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A COMMON GENERALIZATION OF THE POSTMAN, RADIAL AND RIVER METRICS

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Abstract. By using a metric d on a set X, a function φ of X to itself, a metric ρ on the range of φ, and a suitable relation Γ on X^2 to X, we construct a metric $d_{\rho\varphi\Gamma}$ on X. This compound metric includes the postman, radial and river metrics as some very particular cases.

Our construction here closely follows a former one of M. Borkowski, D. Bugajewski and H. Przybycień. Moreover, it may also be compared to that of A. G Aksoy and B. Maurizi. However, instead of a metric projection and a collinearity relation we use the above mentioned φ and Γ.

Introduction

The defining axioms of a metric were abstracted from the well-known properties of the Euclidean distances by M. Fréchet in 1906. The appropriateness of weakening and strengthening of these axioms have later been justified by several authors.

However, in the present paper, we shall adhere to the original axioms. Though, most distance functions occurring in analysis are *extended-valued pseudo-metrics*. Moreover, *semimetrics*, *quasi-metrics*, *ultrametrics*, and *partial metrics* also have several applications.

Thus, now a metric on a set X is a function d of X^2 to $\mathbb{R}$ such that, for any $x, y, z \in X$, we have

(1) $d(x, y) \geq 0$;

(2) $d(x, y) = d(y, x)$;

(3) $d(x, z) \leq d(x, y) + d(y, z)$;

(4) $d(x, y) = 0$ if and only if $x = y$.

Here, (1) and (4) are referred to as the *positive definiteness*, (2) as *symmetry*, and (3) as *the triangle inequality*. Note that, by (3) and (2), we always have

$$d(x, x) \leq d(x, y) + d(y, x) = 2\, d(x, y).$$

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Hence, by using the "*equality implies indistancy*" part of (4), we can infer (1). However, it is usually more convenient to stress *nonnegativity* as a separate axiom.

For any $x, y \in X$, by defining

$$d(x, y) = \begin{cases} 0 & \text{if} \quad x = y, \\ 1 & \text{if} \quad x \neq y, \end{cases}$$

we can at once get an ultrametric d on X. This is called the *discrete metric* on X. Thus, each set can be considered as a discrete metric space. Therefore, the notions of a set and a metric space are actually equivalent.

However, to provide several genuine illustrating examples for a metric, it is best to assume that $X = \mathbb{C}$ with $\mathbb{C} = \mathbb{R}^2$. Thus, for any $x, y \in X$, we may write

$$x = (x_1, x_2), \qquad \bar{x} = (x_1, -x_2);$$
$$x + y = (x_1 + y_1,\ x_2 + y_2), \qquad x y = (x_1 y_1 - x_2 y_2,\ x_1 y_2 + x_2 y_1).$$

Now, each $r \in \mathbb{R}$ can be identified with $(r, 0) \in X$. And each $x \in X$ can be written in the form $x = x_1 + i x_2$ with $i = (0, 1)$.

Moreover, for any $x, y \in X$, we may also write

$$d(x, y) = |x - y| \qquad \text{with} \qquad |z| = (z \bar{z})^{1/2} = \left(z_1^2 + z_2^2 \right)^{1/2}.$$

Now, by using the above operations on complex numbers, it can be easily seen that $|\ |$ is a norm on X, and thus d is a metric on X. This d is called the Euclidean metric on X.

More generally, for any $x, y \in X$ and $p \in [1, \infty]$, we may also naturally define

$$d_p(x, y) = |x - y|_p \qquad \text{with} \qquad |z|_p = \begin{cases} \left(|z_1|^p + |z_2|^p \right)^{1/p} & \text{if} \quad p < \infty, \\ \max\left\{ |z_1|, |z_2| \right\} & \text{if} \quad p = \infty. \end{cases}$$

Now, it is somewhat more difficult to prove that $|\ |_p$ is a norm on X, and thus d_p is a metric on X. In particular d_1 and d_∞ are called the *taxicab* and *supremum metrics* on X, respectively.

Beside the latter two extreme metrics, there are some further curious, but important metrics on X. For instance, for any $x, y \in X$, we may also define

$$\alpha(x, y) = \begin{cases} 0 & \text{if} \quad x = y, \\ |x| + |y| & \text{if} \quad x \neq y; \end{cases}$$

$$\beta(x, y) = \begin{cases} |x - y| & \text{if} \quad x_1 y_2 = x_2 y_1, \\ |x| + |y| & \text{if} \quad x_1 y_2 \neq x_2 y_1; \end{cases}$$

and

$$\gamma(x, y) = \begin{cases} |x_2 - y_2| & \text{if} \quad x_1 = y_1, \\ |x_2| + |x_1 - y_1| + |y_2| & \text{if} \quad x_1 \neq y_1. \end{cases}$$

Thus, by considering several cases, it can be shown that α, β and γ are metrics on X. These are usually called the *postman, radial* and *river metrics* on X, respectively. (See, for instance, [17 , p. 155] and [5 , p. 315] .) Sometimes, the radial metric is also called the hedgehog or French railroad metric.

In the present paper, following the ideas of Borkowski, Bugajewski and Przybycień [3] , we construct a common generalization of the postman, radial and river metrics. Our generalization here may also be compared to that of Aksoy and Maurizi [1] . However, instead of a metric projection and collinearity relation we shall use some more general objects.

More concretely, by assuming that d is a metric on a set X, φ is a function of X to itself, and ρ is a metric on the range $\varphi[X]$ of φ, we define a generalized metric $d_{\rho\varphi}$ on X such that

$$d_{\rho\varphi}(x, y) = d\big(x, \varphi(x)\big) + \rho\big(\varphi(x), \varphi(y)\big) + d\big(\varphi(y), y\big)$$

for all $x, y \in X$. Moreover, by assuming that Γ is a suitable relation on X^2 to X, we define a generalized equivalence relation $Q_{\varphi\Gamma}$ on X such that

$$Q_{\varphi\Gamma} = \big\{ (x, y) \in X^2 : \quad \varphi(x) = \varphi(y) \in \Gamma(x, y) \big\} .$$

Thus, by defining

$$d_{\rho\varphi\Gamma}(x, y) = \begin{cases} d(x, y) & \text{if} \quad (x, y) \in Q_{\varphi\Gamma} , \\ d_{\rho\varphi}(x, y) & \text{if} \quad (x, y) \notin Q_{\varphi\Gamma} , \end{cases}$$

we can get a metric $d_{\rho\varphi\Gamma}$ on X which includes the postman, radial and river metrics as some very particular cases.

For instance, the postman metric α can be immediately obtained from $d_{\varphi\rho\Gamma}$, by letting d to be the Euclidean metric on $X = \mathbb{C}$, and defining

$$\varphi(x) = 0 \qquad \text{and} \qquad \Gamma(x, y) = \begin{cases} X & \text{if} \quad x = y , \\ \{0\}^c & \text{if} \quad x \neq y . \end{cases}$$

for all $x, y \in X$. While, to get the radial metric β, we have to consider the relation Γ defined such that, for all $x, y \in X$, we have $\Gamma(x, y) = X$ if $x = y$, and

$$\Gamma(x, y) = \big\{ z \in X : \quad \exists \ \lambda \in K : \quad z = \lambda x + (1 - \lambda) y \big\} \qquad \text{if} \qquad x \neq y .$$

The latter relation Γ can also be applied to a similar derivation of the river metric γ by the function φ defined such that $\varphi(x) = x_1$ for all $x \in X$.

1. Fixed points and equivalence relations

Notation 1.1. *Let X be a set and φ be a function of X to itself. Define*

$$A_\varphi = \big\{ x \in X : \quad \varphi(x) = x \big\} , \qquad B_\varphi = \big\{ x \in X : \quad \varphi(x) \in A_\varphi \big\} ,$$

$$D_\varphi = \big\{ (x, x) : \quad x \in A_\varphi \big\} , \qquad E_\varphi = \big\{ (x, y) \in X^2 : \quad \varphi(x) = \varphi(y) \big\} .$$

Remark 1.2. Thus, A_φ and B_φ are the family of all fixed and idempotent points of φ, respectively.

Moreover, D_φ is the identity function of A_φ and E_φ is the equivalence relation on X generated by φ.

Remark 1.3. For the identity function Δ_X of X, we have

$$A_{\Delta_X} = B_{\Delta_X} = X \qquad \text{and} \qquad D_{\Delta_X} = E_{\Delta_X} = \Delta_X .$$

Moreover, if $A_\varphi = X$, or equivalently $D_\varphi = \Delta_X$, then we have $\varphi = \Delta_X$.

Simple reformulations of the above definitions yield the following theorems.

