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GALOIS AND PATAKI CONNECTIONS REVISITED

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Abstract. In this paper, for any generalized ordered sets X and Y, and for any functions f of X to Y, g of Y to X, and φ of X to itself, we establish several consequences and equivalents of the following definitions.

The function f is called

(1) increasingly lower g–seminormal if $f(x) \leq y$ implies $x \leq g(y)$,

(2) increasingly lower φ–semiregular if $f(u) \leq f(v)$ implies $u \leq \varphi(v)$.

While, f is called increasingly upper g–seminormal and increasingly upper φ–semiregular if the converse implications hold, respectively.

The results obtained extend some former results of O. Ore and the present author on Galois and Pataki connections. Namely, according to the ideas of J. Schmidt and the present author, the pairs (f, g) and (f, φ) may be called increasing Galois and Pataki connections if the function f is increasingly g–normal and φ-regular, respectively.

1. A few basic fats on relations and functions

A subset F of a product set $X{\times}Y$ is called a *relation on X to Y*. If in particular $F \subset X^2$, with $X^2 = X \times X$, then we may simply say that F is a *relation on X*. In particular, $\Delta_X = \{ (x, x) : x \in X \}$ is called the *identity relation on X*.

If F is a relation on X to Y, then for any $x \in X$ and $A \subset X$ the sets $F(x) = \{ y \in Y : (x, y) \in F \}$ and $F[A] = \bigcup_{a \in A} F(a)$ are called the *images of x and A under F*, respectively. If $(x, y) \in F$, then we may also write $x F y$.

Moreover, the sets $D_F = \{ x \in X : F(x) \neq \emptyset \}$ and $R_F = F[D_F]$ are called the *domain and range of F*, respectively. If in particular $D_F = X$, then we say that F is a *relation of X to Y*, or that F *is a non-partial relation on X to Y*.

If F is a relation on X to Y, then $F = \bigcup_{x \in X} \{x\}{\times}F(x)$. Therefore, the values $F(x)$, where $x \in X$, uniquely determine F. Thus, the *inverse relation F^{-1} of F* can be naturally defined such that $F^{-1}(y) = \{ x \in X : y \in F(x)\}$ for all $y \in Y$.

Moreover, if in addition G is a relation on Y to Z, then the *composition relation $G \circ F$ of G and F* can be naturally defined such that $(G \circ F)(x) = G[F(x)]$ for all $x \in X$. Thus, we also have $(G \circ F)[A] = G[F[A]]$ for all $A \subset X$.

Now, a relation F on X may be called *reflexive, antisymmetric, and transitive* if $\Delta_X \subset F$, $F \cap F^{-1} \subset \Delta_X$, and $F \circ F \subset F$, respectively. Thus, a reflexive and transitive relation is a *preorder*, and an antisymmetric preorder is a *partial order*.

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In particular, a relation f on X to Y is called a *function* if for each $x \in D_f$ there exists $y \in Y$ such that $f(x) = \{y\}$. In this case, by identifying singletons with their elements, we may simply write $f(x) = y$ in place of $f(x) = \{y\}$.

Moreover, a function $\star$ of X to itself is called an unary operation on X. While, a function $*$ of X^2 to X is called a binary operation on X. And, for any $x, y \in X$, we usually write $x^\star$ and $x * y$ instead of $\star(x)$ and $*(x, y)$, respectively.

If F is a relation on X to Y, then a function f of D_F to Y is called a *selection of* F if $f \subset F$, i. e., $f(x) \in F(x)$ for all $x \in D_F$. Thus, the *Axiom of Choice* can be briefly expressed by saying that every relation has a selection.

For any relation F on X to Y, we may naturally define two *set-valued functions*, $F^\star$ on X to $\mathcal{P}(Y)$ and F^* on $\mathcal{P}(X)$ to $\mathcal{P}(Y)$, such that $F^\star(x) = F(x)$ and $F^*(A) = F[A]$ for all $x \in X$ and $A \subset X$.

2. A few basic facts on generalized ordered sets

According to [19], an ordered pair $X(\leq) = \big(X, \leq\big)$, consisting an arbitrary set X and an arbitrary relation $\leq$ on X, will be called *generalized ordered set*, or *an ordered set without axioms*. And, we shall usually write X in place of $X(\leq)$.

In the sequel, a generalized ordered set $X(\leq)$ will, for instance, be called reflexive if the relation $\leq$ is reflexive. Moreover, the generalized ordered set $X'(\leq') = X(\leq^{-1})$ will be called the dual of $X(\leq)$.

Having in mind the terminology of Birkhoff [1, p. 1], a generalized ordered set may be briefly called a *goset*. Moreover, a preordered (partially ordered) set may be call a *proset (poset)*.

Thus, every set X is a proset with the universal relation X^2. Moreover, X is a poset with the identity relation Δ_X. And every subfamily of the power set $\mathcal{P}(X)$ of X is a poset with the ordinary set inclusion $\subset$.

The usual definitions on posets can be naturally extended to gosets [19] (or even to arbitrary *relator spaces* [18] which include *context spaces* [9, p. 17] as an important particular case).

For instance, for any subset A of a goset X, we may naturally define

$$\operatorname{lb}(A) = \big\{ x \in X : \quad \forall\ a \in A : \quad x \leq a \big\},$$

$$\operatorname{ub}(A) = \big\{ x \in X : \quad \forall\ a \in A : \quad a \leq x \big\},$$

and

$$\min(A) = A \cap \operatorname{lb}(A), \qquad \max(A) = A \cap \operatorname{ub}(A),$$

$$\inf(A) = \max\big(\operatorname{lb}(A)\big), \qquad \sup(A) = \min\big(\operatorname{ub}(A)\big).$$

Concerning lower and upper bounds, one can easily prove the following theorems.

Theorem 2.1. *By identifying singletons with their elements, for any subset A of a goset X, we have*

(1) $\operatorname{lb}(A) = \bigcap_{a \in A} \operatorname{lb}(a)$, *(2)* $\operatorname{ub}(A) = \bigcap_{a \in A} \operatorname{ub}(a)$.

Remark 2.2. From this theorem, it is clear

(1) $\operatorname{lb}(\emptyset) = X$ and $\operatorname{ub}(\emptyset) = X$,

(2) $\operatorname{lb}(B) \subset \operatorname{lb}(A)$ and $\operatorname{ub}(B) \subset \operatorname{ub}(A)$ for all $A \subset B \subset X$.

Theorem 2.3. *For any two subsets A and B of a goset X, we have*

$$A \subset \operatorname{lb}(B) \iff B \subset \operatorname{ub}(A).$$

Proof. Note that, by the corresponding definitions, each of the above inclusions is equivalent to the property that $a \leq b$ for all $a \in A$ and $b \in B$, which can be briefly expressed by writing that $A \times B \subset \leq$.

Remark 2.4. From the above theorem, it will follow that

(1) $\operatorname{lb}(A) = \operatorname{lb}\big(\operatorname{ub}\big(\operatorname{lb}(A)\big)\big)$, (2) $\operatorname{ub}(A) = \operatorname{ub}\big(\operatorname{lb}\big(\operatorname{ub}(A)\big)\big)$.

Now, concerning minima and maxima, and infima and suprema, one can also easily prove the following theorems.

Theorem 2.5. *For any subset A of a goset X, we have*

(1) $\min(A) = \big\{x \in A : \ A \subset \operatorname{ub}(x)\big\}$, *(2)* $\max(A) = \big\{x \in A : \ A \subset \operatorname{lb}(x)\big\}$.

