\big)^{-1}(w) = \big(\, S^{-1}\,\big)^c(w) = S^{-1}(w)^c$, we can infer that

$$G(w) \cap R(x)^c = \emptyset \quad\Longrightarrow\; F(x) \cap S^{-1}(w)^c = \emptyset\,,$$

and thus

$$G(w) \subset R(x) \quad\Longrightarrow\quad F(x) \subset S^{-1}(w)\,.$$

Therefore, (2) also holds.

By using Theorem 15.9, we can quite similarly prove the following

Theorem 17.2. *Under the notations of Definition 15.1, the following assertions are equivalent:*

(1) $\mathcal{F}$ *is lower* $\circledast$*–*$\mathcal{G}$*–seminormal with respect to the relators* $\mathcal{R}^c$ *and* $\mathcal{S}^c$*,*

(2) for any $G \in \mathcal{G}$ *and* $R \in \mathcal{R}$ *there exist* $S \in \mathcal{S}$ *and* $F \in \mathcal{F}$ *such that* $F(x) \subset S^{-1}(w)$ *implies* $G(w) \subset R(x)$*.*

By using Theorem 16.7, we can only prove the following

Theorem 17.3. *If in addition the notations of Definition 16.1 we assume that each member of* $\mathcal{F}$ *is a function, then the following assertions are equivalent:*

(1) $\mathcal{F}$ *is upper* $\circledast$*–*Φ*–semiregular with respect to the relators* $\mathcal{R}^c$ *and* $\mathcal{S}^c$*,*

(2) for any $F_1, F_2 \in \mathcal{F}$ *and* $S \in \mathcal{S}$ *there exist* $\phi \in \Phi$ *and* $R \in \mathcal{R}$ *such that* $\phi(v) \subset R(u)$ *implies* $f_2(v) \in S\big(f_1(u)\big)$ *for all* $u, v \in X$*.*

Proof. To prove the implication (1) $\Longrightarrow$ (2), note that if (1) holds, then by Theorem 16.7 for any f_1, $f_2 \in \mathcal{F}$ and $S \in \mathcal{S}$ there exist $\phi \in \Phi$ and $R \in \mathcal{R}$ such that, for any $u, v \in X$,

$$f_2(v) \in S^c\big(f_1(u)\big) \neq \emptyset \quad \Longrightarrow \quad \phi(v) \cap R^c(u) \neq \emptyset\,,$$

and thus

$$\phi(v) \cap R^c(u) = \emptyset \quad \Longrightarrow \quad f_2(v) \notin S^c\big(f_1(u)\big).$$

Hence, by using that $R^c(x) = R(x)^c$ and $S^c\big(f_1(u)\big) = S\big(f_1(u)\big)^c$, we can infer that

$$\phi(v) \cap R(u)^c = \emptyset \quad \Longrightarrow \quad f_2(v) \notin S\big(f_1(u)\big)^c,$$

and thus

$$\phi(v) \subset R(u) \quad \Longrightarrow \quad f_2(v) \in S\big(f_1(u)\big)\,.$$

Therefore, (2) also holds.

By using Theorem 16.9, we can quite similarly prove the following

Theorem 17.4. *If in addition the notations of Definition 16.1 we assume that each member of* $\mathcal{F}$ *is a function, then the following assertions are equivalent:*

(1) $\mathcal{F}$ *is lower* $\circledast$*–*Φ*–semiregular with respect to the relators* $\mathcal{R}^c$ *and* $\mathcal{S}^c$*,*

(2) for any $\phi \in \Phi$ *and* $R \in \mathcal{R}$ *there exist* $f_1, f_2 \in \mathcal{F}$ *and* $S \in \mathcal{S}$ *such that* $f_2(v) \in S\big(f_1(u)\big)$ *implies* $\phi(v) \subset R(u)$ *for all* $u, v \in X$*.*

18. Relationships between seminormal and semiregular relators

The subsequent theorems are straightforward generalization of some fundamental theorems on Galois and Pataki connections established in [**?**] and [**?**].