Theorem 1.4. *For any $x, y \in X$, the following assertions are equivalent:*

(1) $(x, y) \in D_\varphi$ *;*

(2) $x, y \in A_\varphi$ *and* $x = y$ *;*

(3) $x, y \in A_\varphi$ *and* $(x, y) \in E_\varphi$ *;*

(4) $x = \varphi(x), \quad \varphi(x) = \varphi(y), \quad \varphi(y) = y$ *.*

Theorem 1.5. *We have*

(1) $B_\varphi = \varphi^{-1}[A_\varphi] = \{ x \in X : \ \varphi^2(x) = \varphi(x) \}$ *;*

(2) $D_\varphi = \Delta_{A_\varphi} = A_\varphi^2 \cap E_\varphi$ *;* *(3)* $E_\varphi = \varphi^{-1} \circ \varphi$ *;*

(4) $\varphi[A_\varphi] \subset A_\varphi \subset \varphi[X]$ *;* *(5)* $A_\varphi = A_{\varphi^2} \cap B_\varphi$ *.*

Proof. By the corresponding definitions, for any $x \in X$, we have

$$x \in B_\varphi \iff \varphi(x) \in A_\varphi \iff x \in \varphi^{-1}[A_\varphi],$$

$$\begin{aligned} x \in B_\varphi \iff \varphi(x) \in A_\varphi \iff & \varphi(\varphi(x)) = \varphi(x) \\ \iff & (\varphi \circ \varphi)(x) = \varphi(x) \iff \varphi^2(x) = \varphi(x). \end{aligned}$$

Therefore, (1) is true.

By the corresponding definitions, it is clear that $D_\varphi = \Delta_{A_\varphi}$. Moreover, from Theorem 1.4, we can see hat

$$(x, y) \in D_\varphi \iff (x, y) \in A_\varphi^2, \quad (x, y) \in E_\varphi \iff (x, y) \in A_\varphi^2 \cap E_\varphi .$$

Therefore, (2) is true. On the other hand, by the corresponding definitions, we also have

$$\begin{aligned} (x, y) \in E_\varphi \iff \varphi(y) = \varphi(x) \iff & y \in \varphi^{-1}(\varphi(x)) \\ \iff & y \in (\varphi^{-1} \circ \varphi)(x) \iff (x, y) \in \varphi^{-1} \circ \varphi . \end{aligned}$$

Therefore, (3) is also true.

Furthermore, we can also easily see that

$$x \in A_\varphi \implies x = \varphi(x) \implies x \in \varphi[X],$$

$$x \in A_\varphi \implies \varphi(x) = x \implies \varphi(\varphi(x)) = \varphi(x) = x \implies \varphi^2(x) = x \implies x \in A_{\varphi^2}\,,$$

$$x \in A_\varphi \implies \varphi(x) = x \implies \varphi(\varphi(x)) = \varphi(x) \implies \varphi(x) \in A_\varphi \implies x \in B_\varphi\,.$$

Hence, we can infer that

$$A_\varphi \subset \varphi\,[\,X\,]\,, \qquad A_\varphi \subset X_{\varphi^2} \qquad \text{and} \qquad \varphi\,[\,A_\varphi\,] \subset A_\varphi\,, \qquad A_\varphi \subset B_\varphi\,.$$

Therefore, (4) and $A_\varphi \subset A_{\varphi^2} \cap B_\varphi$ is also true.

Now, to prove (5), it remains to note only that

$$\begin{aligned} x \in A_{\varphi^2} \cap B_\varphi &\implies x \in A_{\varphi^2}\,,\ x \in B_\varphi \\ &\implies \varphi^2(x) = x\,, \quad \varphi^2(x) = \varphi(x) \implies \varphi(x) = x \implies x \in A_\varphi\,, \end{aligned}$$

and thus $A_{\varphi^2} \cap B_\varphi \subset A_\varphi$ is also true.

2. Projections, involutions, and injections

Definition 2.1. In the sequel, we shall say that:

(1) φ is a *projection* if $\varphi^2 = \varphi$;

(2) φ is an *involution* if $\varphi^2 = \Delta_X$;

(3) φ is an *injection* if φ is injective.

Remark 2.2. Hence, it is clear that if φ is an involution, then φ is, in particular, also an injection.

Namely, if $x,\, y \in X$ such that $\varphi(x) = \varphi(y)$, then by the corresponding definitions we also have $x = \varphi^2(x) = \varphi(\varphi(x)) = \varphi(\varphi(y)) = \varphi^2(y) = y$.

Remark 2.3. Moreover, by the corresponding definitions, it is clear that the function φ is simultaneously both a projection and an involution if and only if $\varphi = \Delta_X$.

Now, in addition to Remark 1.3 and Theorem 1.5, we can also easily establish the following three theorems.

Theorem 2.4. *The following assertions are equivalent:*

(1) φ *is an injection;* *(2)* φ^{-1} *is a function;* *(3)* $E_\varphi = \Delta_X$.

Proof. If $(x,\, y) \in E_\varphi$, then $\varphi(x) = \varphi(y)$. Hence, if (1) holds, then we can infer that $x = y$, and thus $(x,\, y) \in \Delta_X$. Therefore, $E_\varphi \subset \Delta_X$. Thus, by the reflexivity of E_φ, (3) also holds.

The converse implication is even more obvious. Namely, if $x,\, y \in X$ such that $\varphi(x) = \varphi(y)$, then $(x,\, y) \in E_\varphi$. Hence, if (3) holds, then we can infer that $(x,\, y) \in \Delta_X$, and thus $x = y$. Therefore, (1) also holds.

Remark 2.5. If φ is an injection, then in addition to the above assertions we can also state that $A_\varphi = B_\varphi$.

Namely, by Theorem 1.5, we always have $A_\varphi \subset B_\varphi$. Moreover, if $x \in B_\varphi$, then $\varphi(\varphi(x)) = \varphi(x)$. Hence, by the injectivity of φ, it follows that $\varphi(x) = x$, and thus $x \in A_\varphi$. Therefore, $B_\varphi \subset A_\varphi$, and thus the required equality is also true.

However, the equality $A_\varphi = B_\varphi$ does not, in general, imply the injectivity of φ.

Example 2.6. Namely, if for instance $X = \{0, 1, 2, 3\}$ and φ of X such that $\varphi(0) = 0$, $\varphi(1) = 2$ and $\varphi(2) = \varphi(3) = 1$, then we have $A_\varphi = B_\varphi = \{0\}$, despite that φ is not injective.

Theorem 2.7. *The following assertions are equivalent:*

(1) φ *is an involution;* *(2)* $\varphi = \varphi^{-1}$*;* *(3)* $A_{\varphi^2} = X$.

Proof. Now, to prove the equivalence of (1) and (3), it is enough to note only that, by Remark 1.3, we have $\varphi^2 = \Delta_X$ if and only if $A_{\varphi^2} = X$.

Remark 2.8. If φ is an involution, then by Remarks 2.2, 2.5 and Theorem 2.4 we also have $A_\varphi = B_\varphi$ and $E_\varphi = \Delta_X$.

However, the latter equalities do not, in general, imply that φ is an involution.

Example 2.9. Namely, if for instance $X = \mathbb{R}$ and $\varphi(x) = x/(1 + |x|)$ for all $x \in X$, then it is clear that $A_\varphi = B_\varphi = \{0\}$. Moreover, it can be easily seen that φ is an injection of X onto $]-1, 1[$ such that $\varphi^{-1}(y) = y/(1 - |y|)$ for all $y \in]-1, 1[$. Thus, by the above theorems, $E_\varphi = \Delta_X$, but φ is not an involution.

Theorem 2.10. *The following assertions are equivalent:*

(1) φ *is a projection;* *(2)* $X = B_\varphi$*;* *(3)* $A_\varphi = \varphi[X]$.

Proof. By the corresponding definitions and Theorem 1.5, it is clear that

$$\begin{aligned} \varphi^2 = \varphi \iff \forall\ x \in X:\ \ \varphi^2(x) = \varphi(x) \\ \iff \forall\ x \in X:\ \ x \in B_\varphi \iff X \subset B_\varphi \iff X = B_\varphi. \end{aligned}$$

and

$$\begin{aligned} \varphi^2 = \varphi \iff \forall\ x \in X:\ \ \varphi(\varphi(x)) = \varphi(x) \\ \iff \forall\ x \in X:\ \ \varphi(x) \in A_\varphi \iff \varphi[X] \subset A_\varphi \iff A_\varphi = \varphi[X]. \end{aligned}$$

Therefore, the required equivalences are also true.

Remark 2.11. If φ is a projection, then by Theorems 1.5 and 2.10 we also have $A_\varphi = A_{\varphi^2} \cap B_\varphi = A_{\varphi^2} \cap X = A_{\varphi^2}$.

However, the equality $A_\varphi = A_{\varphi^2}$ does not, in general, imply that φ is a projection even if φ is injective.