Remark 2.6. By this theorem, we may also naturally define

$$\operatorname{lb}^*(A) = \big\{ x \in X : \quad A \cap \operatorname{lb}(x) \subset \operatorname{ub}(x) \big\}.$$

Thus, $\min^*(A) = A \cap \operatorname{lb}^*(A)$ is just the family of all minimal elements of A.

Theorem 2.7. *For any subset A of a goset X, we have*

(1) $\inf(A) = \operatorname{lb}(A) \cap \operatorname{ub}\big(\operatorname{lb}(A)\big)$, *(2)* $\sup(A) = \operatorname{ub}(A) \cap \operatorname{lb}\big(\operatorname{ub}(A)\big)$.

Theorem 2.8. *For any subset A of a goset X, we have*

(1) $\inf(A) = \sup\big(\operatorname{lb}(A)\big)$, *(2)* $\sup(A) = \inf\big(\operatorname{ub}(A)\big)$,

(3) $\min(A) = A \cap \inf(A)$, *(4)* $\max(A) = A \cap \sup(A)$.

Theorem 2.9. *Under the notation $\Phi = \min, \max, \inf$, or $\sup$, for any subset A of an antisymmetric goset X, we have* $\operatorname{card}\big(\Phi(A)\big) \leq 1$.

Remark 2.10. Conversely, one can also easily see that if X is a reflexive goset such that $\operatorname{card}\big(\Phi(A)\big) \leq 1$ for all $A \subset X$, with $\operatorname{card}(A) = 2$, then X is antisymmetric.

In [20], by using the notation $\mathcal{L} = \{A \subset X : \ A \subset \operatorname{lb}(A)\}$, we have first proved that a reflexive goset X is antisymmetric if and only if $\operatorname{card}(A) \leq 1$ for all $A \in \mathcal{L}$.

3. Some basic properties of functions and relations on gosets

Definition 3.1. An operation φ on a goset X is called *extensive (intensive)* if $x \leq \varphi(x)$ $\big(\varphi(x) \leq x\big)$ for all $x \in X$.

Remark 3.2. Thus, φ is extensive (intensive) if and only if $\Delta_X \leq \varphi$ $(\varphi \leq \Delta_X)$ holds in the pointwise inequality of functions.

Definition 3.3. An operation φ on a goset X is called *lower (upper) semiidempotent)* if $\varphi(x) \leq \varphi\big(\varphi(x)\big)$ $\big(\varphi\big(\varphi(x)\big) \leq \varphi(x)\big)$ for all $x \in X$.

Remark 3.4. Thus, φ is lower (upper) semiidempotent if and only if the restriction of φ to its range $\varphi[X]$ is extensive (intensive), or equivalently $\varphi \leq \varphi^2$ $(\varphi^2 \leq \varphi)$.

Definition 3.5. A function f of one goset X to another Y is called *increasing (decreasing)* if $f(u) \leq f(v)$ $\big(f(v) \leq f(u) \big)$ for all $u, v \in X$ with $u \leq v$.

Remark 3.6. Now, an increasing, extensive (intensive) operation may be called a *preclosure (preinterior) operation.* And an upper (lower) semiidempotent preclosure (preinterior) operation may be called a *closure (interior) operation.*

Moreover, an extensive (intensive) upper (lower) semiidempotent operation may be be called a *semiclosure (semiinterior) operation.* And an increasing upper (lower) semiidempotent function may be called an *upper (lower) semimodification operation.*

In the sequel, trusting to the reader's good sense to avoid confusions, we shall also use the following

Definition 3.7. A relation R on a goset X to a set Y is called *increasing (decreasing)* if $R(u) \subset R(v)$ $\big(R(v) \subset R(u) \big)$ for all $u, v \in X$ with $u \leq v$.

Remark 3.8. Note that thus R is increasing (decreasing) if and only if the associated set-valued function $R^\star$ is increasing (decreasing) as a function of the goset X into the poset $\mathcal{P}(Y)$.

By using this definition, one can easily prove the following two theorems.

Theorem 3.9. *If R is an increasing (decreasing) relation on a goset X to a set Y and S is an arbitrary relation on Y to another set Z, then $S \circ R$ is an increasing (decreasing) relation on X to Z.*

Theorem 3.10. *If f is an increasing (decreasing) function of a goset X to another Y and S is an increasing (decreasing) relation on Y to a set Z, then $S \circ f$ is an increasing relation on X to Z.*

Definition 3.11. A subset A of a goset X is called *ascending (descending)* if $\mathrm{ub}(x) \subset A$ $\big(\mathrm{lb}(x) \subset A \big)$ for all $x \in A$.

Remark 3.12. Hence, by noticing that $\mathrm{ub}(x) = \leq (x)$ and $\mathrm{lb}(x) = \leq' (x)$ for all $x \in X$, we can note that the ascending (descending) subsets of X are just the $\leq$–open ($\leq'$–open) subsets of X in the sense of [18].

Definition 3.13. A relation R on a set X to a goset Y is called *ascending (descending) valued* if $R(x)$ is a ascending (descending) subset of Y for all $x \in X$.

By using the above definitions, we can easily prove the following

Theorem 3.14. *For any relation R on a goset X to a set Y, the following assertions are equivalent:*

(1) R is increasing   *(2) R^{-1} is ascending valued.*

Proof. Suppose that $y \in Y$, $x \in R^{-1}(y)$, and $u \in \mathrm{ub}(x)$. Then, $y \in R(x)$ and $x \leq u$. Moreover, if (1) holds, then $R(x) \subset R(u)$. Hence, we can already infer that $y \in R(u)$, and thus $u \in R^{-1}(y)$. Therefore, $\mathrm{ub}(x) \subset R^{-1}(y)$, and thus (2) also holds.

The converse implication $(2) \Longrightarrow (1)$ can be proved quite similarly by reversing the above argument.

From this theorem, by dualization, we can easily get the following

Corollary 3.15. *For any relation* R *on a set* X *to a goset* Y, *the following assertions are equivalent:*

(1) R^{-1} *is decreasing,* *(2)* R *is descending valued.*

Proof. By considering R^{-1} as a relation on Y' to X, Theorem 3.14 can be applied.

4. Proximal interiors induced by functions of sets to gosets

Definition 4.1. For any function f of a set X to a goset Y, we define a relation Int_f on Y to X such that

$$\operatorname{Int}_f(y) = \big\{ x \in X : \quad f(x) \leq y \big\}$$

for all $y \in Y$. The relation Int_f will be called the proximal interior induced by f.

Remark 4.2. Namely, if R is a relation on X to Y, then by the corresponding definitions, for any $A \subset X$ and $B \subset Y$, we have

$$A \in \operatorname{Int}_{R^*}(B) \iff R^*(A) \subset B \iff R[A] \subset B \iff A \in \operatorname{Int}_R(B).$$

Note that if in particular f is a function of a power set $\mathcal{P}(X)$ to a goset Y, then we may naturally define $\operatorname{int}_f(y) = \big\{ x \in X : \ \{x\} \in \operatorname{Int}_f(y) \big\}$ for all $y \in Y$. Thus, int_f is a relation on Y to X which may be called the topological interior induced by f.