Theorem 18.1. *Let* $(X, Y)(\mathcal{R})$ *and* $(Z, W)(\mathcal{S})$ *be relator spaces. Moreover, let* $\mathcal{F}$ *be a relator on* X *to* Z *and* $\mathcal{G}$ *be a relator on* W *to* Y*. Furthermore, assume that* $\square$ *is a direct inversion compatible unary operation for relators.*

Then the following assertions are equivalent:

(1) $\mathcal{F}$ *is lower* $\square$*–*$\mathcal{G}$*–seminormal with respect to the relators* $\mathcal{R}$ *and* $\mathcal{S}$*,*

(2) $\mathcal{G}$ *is upper* $\square$*–*$\mathcal{F}$*–seminormal with respect to the relators* $\mathcal{R}^{-1}$ *and* $\mathcal{S}^{-1}$*.*

Proof. To prove that (1) implies (2), note that if (1) holds, then by Definition 15.1 we have

$$\Big(\big(\mathcal{G}^{\,\Box} \big)^{-1} \circ \mathcal{R}^{\,\Box} \Big)^{\Box} \subset \big(\mathcal{S}^{\,\Box} \circ \mathcal{F}^{\,\Box} \big)^{\Box}.$$

Hence, by using that

$$\bigg(\Big(\big(\mathcal{G}^{\,\Box} \big)^{-1} \circ \mathcal{R}^{\,\Box} \Big)^{\Box} \bigg)^{-1} = \bigg(\Big(\big(\mathcal{G}^{\,\Box} \big)^{-1} \circ \mathcal{R}^{\,\Box} \Big)^{-1} \bigg)^{\Box}$$

$$= \Big(\big(\mathcal{R}^{\,\Box} \big)^{-1} \circ \mathcal{G}^{\,\Box} \Big)^{\Box} = \Big(\big(\mathcal{R}^{\,-1} \big)^{\Box} \circ \mathcal{G}^{\,\Box} \Big)^{\Box}$$

and

$$\Big(\big(\mathcal{S}^{\,\Box} \circ \mathcal{F}^{\,\Box} \big)^{\Box} \Big)^{-1} = \Big(\big(\mathcal{S}^{\,\Box} \circ \mathcal{F}^{\,\Box} \big)^{-1} \Big)^{\Box}$$

$$= \Big(\big(\mathcal{F}^{\,\Box} \big)^{-1} \circ \big(\mathcal{S}^{\,\Box} \big)^{-1} \Big)^{\Box} = \Big(\big(\mathcal{F}^{\,\Box} \big)^{-1} \circ \big(\mathcal{S}^{\,-1} \big)^{\Box} \Big)^{\Box},$$

we can already infer that

$$\Big(\big(\mathcal{R}^{\,-1} \big)^{\Box} \circ \mathcal{G}^{\,\Box} \Big)^{\Box} \subset \Big(\big(\mathcal{F}^{\,\Box} \big)^{-1} \circ \big(\mathcal{S}^{\,-1} \big)^{\Box} \Big)^{\Box}.$$

Thus, by Definition 15.1, assertion (2) also holds.

Remark 18.2. From this theorem, we can at once see that now the following assertions are also equivalent :

(1) $\mathcal{F}$ is upper $\Box$–$\mathcal{G}$–seminormal with respect to the relators $\mathcal{R}$ and $\mathcal{S}$,
(2) $\mathcal{G}$ is lower $\Box$–$\mathcal{F}$–seminormal with respect to the relators $\mathcal{R}^{-1}$ and $\mathcal{S}^{-1}$.