Example 2.12. Namely, if for instance X and φ are as in Example 2.9, then it can be easily seen that $\varphi^2(x) = x/(1 + 2|x|)$ for all $x \in X$. Therefore, $A_{\varphi^2} = \{0\}$ also holds.

Remark 2.13. In this respect, it is also worth noticing that if in particular φ is an involution such that $A_\varphi = A_{\varphi^2}$, then by Theorem 2.7 we also have $A_\varphi = X$. Therefore, by Remark 1.3, $\varphi = \Delta_X$. Thus, in particular φ is a projection.

3. Weak partial pseudo-metrics specified by φ

Definition 3.1. A function d of X^2 to $\mathbb{R}$ is called a *φ–metric* on X if for any $x, y, z \in X$ we have

(1) $d(x, y) \geq 0$;

(2) $d(x, y) = d(y, x)$;

(3) $d(x, z) \leq d(x, y) + d(y, z) - d(y, y)$;

(4) $d(x, y) = 0$ if and only if $(x, y) \in D_\varphi$.

Remark 3.2. A function d of X^2 to $\mathbb{R}$ satisfying conditions (1)–(3) has formerly been called a *weak partial pseudo-metric* by Heckmann [6]. This is a straightforward generalization of the *partial metric* of Matthews [11].

Now, a function d of X^2 to $\mathbb{R}$ may be briefly called a partial metric on X if it is a weak partial pseudo-metric on X such that

(1) $d(x, x) \leq d(x, y)$ for all $x, y \in X$

(2) $d(x, x) = d(x, y) = d(y, y)$ implies $x = y$.

Remark 3.3. Non-zero self-distances were already considered in a 1985 thesis of Matthews. And, the modified triangle inequality (3) was already suggested to Matthews by Wickers [21] in 1987. However, the present definition of a partial metric was only first investigated in the later works [11] and [12].

Partial metrics, being a minimal generalization of metrics allowing non-zero self-distances, were motivated by experience from theoretical computer science. The interested reader can get a rapid overview on the subject by consulting the works [13], [4] and [10], where convincing illustrating examples are also given.

Now, analogously to Propositions 2.2 and 2.4 of Heckmann [6], we can also easily establish the following two theorems.

Theorem 3.4. *For any function d of X^2 to $\mathbb{R}$, the following assertions are equivalent:*

(1) d is a metric on X;

(2) d is a Δ_X–metric on X;

(3) d is a φ–metric on X, for some φ, such that $d(x, x) = 0$ for all $x \in X$.

Proof. Since $D_{\Delta_X} = \Delta_X$, it is clear that (1) $\iff$ (2) $\implies$ (3). Moreover, if (3) holds, then by Definition 3.1 (4) we can see that $\Delta_X \subset D_\varphi$. Hence, since $D_\varphi = \Delta_{A_\varphi}$, it is clear that $A_\varphi = X$. Therefore, $\varphi = \Delta_X$, and thus (2) also holds.

Theorem 3.5. *If p is a φ–metric on X, then for any $x, y \in X$ we have*

(1) $d(x, x) + d(y, y) \leq 2\, d(x, y)$;

(2) $\min\big\{ d(x, x),\ d(y, y)\big\} \leq d(x, y)$;

(3) $d(x, y) = \min\limits_{z \in X} \big(d(x, z) + d(z, y) - d(z, z)\big)$.

Proof. By Definition 3.1 (3) and (2), we have

$$d(x, x) \leq d(x, y) + d(y, x) - d(y, y) = 2\, d(x, y) - d(y, y),$$

and thus (1) is true. Hence, it is clear that either

$$d(x, x) \leq d(x, y) \qquad \text{or} \qquad d(y, y) \leq d(x, y).$$

Therefore, (2) also holds.

Finally, to prove (3), we need only note that, by Definition 3.1 (4), we have

$$d(x, y) \leq d(x, z) + d(z, y) - d(z, z)$$

for all $z \in X$. Moreover, we also have $d(x, y) = d(x, y) + d(y, y) - d(y, y)$.

Remark 3.6. Note that the "small self-distances condition" in Remark 3.2 (1) can also be reformulated by writing that

$$d(x, x) = \min_{y \in X} d(x, y)$$

for all $x \in X$. Therefore, the interesting partial metric axioms are about minima.

Remark 3.7. However, the *ultra-metric triangle inequality* [20] says that

$$d(x, z) \leq \max\big\{ d(x, y),\ d(y, z)\big\}$$

for all $x, y, z \in X$. This is also called the non-Archimedean triangle inequality.

While, the famous *four-point property* [1] says that

$$d(x, y) + d(z, w) \leq \max\big\{ d(x, z) + d(y, w),\ d(x, w) + d(y, z)\big\}$$

for all $x, y, z, w \in X$. This is closely related to the ultra-metric triangle inequality.

Note that, under the usual symmetry condition and the zero self-distances assumption, "the ultra-metric triangle inequality" implies "the four-point property" implies "the ordinary triangle inequality".

Moreover, the ordinary triangle inequality is equivalent to the *rectangle inequality* which says that

$$|\, d(x, y) - d(z, w)\,| \leq d(x, z) + d(y, w)$$

for all $x, y, z, w \in X$. There is a curious *rectrangular inequality* in [2] too.

4. φ-metrics derived from ordinary metrics by φ

Notation 4.1. *Let d and ρ be metrics on X and $\varphi[X]$, respectively. Moreover, for any $x, y \in X$, define*

$$d_{\rho\varphi}(x, y) = d\big(x, \varphi(x)\big) + \rho\big(\varphi(x), \varphi(y)\big) + d\big(\varphi(y), y\big).$$

Remark 4.2. Thus, in particular, we have

$$d_{\rho\Delta_X}(x, y) = d(x, x) + \rho(x, y) + d(y, y) = \rho(x, y)$$

for all $x, y \in X$. Therefore, $d_{\rho\Delta_X} = \rho$.

Our former definitions are mainly motivated by the following

Theorem 4.3. *The function* $d_{\rho\varphi}$ *is a* φ*–metric on* X *such that:*

(1) $d_{\rho\varphi}(x, y) = \rho(x, y)$ *for all* $x, y \in A_{\varphi}$ *;*

(2) $d_{\rho\varphi}(x, x) = 2\, d\big(x, \varphi(x)\big)$ *for all* $x \in X$.

Proof. By Notation 4.1, it is clear that $d_{\rho\varphi}$ is a nonnegative, real-valued function of X^2. Moreover, if $x, y \in X$, then by using the nonegativity and separating properties of d and ρ, and Theorem 1.4, we can easily see that

$$\begin{aligned} d_{\rho\varphi}(x, y) = 0 \quad &\Longleftrightarrow \quad d\big(x, \varphi(x)\big) + \rho\big(\varphi(x), \varphi(y)\big) + d\big(\varphi(y), y\big) = 0 \\ &\Longleftrightarrow \quad d\big(x, \varphi(x)\big) = 0\,, \quad \rho\big(\varphi(x), \varphi(y)\big) = 0\,, \quad d\big(\varphi(y), y\big) = 0 \\ &\Longleftrightarrow \quad x = \varphi(x)\,, \quad \varphi(x) = \varphi(y)\,, \quad \varphi(y) = y \quad \Longleftrightarrow \quad (x, y) \in D_{\varphi}\,. \end{aligned}$$

Furthermore, by the symmetry properties of d and ρ, it is clear that

$$\begin{aligned} d_{\rho\varphi}(y, x) &= d\big(y, \varphi(y)\big) + \rho\big(\varphi(y), \varphi(x)\big) + d\big(\varphi(x), x\big) \\ &= d\big(\varphi(y), y\big) + \rho\big(\varphi(x), \varphi(y)\big) + d\big(x, \varphi(x)\big) \\ &= d\big(x, \varphi(x)\big) + \rho\big(\varphi(x), \varphi(y)\big) + d\big(\varphi(y), y\big) = d_{\rho\varphi}(x, y) \end{aligned}$$

for all $x, y \in X$. Thus, $d_{\rho\varphi}$ is also symmetric.