Concerning the relation Int_f, we can easily prove the following

Theorem 4.3. *If* f *is a function of a set* X *to a goset* Y, *then for any* $y \in Y$ *we have*

$$\operatorname{Int}_f(y) = f^{-1}\big[\operatorname{lb}(y)\big].$$

Proof. By the corresponding definitions, it is clear that

$$x \in \operatorname{Int}_f(y) \iff f(x) \leq y \iff f(x) \in \operatorname{lb}(y) \iff x \in f^{-1}\big[\operatorname{lb}(y)\big]$$

for all $x \in X$. Therefore, the required equality is also true.

Remark 4.4. From the above theorem, we can at once see that

$$\operatorname{Int}_f(y) = f^{-1}\big[\operatorname{lb}(y)\big] = f^{-1}\big[\leq^{-1}(y)\big] = \big(f^{-1} \circ \leq^{-1}\big)(y)$$

for all $y \in Y$. Therefore, $\operatorname{Int}_f = f^{-1} \circ \leq^{-1} = \big(\leq \circ f\big)^{-1}$.

However, the latter observation cannot be used to simplify the proofs of the following two theorems.

Theorem 4.5. *If* f *is a function of a set* X *to a transitive goset* Y, *then*

(1) Int_f *is increasing,* *(2)* $\operatorname{Int}_f^{-1}$ *is ascending valued.*

Proof. If $z, w \in Y$ such that $z \leq w$, then by the definition of lb and the transitivity of Y we also have $\operatorname{lb}(z) \subset \operatorname{lb}(w)$, and thus $f^{-1}\big[\operatorname{lb}(z)\big] \subset f^{-1}\big[\operatorname{lb}(w)\big]$. Therefore, by Theorem 4.3, $\operatorname{Int}_f(z) \subset \operatorname{Int}_f(w)$, and thus (1) is also true.

Moreover, from Theorem 3.14, we can see that (1) and (2) are equivalent even if Y is not assumed to be transitive.

Theorem 4.6. *If f is an increasing function of an arbitrary goset X to a transitive one Y, then*

(1) $\operatorname{Int}_f^{-1}$ *is decreasing,* *(2)* Int_f *is descending valued.*

Proof. If $u, v \in Y$ such that $u \leq v$ and $z \in \operatorname{Int}_f^{-1}(v)$, i. e., $v \in \operatorname{Int}_f(z)$, then by the increasingness of f and the definition of Int_f, we have $f(u) \leq f(v)$ and $f(v) \leq z$. Hence, by the transitivity of Y, we can infer that $f(u) \leq z$, and thus $u \in \operatorname{Int}_f(z)$, i. e., $z \in \operatorname{Int}_f^{-1}(u)$. Therefore, $\operatorname{Int}_f^{-1}(v) \subset \operatorname{Int}_f^{-1}(u)$, and thus (1) is also true.

Moreover, from Corollary 3.15, we can see that (1) and (2) are equivalent even if f and Y are not assumed to be increasing and transitive, respectively.

5. Order relations induced by functions of sets to gosets

Definition 5.1. For any function f of a set X to a goset Y, we define a relation Ord_f on X such that

$$\operatorname{Ord}_f(x) = \big\{ u \in X : \quad f(x) \leq f(u) \big\}$$

for all $x \in X$. The relation Ord_f will be called the natural order induced by f which may also be denoted by $\leq_f$.

Remark 5.2. Note that, analogously to Remark 4.2, for any $A, U \subset X$ we have

$$U \in \operatorname{Ord}_{R^*}(A) \iff R^*(A) \subset R^*(U) \iff R[A] \subset R[U].$$

Therefore, Ord_{R^*} is the natural order induced by R which may be denoted by $\leq_R$.

By the corresponding definitions, we evidently have the following

Theorem 5.3. *If f is a function of a set X to a goset Y, then*

(1) Ord_f *is reflexive if Y is reflexive,*

(2) Ord_f *is transitive if Y is transitive.*

Remark 5.4. Note that if in particular f is onto Y, then the converses of the above statements are also true.

In this respect it is also worth mentioning that the following two theorems are also true.

Theorem 5.5. *If f is an injective function of a set X to an antisymmetric goset Y, then* Ord_f *is also antisymmetric.*

Proof. If $u, v \in X$ such that $v \in \operatorname{Ord}_f(u)$ and $u \in \operatorname{Ord}_f(v)$, then by the definition of Ord_f we have $f(u) \leq f(v)$ and $f(v) \leq f(u)$. Hence, by the antisymmetry of Y, it follows that $f(u) = f(v)$. Thus, by the injectivity of f, we also have $u = v$. Therefore, the required assertion is also true.

Theorem 5.6. *If f is a function of a set X to a goset Y such that the relation* Ord_f *is antisymmetric, then*

(1) f is injective if Y is reflexive,

(2) Y is antisymmetric if f is onto Y.

Proof. If $u, v \in X$ such that $f(u) \leq f(v)$ and $f(v) \leq f(u)$, then by the definition of Ord_f we have $v \in \operatorname{Ord}_f(u)$ and $u \in \operatorname{Ord}_f(v)$. Thus, by the antisymmetry of Ord_f, we necessarily have $u = v$.

Hence, if $Y = f[X]$, it is clear that Y is antisymmetric. While, if Y is reflexive, we can also see that f is injective.

Now, as an immediate consequence of the above two theorems, we can also state

Corollary 5.7. *For a function f of a set X onto a reflexive goset Y, the following assertions are equivalent:*

(1) Ord_f *is antisymmetric,*

(2) f *is injective and Y is antisymmetric.*

Remark 5.8. Note that if f is a function of a set X to a proset Y, then by Theorem 5.3 Ord_f is already a preorder on X.

While, if f is an injective function of a set X to a poset Y, then by the above observation and Theorem 5.5 Ord_f is already a partial order on X.

Theorem 5.9. *If f is a function of a set X to a goset Y, then*

$$\operatorname{Ord}_f = \big(\operatorname{Int}_f \circ f\big)^{-1}.$$

Proof. By the corresponding definitions, it is clear that

$$u \in \operatorname{Ord}_f(x) \iff f(x) \leq f(u) \iff x \in \operatorname{Int}_f\big(f(u)\big)$$
$$\iff x \in \big(\operatorname{Int}_f \circ f\big)(u) \iff u \in \big(\operatorname{Int}_f \circ f\big)^{-1}(x)$$

for all $x, u \in X$. Therefore, $\operatorname{Ord}_f(x) = \big(\operatorname{Int}_f \circ f\big)^{-1}(x)$ for all $x \in X$, and thus the required equality is also true.

From the above theorem, it is clear that we also have

Corollary 5.10. *If f is a function of a set X to a goset Y, then*

(1) $\operatorname{Ord}_f^{-1} = \operatorname{Int}_f \circ f$, (2) $\operatorname{Ord}_f = f^{-1} \circ \operatorname{Int}_f^{-1}$.

Theorem 5.11. *If f is an increasing function f of an arbitrary goset X to a transitive one Y, then*

(1) Ord_f *is decreasing,* (2) Ord_f *is ascending valued,*

(3) $\operatorname{Ord}_f^{-1}$ *is increasing,* (4) $\operatorname{Ord}_f^{-1}$ *is descending valued.*

Proof. From Theorem 4.5, we can see that Int_f is increasing. Moreover, by Corollary 5.10, we have $\operatorname{Ord}_f^{-1} = \operatorname{Int}_f \circ f$. Hence, by Theorem 0.0, we can see that (3) is true.