However, it is now more important to note that we also have the following

Theorem 18.3. *Let $(X, Y)(\mathcal{R})$ and $Z(\mathcal{S})$ be relator spaces. Moreover, let $\mathcal{F}$ be a relator on X to Z and $\mathcal{G}$ be a relator on Z to Y.*

Furthermore, assume that $\Box$ is a direct, increasing, inversion and composition compatible unary operation for relators such that $\mathcal{F}$ is upper $\Box$–$\mathcal{G}$–seminormal.

Then, under the notation $\Phi = \mathcal{G} \circ \mathcal{F}$, the relator $\mathcal{F}$ is upper $\Box$–Φ–semiregular.

Proof. Now, by Theorem 15.4, we have

$$\big(\mathcal{S} \circ \mathcal{F} \big)^{\Box} \subset \big(\mathcal{G}^{\,-1} \circ \mathcal{R} \big)^{\Box}.$$

Hence, by using the definition of the composition of relators, we can infer that

$$\mathcal{F}^{\,-1} \circ \big(\mathcal{S} \circ \mathcal{F} \big)^{\Box} \subset \mathcal{F}^{\,-1} \circ \big(\mathcal{G}^{\,-1} \circ \mathcal{R} \big)^{\Box}.$$

Hence, by using the increasingness of $\Box$, we can infer that

$$\Big(\mathcal{F}^{\,-1} \circ \big(\mathcal{S} \circ \mathcal{F} \big)^{\Box} \Big)^{\Box} \subset \Big(\mathcal{F}^{\,-1} \circ \big(\mathcal{G}^{\,-1} \circ \mathcal{R} \big)^{\Box} \Big)^{\Box}.$$

Hence, by using the left composition compatibility of $\Box$, we can infer that

$$\big(\mathcal{F}^{\,-1} \circ \mathcal{S} \circ \mathcal{F} \big)^{\Box} \subset \big(\mathcal{F}^{\,-1} \circ \mathcal{G}^{\,-1} \circ \mathcal{R} \big)^{\Box}.$$

Hence, by using that $\mathcal{F}^{\,-1} \circ \mathcal{G}^{\,-1} = \big(\mathcal{G} \circ \mathcal{F} \big)^{-1} = \Phi^{-1}$, we can already infer that

$$\big(\mathcal{F}^{\,-1} \circ \mathcal{S} \circ \mathcal{F} \big)^{\Box} \subset \big(\Phi^{-1} \circ \mathcal{R} \big)^{\Box}.$$

Thus, by Theorem 16.4, the required assertion is also true.

Remark 18.4. Beside proving a lower counterpart of this theorem, it is also worth noticing that if $\mathcal{F}$ is lower $\Box$–$\mathcal{G}$–seminormal with respect to the relators $\mathcal{R}$ and $\mathcal{S}$, then by Theorem 18.1 the relator $\mathcal{G}$ is upper $\Box$–$\mathcal{F}$–seminormal with respect to the relators $\mathcal{S}^{-1}$ and $\mathcal{R}^{-1}$. Thus, by Theorem 18.3, the relator $\mathcal{G}$ is upper $\Box$–$\mathcal{F}\circ\mathcal{G}$–semiregular with respect to the relators $\mathcal{S}^{-1}$ and $\mathcal{R}^{-1}$.

However, it is now more important to note that, as a partial converse to Theorem 18.3, we can also prove the following

Theorem 18.5. *Let $(X, Y)(\mathcal{R})$ and $Z(\mathcal{S})$ be relator spaces. Moreover, let $\mathcal{F}$ be a relator on X onto Z and Φ be a relator on X to Y.*

Furthermore, assume that $\Box$ is a direct, increasing, inversion and composition compatible unary operation for relators such that $\mathcal{F}$ is upper $\Box$–Φ–semiregular. Moreover, assume that $\mathcal{F}\circ\mathcal{F}^{-1}=\{\Delta_Z\}$.

Then, for any relator $\mathcal{G}$ on Z to Y with $\Phi=\mathcal{G}\circ\mathcal{F}$, the relator $\mathcal{F}$ is upper $\Box$–$\mathcal{G}$–seminormal.