Moreover, if $x \in X$, then by the symmetry of d, it is clear that

$$d_{\rho\varphi}(x, x) = d\big(x, \varphi(x)\big) + \rho\big(\varphi(x), \varphi(x)\big) + d\big(\varphi(x), x\big) = 2\, d\big(x, \varphi(x)\big)\,.$$

Now, if $x, y, z \in X$, then by using the triangle inequality for ρ and the symmetry of d, we can easily see that

$$\begin{aligned} d_{\rho\varphi}(x, z) &= d\big(x, \varphi(x)\big) + \rho\big(\varphi(x), \varphi(z)\big) + d\big(\varphi(z), z\big) \\ &\leq d\big(x, \varphi(x)\big) + \rho\big(\varphi(x), \varphi(y)\big) + \rho\big(\varphi(y), \varphi(z)\big) + d\big(\varphi(z), z\big) \\ &= d\big(x, \varphi(x)\big) + \rho\big(\varphi(x), \varphi(y)\big) + d\big(\varphi(y), y\big) \\ &\quad + d\big(y, \varphi(y)\big) + \rho\big(\varphi(y), \varphi(z)\big) + d\big(\varphi(z), z\big) - 2\, d\big(y, \varphi(y)\big) \\ &= d_{\rho\varphi}(x, y) + d_{\rho\varphi}(y, z) - d_{\rho\varphi}(y, y)\,. \end{aligned}$$

Thus, we have proved that $d_{\rho\varphi}$ is a φ–metric on X such that (2) holds.

Finally, if $x, y \in A_{\varphi}$, then we can also easily see that

$$\begin{aligned} d_{\rho\varphi}(x, y) &= d\big(x, \varphi(x)\big) + \rho\big(\varphi(x), \varphi(y)\big) + d\big(\varphi(y), y\big) \\ &= d(x, x) + \rho(x, y) + d(y, y) = \rho(x, y)\,, \end{aligned}$$

and thus (1) also holds.

From (1) in Theorem 4.3, by Theorem 2.10 and Remarks 2.5 and 2.8, it is clear that in particular we have the following two corollaries.

Corollary 4.4. *If in particular* φ *is a projection, then* $d_{\rho\varphi}(x, y) = \rho(x, y)$ *for all* $x, y \in \varphi[X]$.

Corollary 4.5. *If in particular* φ *is an injection (involution), then* $d_{\rho\varphi}(x, y) = \rho(x, y)$ *for all* $x, y \in B_{\varphi}$.

Notation 4.6. *In the sequel, we shall simply write* d_{φ} *in place of* $d_{\rho\varphi}$ *whenever* ρ *is the restriction of* d *to* $\varphi[X]^2$.

Remark 4.7. Thus, for any $x, y \in X$, we have

$$d_{\varphi}(x, y) = d\big(x, \varphi(x)\big) + d\big(\varphi(x), \varphi(y)\big) + d\big(\varphi(y), y\big).$$

Moreover, Theorem 4.3 and Corollaries 4.4 and 4.5 can also be specialized to d_{φ}.

5. Some further properties of the derived φ–metrics

The following two theorems will show that $d_{\rho\varphi}$ is, in general, only a weak partial pseudo-metric on X.

Theorem 5.1. *For any* $x, y \in X$, *the following assertions are equivalent:*

(1) $d_{\rho\varphi}(x, x) \leq d_{\rho\varphi}(x, y)$;

(2) $d\big(x, \varphi(x)\big) - d\big(y, \varphi(y)\big) \leq \rho\big(\varphi(x), \varphi(y)\big)$.

Proof. By Theorem 4.3 (2) and Notation 4.1, we have

$$\begin{aligned} & d_{\rho\varphi}(x, x) \leq d_{\rho\varphi}(x, y) \\ & \qquad \iff \quad 2\,d\big(x, \varphi(x)\big) \leq d\big(x, \varphi(x)\big) + \rho\big(\varphi(x), \varphi(y)\big) + d\big(\varphi(y), y\big) \\ & \qquad\qquad\qquad\qquad \iff \quad d\big(x, \varphi(x)\big) - d\big(y, \varphi(y)\big) \leq \rho\big(\varphi(x), \varphi(y)\big). \end{aligned}$$

Hence, by the symmetries of $d_{\rho\varphi}$ and ρ, we can immediately derive

Corollary 5.2. *The following assertions are equivalent:*

(1) $d_{\rho\varphi}(x, x) \leq d_{\rho\varphi}(x, y)$ *for all* $x, y \in X$;

(2) $\big|\, d\big(x, \varphi(x)\big) - d\big(y, \varphi(y)\big) \,\big| \leq \rho\big(\varphi(x), \varphi(y)\big)$ *for all* $x, y \in X$.

Theorem 5.3. *For any* $x, y \in X$, *the following assertions are equivalent:*

(1) $d_{\rho\varphi}(x, x) = d_{\rho\varphi}(x, y) = d_{\rho\varphi}(y, y)$;

(2) $\varphi(x) = \varphi(y)$ *and* $d\big(x, \varphi(x)\big) = d\big(y, \varphi(y)\big)$.

Proof. Quite similarly, as in the proof of Theorem 5.1, we can see that

$$\begin{aligned} & d_{\rho\varphi}(x, x) = d_{\rho\varphi}(x, y) = d_{\rho\varphi}(y, y) \\ & \qquad \iff \quad \rho\big(\varphi(x), \varphi(y)\big) = d\big(x, \varphi(x)\big) - d\big(y, \varphi(y)\big) = 0 \\ & \qquad\qquad\qquad\qquad \iff \quad \varphi(x) = \varphi(y), \quad d\big(x, \varphi(x)\big) = d\big(y, \varphi(y)\big). \end{aligned}$$

Hence, it is clear that in particular we also have

Corollary 5.4. *If in particular φ is an injection, then $d_{\rho\varphi}(x, x) = d_{\rho\varphi}(x, y) = d_{\rho\varphi}(y, y)$ implies $x = y$.*

Remark 5.5. Thus, if φ is an injection, then by Theorem 4.3 and Corollary 5.4 $d_{\rho\varphi}$ is a weak partial metric on X in the sense of Heckmann [6].

The following example shows that d_{φ} need not be a partial metric even if in particular φ is an involution on X.

Example 5.6. If for instance $X = \mathbb{C}$, $\varphi(x) = \bar{x}$ for all $x \in X$, and d is the Euclidean metric on X, then φ is an involution on X such that

$$\begin{aligned} d\big(\varphi(1), \varphi(i)\big) &= |\,1 - (-i)\,| = \sqrt{2} \\ &< 2 = \big|\,|\,1-1\,| - |\,i - (-i)\,|\,\big| = \big|\,d\big(1, \varphi(1)\big) - d\big(i, \varphi(i)\big)\,\big|. \end{aligned}$$

Thus, by Remark 5.5 and Corollary 5.2, d_{φ} is a weak partial metric, but not a partial metric on X.

In addition Theorem 4.3, we can also easily prove the following

Theorem 5.7. *If in particular $d(u, v) \le \rho(u, v)$ for all $u, v \in \varphi[X]$, then for any $x, y \in X$ we have*

(1) $d(x, y) \le d_{\rho\varphi}(x, y)$; *(2)* $d_{\rho\varphi}(x, x) \le d_{\rho\varphi}(x, y) + d(x, y)$.

Proof. By using the triangle inequality for d and the assumption of the theorem, we can easily see that

$$\begin{aligned} d(x, y) &\le d\big(x, \varphi(x)\big) + d\big(\varphi(x), y\big) \\ &\le d\big(x, \varphi(x)\big) + d\big(\varphi(x), \varphi(y)\big) + d\big(\varphi(y), y\big) \\ &\le d\big(x, \varphi(x)\big) + \rho\big(\varphi(x), \varphi(y)\big) + d\big(\varphi(y), y\big) = d_{\rho\varphi}(x, y). \end{aligned}$$

Moreover, by using Theorem 4.3 (2), the symmetry of d, the triangle inequality for d, and the assumption of the theorem, we can also easily see that

$$\begin{aligned} d_{\rho\varphi}(x, x) &= 2\,d\big(x, \varphi(x)\big) = d\big(x, \varphi(x)\big) + d\big(\varphi(x), x\big) \\ &\le d\big(x, \varphi(x)\big) + d\big(\varphi(x), \varphi(y)\big) + d\big(\varphi(y), x\big) \\ &\le d\big(x, \varphi(x)\big) + \rho\big(\varphi(x), \varphi(y)\big) + d\big(\varphi(y), y\big) + d(y, x) \\ &= d_{\rho\varphi}(x, y) + d(x, y). \end{aligned}$$

From this theorem, according to Notation 4.6, we can immediately derive

Corollary 5.8. *For any any $x, y \in X$ we have*

(1) $d(x, y) \le d_{\varphi}(x, y)$; *(2)* $d_{\varphi}(x, x) \le d_{\varphi}(x, y) + d(x, y)$.

6. φ–dominated, φ–equivalence relations

Definition 6.1. A relation Q on X is called *φ–transitive* if

$$(x, y) \in Q, \quad (y, z) \in Q, \quad y \in A_\varphi^c \quad \Longrightarrow \quad (x, z) \in Q.$$

Remark 6.2. Note that thus every transitive relation Q on X is, in particular, φ–transitive.

Moreover, if in particular $A_\varphi = \emptyset$, then every φ–transitive relation Q on X is already transitive.