On the other hand, from Theorem 4.6 we can see that $\operatorname{Int}_f^{-1}$ is decreasing. Moreover, by Corollary 5.10, we have $\operatorname{Ord}_f = f^{-1} \circ \operatorname{Int}_f^{-1}$. Hence, by Theorem 3.9, we can see that (1) is also true.

Moreover, from Theorem 3.14 and Corollary 3.15, we can see that (3) and (1) are are equivalent to (2) and (4), respectively, even if f and Y are not assumed to be increasing and transitive, respectively.

Theorem 5.12. *If f is a function of an arbitrary goset X to a reflexive one Y such that any one of the assertions (1)–(4) of Theorem 5.11 holds, then f is increasing.*

Proof. Suppose that $u, v \in X$ such that $u \leq v$. Then, by Theorem 5.3, we have $u \in \operatorname{Ord}_f(u)$ and $v \in \operatorname{Ord}_f(v)$.

Moreover, if (1) holds, then we also have $\operatorname{Ord}_f(v) \subset \operatorname{Ord}_f(u)$. Hence, by using that $v \in \operatorname{Ord}_f(v)$, we can infer that $v \in \operatorname{Ord}_f(u)$.

While, if (2) holds, then from $u \in \operatorname{Ord}_f(u)$ and $u \leq v$ we can infer that $v \in \operatorname{Ord}_f(u)$.

Hence, by the definition of Ord_f, we can already see that in both cases we have $f(u) \leq f(v)$, and thus f is increasing.

The remaining two cases when either (3) or (4) holds, are now quite obvious from the above observations by Theorem 3.14 and Corollary 3.15.

Now, as an immediate consequence of Theorems 5.11 and 5.12, we can also state

Corollary 5.13. *A function f of a goset X to a proset Y is increasing if and only if any one of the assertions (1)–(4) of Theorem 8.11 holds.*

6. Increasingly seminormal and semiregular functions

According to [23], we may naturally introduce the the following two definitions.

Definition 6.1. A function f of one goset X to another Y is called *increasingly lower (upper) g–seminormal*, for some function g of Y to X, if for any $x \in X$ and $y \in X$

$$f(x) \leq y \implies x \leq g(y) \qquad \big(x \leq g(y) \implies f(x) \leq y \big).$$

Remark 6.2. Now, the function f may be naturally called *increasingly g-normal* if it is both increasingly lower and upper g–seminormal.

Moreover, a function f of one goset X to another Y may, for instance, be naturally called *increasingly normal* if it is increasingly g-normal for some function g of Y to X.

If f is increasingly g–normal function of X to Y, then we may also say that the pair (f, g) is an *increasing Galois connection between X and Y*.

For the origins, developments, and applications of Galois connections, see the papers by Ore [12], Pickert [14], Schmidt [15], Xia [27], Száz [23, 22, 21, 24], and the books by Birkhoff [1], Blyth–Janowitz [2], Gierz at al. [10], Ganter–Wille [9], Davey–Pristley [7], Denecke–Erné–Wismath [6].

Definition 6.3. A function f of one goset X to another Y is called *increasingly lower (upper) φ–semiregular*, for some operation φ on X, if for any $u, v \in X$

$$f(u) \leq f(v) \implies u \leq \varphi(v) \qquad \big(u \leq \varphi(v) \implies f(u) \leq f(v) \big).$$

Remark 6.4. Now, the function f may be naturally called increasingly φ-regular if it is both increasingly lower and upper φ–semiregular.

Moreover, a function f of one goset X to another Y may, for instance, be naturally called *increasingly regular* if it is increasingly φ-regular for some operation φ on X.

If f is an increasingly φ–regular function of X to Y, then we may also say that the pair (f, φ) is an *increasing Pataki connection between X and Y*.

For the origins, developments, and applications of Pataki connections, see the papers by Száz [16], Pataki [13], Száz [23, 22, 21, 24], and Száz–Túri [26]. (The publications of [16] and [13] were rejected by L. Tamássy in 1998 and I. Faragó in 1999 at the Publ. Math. Debrecen and Pure Math. Appl. Budapest, respectively.)

By using the above definitions, we can easily prove the following

Theorem 6.5. *If f is an increasingly lower (upper) g–seminormal function of one goset X to another Y, then $\varphi = g \circ f$ is an operation on X such that f is increasingly lower (upper) φ–semiregular.*

Proof. If f is increasingly lower g–seminormal, then by the corresponding definitions it is clear that

$$f(u) \leq f(v) \implies u \leq g\big(f(v)\big) \implies u \leq (g \circ f)(v) \implies u \leq \varphi(v)$$

for all $u, v \in X$. Therefore, f is increasingly lower φ–semiregular too.

Now, as an immediate consequence of the this theorem, we can also state

Corollary 6.6. *If f is an increasingly g–normal function of one goset X to another Y, then $\varphi = g \circ f$ is an operation on X such that f is φ–regular.*

Remark 6.7. Hence, it is clear that several properties of the increasingly normal functions can be immediately derived from those of the increasingly regular ones.

However, the following partial converses of the above results indicate that the increasingly normal functions are still somewhat more general objects than the increasingly regular ones.

Theorem 6.8. *If f is an increasingly lower (upper) φ–semiregular function of one goset X onto another Y and g is a function of Y to X such that $\varphi = g \circ f$, then f is lower (upper) g–seminormal.*

Proof. Suppose that $x \in X$ and $y \in Y$. Then, since $Y = f[X]$, there exists $v \in X$ such that $y = f(v)$.

Now, if f is increasingly lower φ–semiregular, then we can easily see that

$$\begin{aligned} f(x) \leq y \implies f(x) \leq f(v) \implies x \leq \varphi(v) \\ \implies x \leq \big((g \circ f)(v)\big) \implies x \leq g\big(f(v)\big) \implies x \leq g(y)\,. \end{aligned}$$

Therefore, f is lower g–seminormal too.

Now, as an immediate consequence of this theorem, we can also state

Corollary 6.9. *If f is an increasingly φ–regular function of one goset X onto another Y and g is a function of Y to X such that $\varphi = g \circ f$, then f is g–normal.*

Remark 6.10. Note that if in particular X is transitive, then instead of $\varphi = g \circ f$ it is enough to assume only some inequalities to get the corresponding assertions.

In addition to the above results, we can also easily prove the following

Theorem 6.11. *If f is an increasingly lower (upper) g–seminormal function of one goset X to another Y, then g is a an increasingly upper (lower) f–seminormal function of Y' to X'.*

Proof. If f is an increasingly lower g–seminormal, then by the corresponding definitions it is clear that

$$y \leq' f(x) \implies f(x) \leq y \implies x \leq g(y) \implies g(y) \leq' x$$

for all $y \in Y$ and $x \in X$ Therefore, g is a an increasingly upper f–seminormal as a function of Y' to X'.

Now, as an immediate consequence of this theorem, we can also state

Corollary 6.12. *If f is an increasingly g–normal function of one goset X to another Y, then g is a an increasingly f–normal function of Y' to X'.*

Remark 6.13. By using the latter results, the properties of the functions g and $f \circ g$ can be easily derived from those of f and $g \circ f$. However, it is sometimes more convenient to apply direct proofs.

7. Some basic properties of increasingly semiregular functions

Theorem 7.1. *If f is an increasingly lower φ–semiregular function of an arbitrary goset X to a reflexive one Y, then φ is extensive.*

Proof. Because of the reflexivity of Y, for any $x \in X$, we have $f(x) \leq f(x)$. Hence, by the assumed semiregularity of f, we can infer that $x \leq \varphi(x)$. Therefore, the required assertion is also true.