Proof. Now, by Theorem 16.4 and the equality $\Phi^{-1}=\mathcal{F}^{-1}\circ\mathcal{G}^{-1}$, we have

$$\big(\mathcal{F}^{-1}\circ\mathcal{S}\circ\mathcal{F}\big)^{\Box}\subset\big(\Phi^{-1}\circ\mathcal{R}\big)^{\Box}=\big(\mathcal{F}^{-1}\circ\mathcal{G}^{-1}\circ\mathcal{R}\big)^{\Box}.$$

Hence, quite similarly as in the proof of Theorem 18.3, we can infer that

$$\mathcal{F}\circ\big(\mathcal{F}^{-1}\circ\mathcal{S}\circ\mathcal{F}\big)^{\Box}\subset\mathcal{F}\circ\big(\mathcal{F}^{-1}\circ\mathcal{G}^{-1}\circ\mathcal{R}\big)^{\Box},$$

and hence

$$\Big(\mathcal{F}\circ\big(\mathcal{F}^{-1}\circ\mathcal{S}\circ\mathcal{F}\big)^{\Box}\Big)^{\Box}\subset\Big(\mathcal{F}\circ\big(\mathcal{F}^{-1}\circ\mathcal{G}^{-1}\circ\mathcal{R}\big)^{\Box}\Big)^{\Box},$$

and hence

$$\big(\mathcal{F}\circ\mathcal{F}^{-1}\circ\mathcal{S}\circ\mathcal{F}\big)^{\Box}\subset\big(\mathcal{F}\circ\mathcal{F}^{-1}\circ\mathcal{G}^{-1}\circ\mathcal{R}\big)^{\Box}.$$

Hence, by using that $\mathcal{F}\circ\mathcal{F}^{-1}\circ\mathcal{U}=\{\Delta_Z\}\circ\mathcal{U}=\mathcal{U}$ for any relator $\mathcal{U}$ on X to Z, we can already infer that

$$\big(\mathcal{S}\circ\mathcal{F}\big)^{\Box}\subset\big(\mathcal{G}^{-1}\circ\mathcal{R}\big)^{\Box}.$$

Thus, by Theorem 15.4, the required assertion is also true.

Remark 18.6. To see the limited range of applicability of this theorem, note that if $\mathcal{F}\circ\mathcal{F}^{-1}=\{\Delta_Z\}$, then for any $F_1, F_2\in\mathcal{F}$ we have $F_1\circ F_2^{-1}=\Delta_Z$. That is,

$$F_1\,[\,F_2^{-1}(z)\,]=\big(F_1\circ F_2^{-1}\big)(z)=\Delta_Z(z)=\{z\}$$

for all $z\in Z$.

Thus, in particular, for each $z\in Z$, we have

$$z\in F_1\,[\,F_2^{-1}(z)\,],\qquad\text{and hence}\qquad F_1^{-1}(z)\cap F_2^{-1}(z)\neq\emptyset\,.$$

Therefore, each member of $\mathcal{F}$ is onto Z.

Moreover, for each $z\in Z$, we have $F_1\,[\,F_2^{-1}(z)\,]\subset\{z\}$. Therefore,

$$z\in F_2(x)\,,\qquad w\in F_1(x)\quad\Longrightarrow\quad z=w\,.$$

for all $x\in X$. Hence, we can see that $F_2(x)=F_1(x)$ for all $x\in D_{F_1}\cap D_{F_2}$. Moreover, by taking $F_2=F_1$, we can see that each member of $\mathcal{F}$ is a function.

Conversely, we can also easily check that if each member of $\mathcal{F}$ is a function on X onto Z such that $f_1(x)=f_2(x)$ for all $f_1, f_2\in\mathcal{F}$ and $x\in D_{f_1}\cap D_{f_2}$, then $\mathcal{F}\circ\mathcal{F}^{-1}=\{\Delta_Z\}$.

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