While, if in particular $A_\varphi = X$, or equivalently $\varphi = \Delta_X$, then every relation Q on X is φ–transitive.

By using the composition and box products of relations, we can easily establish some concise characterizations of φ–transitive relations.

Definition 6.3. For any two relations F and G on X, the relation $F \boxtimes G$ on X^2, defined such that

$$\big(F \boxtimes G\big)(x, y) = F(x) \times G(y)$$

for all $x, y \in X$, is called the *box product* of the relations F and G.

Remark 6.4. Note that, in contrast to the composition, the box product of two relations can be easily extended to arbitrary family of relations.

However, in the sequel we shall only need the box product of two relations which is closely related to the composition of relations by the following

Lemma 6.5. *If F and G are relations on X, then for any $A \subset X^2$ we have*

$$\big(F \boxtimes G\big)[A] = G \circ A \circ F^{-1}.$$

Proof. If $(z, w) \in \big(F \boxtimes G\big)[A]$, then by the corresponding definitions there exists $(x, y) \in A$ such that $(z, w) \in \big(F \boxtimes G\big)(x, y)$, and thus $(z, w) \in F(x) \times G(y)$. Hence, we can infer that $z \in F(x)$ and $w \in G(y)$, and thus $(x, z) \in F$ and $(y, w) \in G$. Now, by using that $(z, x) \in F^{-1}$ and $(x, y) \in A$, we can see that $(z, y) \in A \circ F^{-1}$. Hence, by using that $(y, w) \in G$, we can already see that $(z, w) \in G \circ \big(A \circ F^{-1}\big)$. This shows that $\big(F \boxtimes G\big)[A] \subset G \circ \big(A \circ F^{-1}\big)$.

The converse inclusion can be proved quite similarly. Hence, by the associativity of the composition, it is clear that the required equality can also be stated.

Remark 6.6. From the above lemma, we can immediately infer that

$$\big(F \boxtimes G\big)(x, y) = G \circ \{(x, y)\} \circ F^{-1}$$

for all $x, y \in X$. Moreover, by taking F^{-1} in place of F, we can also see that

$$G \circ F = G \circ \Delta_X \circ F = \big(F^{-1} \boxtimes G\big)[\Delta_X].$$

Therefore, the composition and the box products are actually equivalent tools.

However, it is now more important to note that, by using Lemma 6.5, we can also easily prove the following

Theorem 6.7. *For a relation Q on X, the following assertions are equivalent:*

(1) Q *is* φ*-transitive;*

(2) $Q \circ \Delta_{A_{\varphi}^{c}} \circ Q \subset Q$; *(3)* $\big(Q^{-1} \boxtimes Q \big) [\Delta_{A_{\varphi}^{c}}] \subset Q$.

Proof. Note that

$$\begin{aligned}(x, z) \in Q^{-1}(y) \times Q(y) \quad &\Longrightarrow \quad x \in Q^{-1}(y), \quad z \in Q(y) \\ &\Longrightarrow \quad y \in Q(x), \quad z \in Q(y) \quad \Longrightarrow \quad (x, y) \in Q, \quad (y, z) \in Q\end{aligned}$$

for all $x, y, z \in X$. Therefore, if (1) holds, then we have

$$\big(Q^{-1} \boxtimes Q \big)(y, y) = Q^{-1}(y) \times Q(y) \subset Q$$

for all $y \in A_{\varphi}^{c}$. Hence, it is clear that

$$\big(Q^{-1} \boxtimes Q \big) [\Delta_{A_{\varphi}^{c}}] = \bigcup \big\{ \big(Q^{-1} \boxtimes Q \big)(y, y) : \quad y \in A_{\varphi}^{c} \big\} \subset Q,$$

and thus (3) also holds.

The converse implication $(3) \Longrightarrow (1)$ can be proved quite similarly. Moreover, by using Lemma 6.5, we can see that

$$\big(Q^{-1} \boxtimes Q \big) [\Delta_{A_{\varphi}^{c}}] = Q \circ \Delta_{A_{\varphi}^{c}} \circ Q.$$

Therefore, inclusions (2) and (3) are also equivalent.

Remark 6.8. Note that, under the notation $\Theta_{\varphi} = X \times A_{\varphi}^{c}$, we have

$$\big(\Delta_{A_{\varphi}^{c}} \circ Q \big)(x) = \Delta_{A_{\varphi}^{c}} [Q(x)] = Q(x) \cap A_{\varphi}^{c} = Q(x) \cap \Theta_{\varphi}(x) = \big(Q \cap \Theta_{\varphi} \big)(x)$$

for all $x \in X$. Therefore, $\Delta_{A_{\varphi}^{c}} \circ Q = Q \cap \Theta_{\varphi}$ is also true.

From Theorem 6.7, we can immediately get the nontrivial part of the following

Corollary 6.9. *For a relation Q on X, the following assertions are equivalent:*

(1) Q *is transitive;*

(2) $Q \circ Q \subset Q$; *(3)* $\big(Q^{-1} \boxtimes Q \big) [\Delta_{X}] \subset Q$.

Proof. If X is not a singleton, then by the Axiom of Choice there exists a function φ of X to itself such that $\varphi(x) \in X \setminus \{x\}$ for all $x \in X$, and thus $A_{\varphi} = \emptyset$. Therefore, Theorem 6.7 can be applied.

While, if in particular X is a singleton, then $Q_0 = \emptyset$ and $Q_1 = X^2$ are the only relations on X. Moreover, these two extreme relations trivially satisfy conditions (1)–(3) even if X is not a singleton.

Definition 6.10. A tolerance (reflexive and symmetric) relation Q on X is called a φ*–equivalence* if it is φ–transitive.

Remark 6.11. Quite similarly, an intolerance (reflexive and antisymmetric) relation Q on X may be called a *φ-partial order* if it is φ-transitive.

In the sequel, we shall also need the following

Definition 6.12. A relation Q on X is called *φ-dominated* if $Q \subset E_\varphi$. (That is, $(x, y) \in Q$ implies $\varphi(x) = \varphi(y)$.)

Remark 6.13. Note that if in particular φ is injective or equivalently $E_\varphi = \Delta_X$, then Δ_X is the only φ–dominated reflexive relation on X.

Moreover, note that Δ_X is actually an equivalence relation on X such that $\Delta_X \subset E_\varphi$. Thus, in particular, it is a φ–dominated φ–equivalence relation on X for any function φ of X to itself.

7. A metric derived from d and $d_{\rho\varphi}$ by Q

Notation 7.1. *In addition to Notation 1.1 and 4.1, assume now that Q is a φ–dominated, φ–equivalence relation on X.*

Moreover, for any $x, y \in X$, define

$$d_{\rho\varphi Q}(x, y) = \begin{cases} d(x, y) & \text{if} \quad (x, y) \in Q, \\ d_{\rho\varphi}(x, y) & \text{if} \quad (x, y) \notin Q. \end{cases}$$

Remark 7.2. Note that if in particular $\varphi = \Delta_X$, then by Remarks 4.2 and 6.13 we have $d_{\rho\varphi} = \rho$ and $Q = \Delta_X$.

Therefore, in this particular case, $d_{\rho\varphi Q}(x, y) = d(x, y)$ for $(x, y) \in \Delta_X$ and $d_{\rho\varphi Q}(x, y) = \rho(x, y)$ for $(x, y) \notin \Delta_X$. Hence, since $d(x, y) = 0 = \rho(x, y)$ if $(x, y) \in \Delta_X$, we can already see that $d_{\rho\varphi Q} = \rho$.

Our former definitions are mainly motivated by the following

Theorem 7.3. *The function $d_{\rho\varphi Q}$ is a metric on X such that*

(1) $d_{\rho\varphi Q}(x, y) = \rho(x, y)$ for all $x, y \in A_\varphi$;

(2) $d(x, y) \le d_{\rho\varphi Q}(x, y)$ for all $x, y \in X$ whenever $d(u, v) \le \rho(u, v)$ for all $u, v \in \varphi[X]$.

Proof. For the sake of brevity, define $\delta = d_{\rho\varphi}$ and $\sigma = d_{\rho\varphi Q}$. Then by the corresponding definitions, for any $x, y \in X$, we have

$$\sigma(x, y) = \begin{cases} d(x, y) & \text{if} \quad (x, y) \in Q, \\ \delta(x, y) & \text{if} \quad (x, y) \notin Q, \end{cases}$$

with

$$\delta(x, y) = d\big(x, \varphi(x)\big) + \rho\big(\varphi(x), \varphi(y)\big) + d\big(\varphi(y), y\big)$$

Moreover, by Theorems 4.3, 1.4 and 5.7, δ is a symmetric, nonnegative, real-valued function of X^2, satisfying the triangle inequality, such that

(a) $\delta(x, y) = \rho(x, y)$ for all $x, y \in A_\varphi$;

(b) $\delta(x, y) = 0$ is equivalent to $x = \varphi(x) = \varphi(y) = y$ for all $x, y \in X$;

(c) $d(x, y) \leq \delta(x, y)$ for all $x, y \in X$ whenever $d(u, v) \leq \rho(u, v)$ for all $u, v \in \varphi[X]$.