Hence, in particular we can immediately derive the following

Corollary 7.2. *If f is an increasingly lower φ–semiregular function of an arbitrary goset X to a reflexive one Y such that f is increasing, then $f \leq f \circ \varphi$.*

Proof. By Theorem 7.1, for any $x \in X$, we have $x \leq \varphi(x)$. Hence, by the assumed increasingness of f, it follows that $f(x) \leq f\big(\varphi(x)\big)$, and thus $f(x) \leq \big(f \circ \varphi\big)(x)$. Therefore, the required inequality is also true.

On the other hand, to get the converse inequality, we can also easily prove

Theorem 7.3. *If f is an increasingly upper φ–semiregular function of a reflexive goset X to an arbitrary one Y, then $f \circ \varphi \leq f$.*

Proof. Because of the reflexivity of X, for any $x \in X$, we have $\varphi(x) \leq \varphi(x)$. Hence, by the assumed semiregularity of f, we can infer that $f\big(\varphi(x)\big) \leq f(x)$, and thus $(f \circ \varphi)(x) \leq f(x)$. Therefore, the required inequality is also true.

From this theorem, by Corollary 7.2, it is clear that in particular we also have

Corollary 7.4. *If f is an increasingly φ–regular function of a reflexive goset X to a reflexive, antisymmetric one Y such that f is increasing, then $f = f \circ \varphi$.*

Moreover, by using Theorem 7.3, we can also easily prove the following

Theorem 7.5. *If f is an increasingly φ–regular function of a reflexive goset X to a transitive one Y, then φ is upper semiidempotent.*

Proof. By Theorem 7.3, we have $f \circ \varphi \leq f$. Hence, by using the corresponding definitions, we can infer that $f \circ \varphi^2 \leq f \circ \varphi$. Now, by the transitivity of Y, it is clear that $f \circ \varphi^2 \leq f$ also holds. Therefore, for any $x \in X$, we have $f\big(\varphi^2(x)\big) \leq f(x)$. Hence, by using the increasing lower φ–semiregularity of f, we can infer that $\varphi^2(x) \leq \varphi(x)$. Therefore, the required assertion is also true.

From this theorem, by Theorem 7.1, it is clear that in particular we also have

Corollary 7.6. *If f is an increasingly φ–regular function of a reflexive goset X to a proset Y, then φ is a semiclosure operation on X.*

Moreover, by using Theorem 7.1 and Corollary 7.2, we can also easily prove

Theorem 7.7. *If f is an increasingly φ–regular function of a transitive goset X to a reflexive one Y, then*

(1) f is increasing, *(2) $f \leq f \circ \varphi$.*

Proof. By Theorem 7.1, we have $x \leq \varphi(x)$ for all $x \in X$. Therefore, if $u, v \in X$ such that $u \leq v$, then by the inequality $v \leq \varphi(v)$ and the transitivity of X, we also have $u \leq \varphi(v)$. Hence, by using the increasing upper φ–semiregularity of f, we can infer that $f(u) \leq f(v)$. Therefore, assertion (1) is true. Hence, by Corollary 7.2, we can see that (2) is also true.

From this theorem, by Theorem 7.3, it is clear that in particular we also have

Corollary 7.8. *If f is an increasingly φ–regular function of a proset X to a reflexive, antisymmetric goset Y, then $f = f \circ \varphi$.*

From Corollaries 7.4 and 7.8, it is clear that in particular we also have

Theorem 7.9. *If f is an injective, increasingly φ–regular function of a reflexive goset X to a reflexive, antisymmetric goset Y such that either f is increasing or X is transitive, then $\varphi = \Delta_X$.*

Moreover, as a partial converse to this theorem, we can also easily prove

Theorem 7.10. *If f is an increasingly lower Δ_X–semiregular function of an antisymmetric goset X to a reflexive goset Y, then f is injective.*

Proof. If $u, v \in X$ such that $f(u) = f(v)$, then by the reflexivity of Y we also have $f(u) \leq f(v)$ and $f(v) \leq f(u)$. Hence, by using the lower Δ_X–semiregularity of f, we can infer that $u \leq \Delta_X(v)$ and $v \leq \Delta_X(u)$, and thus $u \leq v$ and $v \leq u$. Hence, by the antisymmetry of X, it follows that $u = v$, and thus the required assertion is also true.

Now, as an immediate consequence of Theorems 7.9 and 7.10, we can also state

Corollary 7.11. *If f is an increasingly φ–regular function f of one reflexive, antisymmetric goset X to another Y such that either f is increasing or X is transitive, then the following assertions are equivalent;*

(1) f injective; *(2) $\varphi = \Delta_X$.*

Moreover, by using Corollary 7.6 and Theorems 7.7 and 7.3, we can also prove

Theorem 7.12. *If f is an increasingly φ–regular function of one proset X to another Y, then φ is a closure operation on X.*

Proof. By Corollary 7.6, we need only show that φ is also increasing. For this, note that if $u, v \in X$ such that $u \leq v$, then by Theorem 7.7 we have $f(u) \leq f(v)$. Moreover, by Theorem 7.3, we have $f\big(\varphi(u)\big) \leq f(u)$. Thus, by the transitivity of Y, we also have $f\big(\varphi(u)\big) \leq f(v)$. Hence, by using the increasing lower φ–semiregularity of f, we can already infer that $\varphi(u) \leq \varphi(v)$. Therefore, the required assertion is also true.

On the other hand, as a partial converse to Theorems 7.1, we can also prove

Theorem 7.13. *If φ is an extensive operation on a transitive goset X, then φ is increasingly lower φ–semiregular.*

Proof. If $u \in X$, then by the extensivity of φ we have $u \leq \varphi(u)$. Therefore, if $v \in X$ such that $\varphi(u) \leq \varphi(v)$, then by the transitivity of X we also have $u \leq \varphi(v)$. Thus, the required assertion is also true.

Now, as an immediate consequence of Theorems 7.13 and 7.1, we can also state

Corollary 7.14. *For an operation φ on a proset X, the following assertions are equivalent:*

(1) φ is extensive, *(2) φ is increasingly lower φ–semiregular.*

Moreover, as a partial converse to Theorems 7.5 and 7.7, we can also prove

Theorem 7.15. *If φ is an upper semimodification operation on a transitive goset X, then φ is increasingly upper φ–semiregular.*

Proof. If $u, v \in X$ such that $u \leq \varphi(v)$, then by the increasingness of φ we also have $\varphi(u) \leq \varphi\big(\varphi(v)\big)$. Moreover, by the upper semiidempotency of φ, we also have $\varphi\big(\varphi(v)\big) \leq \varphi(v)$. Hence, by the transitivity of X, it already follows that $\varphi(u) \leq \varphi(v)$. Therefore, the required assertion is also true.

Now, as an immediate consequence of Theorems 7.15 and 7.13, we can also state

Corollary 7.16. *If φ is a closure operation on transitive goset X, then φ is increasingly φ–regular.*

Moreover, combining this corollary with Theorem 7.12, we can also at once state

Theorem 7.17. *For an operation φ on a proset X, the following assertions are equivalent:*

(1) φ is a closure operation, *(2) φ is increasingly φ–regular,*

(3) there exists an increasingly φ–regular function f of X to a proset Y.