Now, from the definition of σ, it is clear that σ is a nonnegative, real-valued function of X^2. Moreover, if $x \in X$, then from the reflexivity of Q we can at once see that $(x, x) \in Q$, and thus $\sigma(x, x) = d(x, x) = 0$.

On the other hand, if $x, y \in X$ such that $\sigma(x, y) = 0$, then in the case $(x, y) \in Q$ we can see that $d(x, y) = \sigma(x, y) = 0$, and thus $x = y$. While, in the case $(x, y) \notin Q$ we can see that $\delta(x, y) = \sigma(x, y) = 0$, and thus $x = y$. Therefore, $\sigma(x, y) = 0$ if and only if $x = y$.

Moreover, if $x, y \in X$, then in the case $(x, y) \in Q$ we can see that $(y, x) \in Q$, and thus $\sigma(x, y) = d(x, y) = d(y, x) = \sigma(y, x)$. While, in the case $(x, y) \notin Q$ we can see that $(y, x) \notin Q$, and thus $\sigma(x, y) = \delta(x, y) = \delta(y, x) = \sigma(y, x)$. Therefore, σ is also symmetric function of X^2.

On the other hand, if $x, y \in A_\varphi$, then in the case $(x, y) \in E_\varphi$, we can see that $x = \varphi(x) = \varphi(y) = y$. Hence, by the reflexivity of Q, it follows that $(x, y) \in Q$. Therefore, $\sigma(x, y) = d(x, y) = 0 = \rho(x, y)$ because of $x = y$. While, in the case $(x, y) \notin E_\varphi$, we can see that $(x, y) \notin Q$. Therefore, $\sigma(x, y) = \delta(x, y) = \rho(x, y)$ is also true by (a). This proves (1).

Moreover, if $d(u, v) \leq \rho(u, v)$ for all $u, v \in \varphi[X]$, then by (c) we have $d(x, y) \leq \delta(x, y)$ for all $x, y \in X$. Hence, since $\sigma(x, y)$ is either $d(x, y)$ or $\delta(x, y)$, it is clear that $\sigma(x, y) \leq \delta(x, y)$ also holds for all $x, y \in X$. Therefore, (2) is also true.

Now, to complete the proof, it remains only to prove that σ also satisfies the triangle inequality. This nontrivial fact will be proved in the next section by considering several cases.

From (1) in Theorem 7.3, by Theorem 2.10 and Remarks 2.5 and 2.8, it is clear that in particular we have the following two corollaries.

Corollary 7.4. *If in particular φ is a projection, then $d_{\rho\varphi Q}(x, y) = \rho(x, y)$ for all $x, y \in \varphi[X]$.*

Corollary 7.5. *If in particular φ is an injection (involution), then $d_{\rho\varphi Q}(x, y) = \rho(x, y)$ for all $x, y \in B_\varphi$.*

Notation 7.6. *In the sequel, analogously to Notation 4.6, we shall simply write $d_{\varphi Q}$ in place of $d_{\rho\varphi Q}$ whenever ρ is the restriction of d to $\varphi[X]^2$.*

Remark 7.7. Thus, for any $x, y \in X$, we have

$$d_{\varphi Q}(x, y) = \begin{cases} d(x, y) & \text{if} \quad (x, y) \in Q, \\ d_\varphi(x, y) & \text{if} \quad (x, y) \notin Q. \end{cases}$$

Moreover, Theorem 7.3 can be specialized in the following form.

Theorem 7.8. *The function $d_{\varphi Q}$ is a metric on X such that*

(1) $d_{\varphi Q}(x, y) = \rho(x, y)$ for all $x, y \in A_\varphi$;

(2) $d(x, y) \leq d_{\varphi Q}(x, y)$ for all $x, y \in X$.

8. The proof of the triangle inequality for $\sigma = d_{\rho\varphi Q}$

To complete the proof of Theorem 7.3, it has remained to show that, for any $x, y, z \in X$, we have

$$\sigma(x, z) \leq \sigma(x, y) + \sigma(y, z).$$

For this, according to definition of σ and the positions of the points $(x, z), (x, y)$ and (y, z) with respect to Q, we have to consider several cases.

Note that if each of the above tree points is in Q, then by the definition of σ and the triangle inequality for d we evidently have

$$\sigma(x, z) = d(x, z) \leq d(x, y) + d(y, z) = \sigma(x, y) + \sigma(y, z).$$

While, if none of the above three points is in Q, then by the definition of σ and the triangle inequality for δ we evidently have

$$\sigma(x, z) = \delta(x, z) \leq \delta(x, y) + \delta(y, z) = \sigma(x, y) + \sigma(y, z).$$

Assume now that $(x, z) \notin Q$, but (x, y), $(y, z) \in Q$. Then, by $Q \subset E_\varphi$ and the definition of E_φ, we have $\varphi(x) = \varphi(y) = \varphi(z)$. Moreover, by the φ–transitivity of Q, we also have $y \notin A_\varphi^c$. Therefore, by the definition of A_φ, we have $\varphi(y) = y$, and thus also $\varphi(x) = y$ and $\varphi(z) = y$. Hence, by the definitions of σ and δ and the zero self-distance property of ρ, it is clear that

$$\begin{aligned}\sigma(x, z) &= \delta(x, z) = d\big(x, \varphi(x)\big) + \rho\big(\varphi(x), \varphi(z)\big) + d\big(\varphi(z), z\big)\\ &= d(x, y) + \rho(y, y) + d(y, z) = d(x, y) + d(y, z) = \sigma(x, y) + \sigma(y, z).\end{aligned}$$

Therefore, instead of the required inequality, the corresponding equality is also true.

Next, assume that $(x, z) \in Q$, $(x, y) \notin Q$, but $(y, z) \in Q$. Then, by $Q \subset E_\varphi$ and the definition of E_φ, we have $\varphi(x) = \varphi(z)$ and $\varphi(y) = \varphi(z)$. Moreover, by the symmetry of Q, we also have $(z, y) \in Q$. Hence, by the φ–transitivity of Q, it is clear that $z \notin A_\varphi^c$. Therefore, by the definition of A_φ, we have $\varphi(z) = z$, and thus also $\varphi(x) = z$ and $\varphi(y) = z$. Now, by the definitions of σ and δ, and the zero self-distance property of ρ and the symmetry of d, we can see that

$$\begin{aligned}\sigma(x, y) &= \delta(x, y) = d\big(x, \varphi(x)\big) + \rho\big(\varphi(x), \varphi(y)\big) + d\big(\varphi(y), y\big)\\ &= d(x, z) + \rho(z, z) + d(z, y) = d(x, z) + d(y, z) = \sigma(x, z) + \sigma(y, z),\end{aligned}$$

and hence

$$\sigma(x, z) = \sigma(x, y) - \sigma(y, z) = \sigma(x, y) + \sigma(y, z) - 2\,\sigma(y, z).$$

Therefore, by the nonnegativity of σ, the required inequality is also true.

Quite similarly, if (x, z), $(x, y) \in Q$, but $(y, z) \notin Q$, then by $Q \subset E_\varphi$ and the definition of E_φ we can see that $\varphi(x) = \varphi(z)$ and $\varphi(x) = \varphi(y)$. Moreover, by the symmetry of Q, we also have $(y, x) \in Q$. Hence, by the φ–transitivity of

Q, it is clear that $x \notin A_{\varphi}^{c}$. Therefore, by the definition of A_{φ}, we have $\varphi(x) = x$, and thus also $\varphi(y) = x$ and $\varphi(z) = x$. Now, by the definitions of σ and δ, and the symmetry of d and the zero self-distance property of ρ, we can see that

$$\begin{aligned}\sigma(y, z) &= \delta(y, z) = d\big(y, \varphi(y)\big) + \rho\big(\varphi(y), \varphi(z)\big) + d\big(\varphi(z), z\big) \\ &= d(y, x) + \rho(x, x) + d(x, z) = d(x, y) + d(x, z) = \sigma(x, y) + \sigma(x, z),\end{aligned}$$

and hence

$$\sigma(x, z) = \sigma(y, z) - \sigma(x, y) = \sigma(x, y) + \sigma(y, z) - 2\,\sigma(x, y).$$

Therefore, by the nonnegativity of σ, the required inequality is also true.