Remark 7.18. According to Erné [6, p. 50], the origins of the equivalence of (1) and (2) go back to R. Dedekind. For some closely related observations see also Everett [8, p. 515] and Meyer and Nieger [11, p. 343].

A simple application of the above theorem gives the following

Corollary 7.19. *For a function f of one proset X to another Y and an operation φ on X, the following assertions are equivalent:*

(1) f is increasingly φ–regular,

(2) φ is a closure operation and $\mathrm{Ord}_f = \mathrm{Ord}_\varphi$.

Proof. Note that, by Definition 5.1, $\mathrm{Ord}_f = \mathrm{Ord}_\varphi$ if and only if, for any $u, v \in X$, we have $f(u) \le f(v) \iff \varphi(u) \le \varphi(v)$.

Moreover, by Theorem 7.17, φ is a closure operation if and only if, for any $u, v \in X$, we have $\varphi(u) \le \varphi(v) \iff u \le \varphi(v)$.

Remark 7.20. Hence, we can also easily see that φ is a closure operation if and only if $\mathrm{Ord}_\varphi(x) = \varphi^{-1}[\mathrm{ub}(x)]$ for all $x \in X$.

8. Some basic properties of increasingly seminormal functions

From the corresponding results of Section 7, by using Theorem 6.5 and it corollary, we can immediately derive the following assertions.

Theorem 8.1. *If f is an increasingly lower g–seminormal function of an arbitrary goset X to a reflexive one Y, then $g \circ f$ is an extensive operation on X.*

Corollary 8.2. *If f is an increasingly lower g–seminormal function of an arbitrary goset X to a reflexive one Y such that f is increasing, then $f \le f \circ g \circ f$.*

Theorem 8.3. *If f is an increasingly upper g–seminormal function of a reflexive goset X to an arbitrary one Y, then $f \circ g \circ f \le f$.*

Corollary 8.4. *If f is an increasingly g–normal function of a reflexive goset X to a reflexive, antisymmetric one Y such that f is increasing, then $f = f \circ g \circ f$.*

Theorem 8.5. *If f is an increasingly g–normal function of a reflexive goset X to a transitive one Y, then $g \circ f$ is an upper semiidempotent operation on X.*

Corollary 8.6. *If f is an increasingly g–normal function of a reflexive goset X to a proset Y, then $g \circ f$ is a semiclosure operation on X.*

Theorem 8.7. *If f is an increasingly g–normal function of a transitive goset X to a reflexive one Y, then*

(1) f is increasing, *(2) $f \le f \circ g \circ f$.*

Corollary 8.8. *If f is an increasingly g–normal function of a proset X to a reflexive, antisymmetric goset Y, then $f = f \circ g \circ f$.*

Theorem 8.9. *If f is an injective, increasingly g–normal function of a reflexive goset X to a reflexive, antisymmetric one Y such that either f is increasing or X is transitive, then $g \circ f = \Delta_X$, and thus g is onto X.*

Remark 8.10. Note that if f is a function of one set X to another Y and g is a function of Y to X such that $g \circ f = \Delta_X$, then f is already injective and g is onto X.

Moreover, by using Corollaries 8.4 and 8.8, we can also prove the following

Theorem 8.11. *If f is an increasingly g–normal function of a reflexive goset X onto a reflexive, antisymmetric one Y such that either f is increasing or X is transitive, then $f \circ g = \Delta_Y$, and thus g is injective.*

Proof. By Corollaries 8.4 and 8.8, we have $f(x) = \big(f \circ g \circ f\big)(x)$, and thus $f(x) = f\left(g\big(f(x)\big)\right)$ for all $x \in X$. Hence, by using that $f[X] = Y$, we can infer that $y = f\big(g(y)\big)$, and thus $\Delta_Y(y) = (f \circ g)(y)$ for all $y \in Y$. Therefore, the required equality is also true.

Remark 8.12. Note that if f is a function of one set X to another Y and g is a function of Y to X such that $f \circ g = \Delta_Y$, then g is already injective and f is onto Y.

From Theorem 7.12, by Corollary 6.6, it is clear that in particular we also have

Theorem 8.13. *If f is an increasingly g–normal function of one proset X to another Y, then $g \circ f$ is a closure operation on X.*

Now, from the above results, by using Theorem 6.11 and its corollary, we can immediately derive the following assertions.

Theorem 8.14. *If f is an increasingly upper g–seminormal function of a reflexive goset X to an arbitrary one Y, then $f \circ g$ is an intensive operation on Y.*

Corollary 8.15. *If f is an increasingly upper g–seminormal function of a reflexive goset X to an arbitrary one Y such that g is increasing, then $g \circ f \circ g \leq g$.*

Theorem 8.16. *If f is an increasingly lower g–seminormal function of an arbitrary goset X to a reflexive one Y, then $g \leq g \circ f \circ g$.*

Corollary 8.17. *If f is an increasingly g–normal function of a reflexive, antisymmetric goset X to a reflexive one Y such that g is increasing, then $g = g \circ f \circ g$.*

Theorem 8.18. *If f is an increasingly g–normal function of a transitive goset X to a reflexive one Y, then $f \circ g$ is a lower semiidempotent operation on Y.*

Corollary 8.19. *If f is an increasingly g–normal function of a proset X to a reflexive goset Y, then $f \circ g$ is a semiinterior operation on Y.*

Theorem 8.20. *If f is an increasingly g–normal function of a reflexive goset X to a transitive one Y, then*

(1) g is increasing, *(2) $g \circ f \circ g \leq g$.*

Corollary 8.21. *If f is an increasingly g–normal function of a reflexive, antisymmetric goset X to a proset Y, then $g = g \circ f \circ g$.*

Theorem 8.22. *If f is an increasingly g–normal function of a reflexive, antisymmetric goset X to a reflexive one Y such that g is injective and either g is increasing or Y is transitive, then $f \circ g = \Delta_Y$, and thus f is onto Y.*

Theorem 8.23. *If f is an increasingly g–normal function of a reflexive, antisymmetric goset X to a reflexive one Y such that g is onto X and either g is increasing or Y is transitive, then $g \circ f = \Delta_X$, and thus f is injective.*

Theorem 8.24. *If f is an increasingly g–normal function of one proset X to another Y, then $f \circ g$ is an interior operation on X.*

Moreover, as a partial converse to Theorem 8.1 and 8.7, we can also prove the following

Theorem 8.25. *If f is an arbitrary function of a transitive goset X to an arbitrary one Y and g is an increasing function of Y to X such that the operation $g \circ f$ is extensive, then f is increasingly lower g–seminormal.*

Proof. If $x \in X$ and $y \in Y$ such that $f(x) \leq y$, then by the increasingness of g we also have $g\big(f(x)\big) \leq g(y)\big)$, and thus $\big(g \circ f\big)(x) \leq g(y)$. Moreover, by the extensiveness of the operation $g \circ f$, we also have $x \leq (f \circ g)(x)$. Hence, by the transitivity of X, it follows that $x \leq g(y)$. Therefore, the required assertion is also true.

Now, as an immediate consequence of Theorems 8.25 and 8.1, we can also state

Corollary 8.26. *For an arbitrary function f on a transitive goset X to a reflexive one Y and an increasing function g of Y to X, the following assertions are equivalent:*

(1) $g \circ f$ is extensive, *(2) f is increasingly lower g–seminormal.*

Analogous to Theorem 8.24, we can also easily prove the following partial converse to Theorems 8.14 and 8.20, which can also be immediately derived from Theorem 8.25 by using Theorem 6.11.