To continue the proof, assume now that (x, z), $(x, y) \notin Q$, but $(y, z) \in Q$. Then, by $Q \subset E_{\varphi}$ and the definition of E_{φ}, we have $\varphi(y) = \varphi(z)$. Moreover, by the definitions of σ and δ, and the triangle inequality for d, we can see that

$$\begin{aligned}\sigma(x, z) = \delta(x, z) &= d\big(x, \varphi(x)\big) + \rho\big(\varphi(x), \varphi(z)\big) + d\big(\varphi(z), z\big) \\ &= d\big(x, \varphi(x)\big) + \rho\big(\varphi(x), \varphi(y)\big) + d\big(\varphi(y), z\big) \\ &\leq d\big(x, \varphi(x)\big) + \rho\big(\varphi(x), \varphi(y)\big) + d\big(\varphi(y), y\big) + d(y, z) \\ &= \delta(x, y) + d(y, z) = \sigma(x, y) + \sigma(y, z).\end{aligned}$$

Therefore, the required inequality is again true.

Quite similarly, if $(x, z) \notin Q$, $(x, y) \in Q$, but $(y, z) \notin Q$, then by $Q \subset E_{\varphi}$ and the definition of E_{φ} we have $\varphi(x) = \varphi(y)$. Moreover, by the definitions of σ and δ, and the triangle inequality for d, we can see that

$$\begin{aligned}\sigma(x, z) = \delta(x, z) &= d\big(x, \varphi(x)\big) + \rho\big(\varphi(x), \varphi(z)\big) + d\big(\varphi(z), z\big) \\ &= d\big(x, \varphi(y)\big) + \rho\big(\varphi(y), \varphi(z)\big) + d\big(\varphi(z), z\big) \\ &\leq d(x, y) + d\big(y, \varphi(y)\big) + +\rho\big(\varphi(y), \varphi(z)\big) + d\big(\varphi(z), z\big) \\ &= d(x, y) + \delta(y, z) = \sigma(x, y) + \sigma(y, z).\end{aligned}$$

Therefore, the required inequality is again true.

Finally, to complete the proof, assume now that $(x, z) \in Q$, but $(x, y) \notin Q$ and $(y, z) \notin Q$. Then, by $Q \subset E_{\varphi}$ and the definition of E_{φ}, we have $\varphi(x) = \varphi(z)$. Moreover, by the definitions of σ and δ, the zero self-distance property of ρ, the triangle inequality for d and ρ, and the nonnegativity of d, we can see that

$$\begin{aligned}\sigma(x, z) &= d(x, z) = d(x, z) + \rho\big(\varphi(x), \varphi(z)\big) \\ &\leq d\big(x, \varphi(x)\big) + d\big(\varphi(x), z\big) + \rho\big(\varphi(x), \varphi(y)\big) + \rho\big(\varphi(y), \varphi(z)\big) \\ &= d\big(x, \varphi(x)\big) + \rho\big(\varphi(x), \varphi(y)\big) + \rho\big(\varphi(y), \varphi(z)\big) + d\big(\varphi(z), z\big) \\ &\leq d\big(x, \varphi(x)\big) + \rho\big(\varphi(x), \varphi(y)\big) + d\big(\varphi(y), y\big) \\ &\quad + d\big(y, \varphi(y)\big) + \rho\big(\varphi(y), \varphi(z)\big) + d\big(\varphi(z), z\big) \\ &= \delta(x, y) + \delta(y, z) = \sigma(x, y) + \sigma(y, z).\end{aligned}$$

Therefore, the required inequality is again true.

9. Collinearity-like relations

To construct φ–dominated, φ–equivalence relations on X, we shall need the following assumptions.

Notation 9.1. *Suppose that* Γ *is a relation on* X^2 *to* X *such that, for any* $x, y, z \in X$, *we have*

(1) $\Gamma(x, x) = X$;

(2) $\Gamma(x, y) = \Gamma(y, x)$;

(3) $\Gamma(x, y) \cap \Gamma(y, z) \cap \{y\}^c \subset \Gamma(x, z)$.

Remark 9.2. Note that, to guarantee property (2), it is enough to assume only that $\Gamma(x, y) \subset \Gamma(y, x)$ for all $x, y \in X$ with $x \neq y$.

Remark 9.3. While, to guarantee property (3), it is enough to assume only that $w \in \Gamma(x, y)$ and $w \in \Gamma(y, z)$ imply $w \in \Gamma(x, z)$ for all $x, y, z \in X$ with $x \neq y$, $y \neq z$ and $y \neq w$.

Namely, if $x = y$, then $w \in \Gamma(y, z)$ already implies that $w \in \Gamma(x, z)$. While, if $y = z$, then $w \in \Gamma(x, y)$ already implies that $w \in \Gamma(x, z)$.

Remark 9.4. Moreover, it is also worth noticing that if (1) is already assumed, then we may also suppose that $x \neq z$.

Namely, if $x = z$, then by (1) we have $\Gamma(x, z) = X$, and thus $w \in \Gamma(x, z)$ trivially holds.

Definition 9.5. If Γ is as in Notation 9.1, then we say that Γ is a *pre-collinearity relation* for X.

While, if Γ is a pre-collinearity relation for X such that for any $x, y, z \in X$

(4) $z \in \Gamma(x, y) \implies x \in \Gamma(y, z)$,

then we say that Γ is a *collinearity relation* for X.

Remark 9.6. Note that, if (1) is already assumed, then to guarantee (4) it is enough to suppose only that $z \in \Gamma(x, y)$ implies $x \in \Gamma(y, z)$ for all $x, y, z \in X$ with $y \neq z$.

Namely, if $y = z$, then by (1) we have $\Gamma(y, z) = X$, and thus $x \in \Gamma(y, z)$ trivially holds.

Remark 9.7. While, if (1), (2) and (4) are already assumed, then to guarantee (3) it is enough to suppose only that $w \in \Gamma(x, y)$ and $w \in \Gamma(y, z)$ imply $w \in \Gamma(x, z)$ for any four, pairwise distinct points x, y, z and w of X.

Namely, if $x = w$, then by (1) we have $\Gamma(w, x) = X$, and thus $z \in \Gamma(w, x)$ trivially holds. Hence, by (4), it follows that $w \in \Gamma(x, z)$. While, if $z = w$, then by (1) we have $\Gamma(w, z) = X$, and thus $x \in \Gamma(w, z)$ trivially holds. Hence, by (4) and (2), it follows that $w \in \Gamma(z, x) = \Gamma(x, z)$.

Remark 9.8. In connection with hypothesis (4), it is also worth noticing that if $z \in \Gamma(x, y)$, then by (2) we also have $z \in \Gamma(y, x)$. Hence, by (4), we can infer that $y \in \Gamma(x, z)$. Thus, by (2), the inclusion $y \in \Gamma(z, x)$ also holds.

Example 9.9. For any $x, y \in X$, define $\Gamma(x, y) = X$. Then, Γ is a collinearity relation for X such that instead of property (3) we actually have $\Gamma(x, y) \cap \Gamma(y, z) = \Gamma(x, z)$ for all $x, y, z \in X$.

Example 9.10. Let 0 be a fixed element of X, and for any $x, y \in X$ define

$$\Gamma(x, y) = \begin{cases} X & \text{if} \quad x = y\,, \\ \{0\}^c & \text{if} \quad x \neq y\,. \end{cases}$$

Then, Γ is a pre-collinearity relation on X such that instead of property (3) we actually have $\Gamma(x, y) \cap \Gamma(y, z) \subset \Gamma(x, z)$ for all $x, y, z \in X$.

Proof. Now, properties (1) and (2) are trivially satisfied. Moreover, if $w \in \Gamma(x, y)$ and $w \in \Gamma(y, z)$, then we can easily see that $w \in \Gamma(x, z)$.

Namely, if $w \neq 0$, then $w \in \{0\}^c \subset \Gamma(x, z)$. While, if $w = 0$, then $w \in \Gamma(x, y)$ and $w \in \Gamma(y, z)$ imply that $x = y$ and $y = z$. Hence, it follows that that $x = z$, and thus $\Gamma(x, z) = X$. Therefore, $w \in \Gamma(x, z)$ trivially holds.

Remark 9.11. Note that now we actually have $\Gamma(x, y) \cap \Gamma(y, z) = \Gamma(x, z)$ for any three, pairwise distinct points x, y and z of X.

However, if for instance $x, y \in X$ such that $x \neq y$, then in contrast to Example 9.9 we have $\Gamma(x, y) \cap \Gamma(y, x) = \{0\}^c \neq X = \Gamma(x, x)$.

Remark 9.12. Moreover, it is also worth noticing that if $y, z \in \{0\}^c$ such that $y \neq z$, then $z \in \{0\}^c = \Gamma(0, y)$, but $0 \notin \{0\}^c = \Gamma(y, z)$.