Theorem 8.27. *If f is an increasing function of an arbitrary goset X to a transitive one Y and g is an arbitrary function of Y to X such that the operation $f \circ g$ is intensive, then f is increasingly upper g–seminormal.*

Now, as an immediate consequence of Theorems 8.27 and 8.14, we can also state

Corollary 8.28. *For an increasing function f on a reflexive goset X to a transitive one Y and an arbitrary function g of Y to X, the following assertions are equivalent:*

(1) $f \circ g$ is intensive, *(2) f is increasingly upper g–seminormal.*

Moreover, combining Theorems 8.25 and 8.27, we can also state

Corollary 8.29. *If f is an increasing function of one transitive goset X to another Y and g is an increasing function of Y to X such that the operation $g \circ f$ is extensive and the operation $f \circ g$ is intensive, then f is increasingly g–normal.*

Hence, by Theorem 8.13, it is clear that in particular we also have the following

Theorem 8.30. *For a function f on one proset X to another Y and a function g on Y to X, the following assertions are equivalent:*

(1) f is increasingly g–normal,

(2) f and g are increasing, $g \circ f$ is extensive, and $f \circ g$ is intensive.

9. Characterizations of increasingly seminormal functions

Theorem 9.1. *For any function f of one goset X to another Y and any function g of Y to X, the following assertions are equivalent:*

(1) $\operatorname{lb}\big(g(y)\big) \subset \operatorname{Int}_f(y)\big)$ for all $y \in Y$,

(2) f is an increasingly upper g–seminormal.

Proof. If (1) holds, then by the corresponding definitions it is clear that

$$x \leq g(y) \quad \Longrightarrow \quad x \in \operatorname{lb}\big(g(y)\big) \quad \Longrightarrow \quad x \in \operatorname{Int}_f(y) \quad \Longrightarrow \quad f(x) \leq y$$

for all $x \in X$ and $y \in Y$. Therefore, (2) also holds.

The converse implication $(2) \Longrightarrow (1)$ can be proved quite similarly by reversing the above argument.

Theorem 9.2. *For any function f of one goset X to another Y and any function g of Y to X, the following assertions are equivalent:*

(1) $\operatorname{Int}_f(y) \subset \operatorname{lb}\big(g(y)\big)$ *for all* $y \in Y$*,*

(2) $g(y) \in \operatorname{ub}\big(\operatorname{Int}_f(y)\big)$ *for all* $y \in Y$*,*

(3) f is an increasingly lower g–seminormal.

Proof. If (3) holds, then by the corresponding definitions it is clear that

$$x \in \operatorname{Int}_f(y) \quad \Longrightarrow \quad f(x) \leq y \quad \Longrightarrow \quad x \leq g(y) \quad \Longrightarrow \quad x \in \operatorname{lb}\big(g(y)\big)$$

for all $x \in X$ and $y \in Y$. Therefore, (1) also holds.

The converse implication $(1) \Longrightarrow (3)$ can be proved quite similarly by reversing the above argument. Moreover, from Theorem 2.3, we can see that (1) and (2) are also equivalent.

Now, as an immediate consequence of the above two theorems, we can also state

Corollary 9.3. *For any function f of one goset X to another Y and any function g of Y to X, the following assertions are equivalent:*

(1) f is an increasingly g–normal, *(2)* $\operatorname{Int}_f(y) = \operatorname{lb}\big(g(y)\big)$ *for all* $y \in Y$*.*

Concerning increasing upper seminormality, we can also prove the following

Theorem 9.4. *If f is an increasingly upper g–seminormal function of a reflexive goset X to an arbitrary one Y, then for any $y \in Y$ we have*

$$g(y) \in \operatorname{Int}_f(y)\,.$$

Proof. By Theorem 8.14, we have $(f \circ g)(y) \leq y$, and thus $f\big(g(y)\big) \leq y$. Therefore, by the definition of Int_f, the required assertion is also true.

Now, as an immediate consequence of Theorems 9.2, 9.4 and 2.5, we can state

Corollary 9.5. *If f is an increasingly g–normal function of a reflexive goset X to an arbitrary one Y, then for any $y \in Y$ we have*

(1) $g(y) \in \max\big(\operatorname{Int}_f(y)\big)$*,* *(2)* $g(y) \in \operatorname{Int}_f(y) \subset \operatorname{lb}\big(g(y)\big)$*.*

Remark 9.6. Note that if in particular X is antisymmetric, then by Theorem 2.9 in (1) we may write $g(y) = \max\big(\operatorname{Int}_f(y)\big)$.

Moreover, as a certain converse to Theorem 9.4, we can also prove the following

Theorem 9.7. *If f is an increasing function of an arbitrary goset X to a transitive one Y and g is a function of Y to X such that*

$$g(y) \in \operatorname{Int}_f(y)$$

for all $y \in Y$, then f is increasingly upper g–seminormal.

Proof. If $x \in X$ and $y \in Y$ such that $x \leq g(y)$, then by the increasingness f we also have $f(x) \leq f\big(g(y)\big)$. Moreover, by the assumption $g(y) \in \operatorname{Int}_f(y)$, we also have $f\big(g(y)\big) \leq y$. Hence, by using the transitivity of Y, we can already infer that $f(x) \leq y$. Therefore, the required assertion is also true.

Now, as an immediate consequence of Theorems 9.4 and 9.7, we can also state

Corollary 9.8. *For an increasing function f on a reflexive goset X to a transitive one Y and an arbitrary function g of Y to X, the following assertions are equivalent:*

(1) $g(y) \in \operatorname{Int}_f(y)$ *for all* $y \in Y$,

(2) f *is increasingly upper* g*–seminormal.*

Moreover, by using Theorems 8.7, 9.2 and 9.7 and Corollary 9.5, we can prove

Theorem 9.9. *For a function f of one proset to another Y and a function g of Y to X, the following assertions are equivalent:*

(1) f *is increasingly* g*–normal,*

(2) f *is increasing and* $g(y) \in \max\big(\operatorname{Int}_f(y)\big)$ *for all* $y \in Y$.

Proof. If (1) holds, then by Theorem 8.7 f is increasing. Moreover, by Corollary 9.5, $g(y) \in \max\big(\operatorname{Int}_f(y)\big)$ for all $y \in Y$. Therefore, (2) also holds.

While, if (2) holds, then by Theorem 9.7 f is increasingly upper g–seminormal. Moreover, by Theorem 9.2, f is increasingly lower g–seminormal. Therefore, (1) also holds.

Remark 9.10. Note that if in particular X is antisymmetric, then by Theorem 2.9 in (2) we may write $g(y) = \max\big(\operatorname{Int}_f(y)\big)$ for all $y \in Y$.

From the above results, by using the Axiom of Choice, we can immediately derive several useful characterizations of increasingly lower and upper seminormal functions.

For instance, from Theorem 9.9 we can immediately derive the following

Theorem 9.11. *For a function f of one proset X to another Y, the following assertions are equivalent:*

(1) f *is increasingly normal,*

(2) f *is increasing and* $\max\big(\operatorname{Int}_f(y)\big) \neq \emptyset$ *for all* $y \in Y$.

10. Characterizations of increasingly semiregular functions

Theorem 10.1. *For any function f of one goset X to another Y and operation φ on X, the following assertions are equivalent:*

(1) $\operatorname{lb}\big(\varphi(x)\big) \subset \operatorname{Ord}_f^{-1}(x)$ *for all* $x \in X$,

(2) f *is increasingly upper* φ*–semiregular.*

Proof. If (1) holds, then by the corresponding definitions it is clear that

$$u \leq \varphi(x) \implies u \in \mathrm{lb}\big(\varphi(x)\big) \implies u \in \mathrm{Ord}_f^{-1}(x) \implies x \in \mathrm{Ord}_f(u) \implies f(u) \leq f(x)$$

for all $x, u \in X$. Therefore, (2) also holds.

The converse implication $(2) \Longrightarrow (1)$ can be proved quite similarly by reversing the above argument.

Theorem 10.2. *For any functions f of one goset X to another Y and operation φ on X, the following assertions are equivalent:*

(1) $\mathrm{Ord}_f^{-1}(x) \subset \mathrm{lb}\big(\varphi(x)\big)$ *for all* $x \in X$,

(2) $\varphi(x) \in \mathrm{ub}\big(\mathrm{Ord}_f^{-1}(x)\big)$ *for all* $x \in X$;

(3) f *is increasingly lower* φ*–semiregular.*

Proof. If (3) holds, then by the corresponding definitions it is clear that

$$u \in \mathrm{Ord}_f^{-1}(x) \implies x \in \mathrm{Ord}_f(u) \implies f(u) \leq f(x) \implies u \leq \varphi(x) \implies u \in \mathrm{lb}\big(\varphi(x)\big)$$

for all $x, u \in X$. Therefore, (1) also holds.

The converse implication $(1) \Longrightarrow (3)$ can be proved quite similarly by reversing the above argument. Moreover, from Theorem 2.3 we can see that (1) and (2) are also equivalent.

Now, as an immediate consequence of the above two theorems, we can also state

Corollary 10.3. *For any functions f of one goset X to another Y and φ of X to itself, the following assertions are equivalent:*

(1) f *is increasingly* φ*–regular,* *(2)* $\mathrm{Ord}_f^{-1}(x) = \mathrm{lb}\big(\varphi(x)\big)$ *for all* $x \in X$.

Concerning increasing upper semiregularity, we can also prove the following

Theorem 10.4. *If f is an increasingly upper φ–semiregular function of a reflexive goset X to an arbitrary one Y, then for any $x \in X$ we have*

$$\varphi(x) \in \mathrm{Ord}_f^{-1}(x)\,.$$

Proof. By Theorem 7.3, we have $\big(f \circ \varphi\big)(x) \leq f(x)$, and thus $f\big(\varphi(x)\big) \leq f(x)$. Hence, by the definition of Ord_f, it follows that $x \in \mathrm{Ord}_f\big(\varphi(x)\big)$, and thus the required assertion is also true.

Now, as an immediate consequence of Theorems 10.2, 10.4 and 2.5, we can state

Corollary 10.5. *If f is an increasingly φ–regular function of a reflexive goset X to an arbitrary one Y, then for any $x \in X$ we have*

(1) $\varphi(x) \in \max\big(\mathrm{Ord}_f^{-1}(x)\big)$, *(2)* $\varphi(x) \in \mathrm{Ord}_f^{-1}(x) \subset \mathrm{lb}\big(\varphi(x)\big)$.

Remark 10.6. Note that if in particular X is antisymmetric, then by Theorem 2.9 in (2) we may write $\varphi(x) = \max\big(\mathrm{Ord}_f^{-1}(x)\big)$.

Moreover, as a certain converse to Theorem 10.4, we can also prove the following

Theorem 10.7. *If f is an increasing function of an arbitrary goset X to a transitive one Y and φ is an operation on X such that*

$$\varphi(x) \in \operatorname{Ord}_f^{-1}(x)$$

for all $x \in X$, then f is increasingly upper φ–semiregular.

Proof. If $u, v \in X$ such that $u \leq \varphi(v)$, then by the increasingness f we also have $f(u) \leq f\big(\varphi(v)\big)$. Moreover, by the assumption $\varphi(v) \in \operatorname{Ord}_f^{-1}(v)$, i. e., $v \in \operatorname{Ord}_f\big(\varphi(v)\big)$, we also have $f\big(\varphi(v)\big) \leq f(v)$. Hence, by using the transitivity of Y, we can already infer that $f(u) \leq f(v)$. Therefore, the required assertion is also true.

Now, as an immediate consequence of Theorems 10.4 and 10.7, we can also state

Corollary 10.8. *For any increasing function f of a reflexive goset X to a transitive one Y and operation φ on X, the following assertions are equivalent:*

(1) $\varphi(x) \in \operatorname{Ord}_f^{-1}(x)$ *for all* $x \in X$,

(2) *f is increasingly upper φ–semiregular.*

Moreover, by using Theorems 7.7 and 10.2 and 10.7 and Corollary 10.5, we can also prove

Theorem 10.9. *For any function f of one proset X to another Y and operation φ on X, the following assertions are equivalent:*

(1) *f is increasingly φ–regular,*

(2) *f is increasing and $\varphi(x) \in \max\big(\operatorname{Ord}_f^{-1}(x)\big)$ for all $x \in X$.*

Proof. If (1) holds, then by Theorem 7.7 f is increasing. Moreover, by Corollary 10.5, $\varphi(x) \in \max\big(\operatorname{Ord}_f^{-1}(x)\big)$ for all $x \in X$. Therefore, (2) also holds.

While, if (2) holds, then by Theorem 10.7 f is increasingly upper φ–semiregular. Moreover, by Theorem 10.2, f is increasingly lower φ–semiregular. Therefore, (1) also holds.

Remark 10.10. Note that if in particular X is antisymmetric, then by Theorem 2.9 in (2) we may write $\varphi(x) = \max\big(\operatorname{Ord}_f^{-1}(x)\big)$ for all $x \in X$.

From the above results, by using the Axiom of Choice, we can immediately derive several useful characterizations of increasingly lower and upper semiregular functions.

For instance, from Theorem 10.9, it is clear that we have the following

Theorem 10.11. *For a function f of one goset X to another Y, the following assertions are equivalent:*

(1) *f is increasingly regular,*

(2) *f is increasing $\max\big(\operatorname{Ord}_f^{-1}(x)\big) \neq \emptyset$ for all $x \in X$.*

Now, as an immediate consequence of Corollaries 6.6 and 5.10 and Theorems 10.11 and 9.11, we can also state

Corollary 10.12. *For a function f of one goset X onto another Y, the following assertions are equivalent:*

(1) *f is increasingly normal,* (2) *f is increasingly regular.*

Proof. If (2) holds, then by Theorem 10.11 we can see that f is increasing and $\max\big(\operatorname{Ord}_f^{-1}(x)\big) \neq \emptyset$ for all $x \in X$. Hence, by Corollary 5.10, we can infer that

$$\max\big(\operatorname{Int}_f\big(f(x)\big)\big) = \max\big(\big(\operatorname{Int}_f \circ f\big)(x)\big) = \max\big(\operatorname{Ord}_f^{-1}(x)\big) \neq \emptyset$$

for all $x \in X$. Now, because of $Y = f\,[\,X\,]$, it is clear that $\max\big(\operatorname{Int}_f(y)\big) \neq \emptyset$ for all $y \in Y$. Thus, by Theorem 9.11, (1) also holds.

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