Therefore, in contrast to Example 9.9 and our forthcoming examples, the relation Γ considered in Example 9.10 is not, in general, a collinearity relation for X.

10. Two more natural examples for collinearity relations

Example 10.1. For any $x, y \in X$, define

$$\Gamma(x, y) = \begin{cases} X & \text{if} \quad x = y\,, \\ \{x, y\} & \text{if} \quad x \neq y\,. \end{cases}$$

Then, Γ is collinearity relation for X.

Proof. Properties 9.1 (1) and (2) are again trivially satisfied. Moreover, if $w \in \Gamma(x, y)$ and $w \in \Gamma(y, z)$ such that $x \neq y$, $y \neq z$ and $y \neq w$, then we can note that $w = x$ and $w = z$. Hence, it follows that $x = z$, and thus $\Gamma(x, z) = X$. Therefore, $w \in \Gamma(x, z)$ trivially holds. Hence, by Remark 9.3, we can see that 9.1 (3) also holds.

Finally, to prove property 9.5(4), we can note that if $x = y$, then $x \in \{y, z\} \subset \Gamma(y, z)$. While, if $x \neq y$, then $z \in \Gamma(x, y)$, with $z \neq y$, implies that $z = x$. Therefore, $x \in \{y, z\} = \Gamma(y, z)$. Thus, by Remark 9.6, property 9.5 (4) also holds.

Remark 10.2. Note that if x, y and z are pairwise distinct points of X, then

$$y \in \{x, y\} = \Gamma(x, y) \quad \text{and} \quad y \in \{y, z\} = \Gamma(y, z)\,, \quad \text{but} \quad y \notin \{x, z\} = \Gamma(x, z)\,.$$

Therefore, in contrast to Example 9.9 and Remark 9.11, we now have $\Gamma(x, y) \cap \Gamma(y, z) \not\subset \Gamma(x, z)$.

Example 10.3. Let X be a vector space over K, and for any $x, y \in X$, define $\Gamma(x, x) = X$ if $x = y$, and

$$\Gamma(x, y) = \big\{ z \in X : \quad \exists\ \lambda \in K : \quad z = \lambda x + (1-\lambda) y \big\} \qquad \text{if} \qquad x \neq y .$$

Then, Γ is a collinearity relation for X.

Proof. If $z \in \Gamma(x, y)$ such that $x \neq y$, then there exists $\lambda \in K$ such that $z = \lambda x + (1-\lambda) y$. Hence, by taking $\mu = 1 - \lambda$, we can see that

$$z = (1-\lambda) y + \lambda x = \mu y + (1-\mu) x .$$

Therefore, $z \in \Gamma(y, x)$ also holds. This shows that $\Gamma(x, y) \subset \Gamma(y, x)$ whenever $x \neq y$. Thus, by Remark 9.2, property 9.1 (2) also holds.

Moreover, if $\lambda \neq 0$, then by taking $\nu = 1 - 1/\lambda$, we can see that

$$x = \big(1 - 1/\lambda\big) y + \big(1/\lambda\big) z = \nu y + (1-\nu) z .$$

Therefore, $x \in \Gamma(y, z)$. Now, to see that property 9.5 (4) also holds, it remains to note only that if $\lambda = 0$, then $z = y$. Therefore, $\Gamma(y, z) = X$, and thus $x \in \Gamma(y, z)$ trivially holds.

Finally, to prove property 9.1 (3), note that if $w \in \Gamma(x, y) \cap \Gamma(y, z) \cap \{y\}^c$, then

$$w \in \Gamma(x, y), \qquad w \in \Gamma(y, z) \qquad \text{and} \qquad y \neq w .$$

Moreover, by Remarks 9.3 and 9.4, we may suppose that $x \neq y$, $y \neq z$ and $x \neq z$ also hold. Now, by the above assumptions, we can state that there exist $\lambda, \mu \in K$ such that

$$w = \lambda x + (1-\lambda) y \qquad \text{and} \qquad w = \mu y + (1-\mu) z$$

Hence, by using that $y \neq w$ and $x \neq z$, we can infer that $\lambda \neq 0$ and $\lambda + \mu \neq 1$.

Namely, if $\lambda + \mu = 1$, then we also have

$$w = (1-\lambda) y + \lambda z , \qquad \text{and thus} \qquad \lambda x + (1-\lambda) y = (1-\lambda) y + \lambda z .$$

Hence, we can infer that $\lambda x = \lambda z$, and thus $x = z$ since $\lambda \neq 0$. And this contradicts the assumption that $x \neq z$.

From the above equations on w, we can also infer that

$$\mu w = \lambda \mu x + (1-\lambda) \mu y \qquad \text{and} \qquad (1-\lambda) w = (1-\lambda) \mu y + (1-\lambda)(1-\mu) z ,$$

and thus

$$(\lambda + \mu - 1) w = \mu w - (1-\lambda) w = \lambda \mu x - (1-\lambda)(1-\mu) z = (\lambda + \mu - 1 - \lambda \mu) z .$$

Hence, since $\lambda + \mu \neq 1$, we can already see that

$$w = \frac{\lambda \mu}{\lambda + \mu - 1}\, x + \left(1 - \frac{\lambda \mu}{\lambda + \mu - 1} \right) z .$$

Therefore, $w \in \Gamma(x, z)$, and thus property 9.5 (4) also holds.

Remark 10.4. Note that if in particular $X = K$, then for any $x, y, z \in X$, with $x \neq y$, we have

$$z = \frac{z-y}{x-y}\, x + \left(1 - \frac{z-y}{x-y}\right) y\,.$$

Therefore, in this particular case $\Gamma(x, y) = X$ also holds for all $x, y \in X$ with $x \neq y$.

Remark 10.5. However, if in particular $X = \mathbb{C}$ and $K = \mathbb{R}$, then we have

$$1 = 0 \cdot 0 + (1-0)\,1\,, \qquad 1 = 1 \cdot 1 + (1-1)\,i\,, \qquad \text{and} \qquad 1 \neq \lambda\, 0 + (1-\lambda)\, i$$

for all $\lambda \in K$. Therefore,

$$1 \in \Gamma(0, 1)\,, \qquad 1 \in \Gamma(1, i)\,, \qquad \text{but} \qquad 1 \notin \Gamma(0, i)\,.$$

Thus, in contrast to Example 9.10 and Remark 9.11, we now have $\Gamma(0,1) \cap \Gamma(1,i) \not\subset \Gamma(0,i)$. This show that the set $\{y\}^c$ cannot be omitted from assumption 9.1 (3).

11. A φ–dominated, φ-equivalence defined by φ and Γ

Notation 11.1. *Define*

$$Q_{\varphi\,\Gamma} = \big\{ (x, y) \in E_\varphi : \quad \varphi(x) \in \Gamma(x, y) \big\}\,.$$

Remark 11.2. Hence, because of $\Gamma(x, x) = X$, it is clear that

$$(x, y) \in Q_{\Delta_X\,\Gamma} \iff x = y\,, \quad x \in \Gamma(x, y) \iff x = y\,.$$

Therefore, in particular we have $Q_{\Delta_X\,\Gamma} = \Delta_X$.

Our former definitions have been mainly motivated by the following

Theorem 11.3. *$Q_{\varphi\,\Gamma}$ is a φ–dominated, φ–equivalence relation on X.*

Proof. If $x \in X$, then because of $(x, x) \in E_\varphi$ and $\varphi(x) \in X = \Gamma(x, x)$ we also have $(x, x) \in Q_{\varphi\,\Gamma}$. Therefore, $Q_{\varphi\,\Gamma}$ is also reflexive on X.

Moreover, if $(x, y) \in Q_{\varphi\,\Gamma}$, then $(x, y) \in E_\varphi$ and $\varphi(x) \in \Gamma(x, y)$. Hence, by the symmetry of E_φ and Γ and the definition of E_φ, we can see that $(y, x) \in E_\varphi$ and $\varphi(y) = \varphi(x) \in \Gamma(x, y) = \Gamma(y, x)$ also hold. Therefore, $(y, x) \in Q_{\varphi\,\Gamma}$, and thus $Q_{\varphi\,\Gamma}$ is also symmetric.

Now, since $Q_{\varphi\,\Gamma}$ is evidently φ-dominated, it remains to prove only that $Q_{\varphi\,\Gamma}$ is φ–transitive too. For this, assume that $(x, y) \in Q_{\varphi\,\Gamma}$, $(y, z) \in Q_{\varphi\,\Gamma}$ and $y \in A_\varphi^c$. Then, by the corresponding definitions, we have

$$(x, y) \in E_\varphi\,, \qquad \varphi(x) \in \Gamma(x, y) \qquad \text{and} \qquad (y, z) \in E_\varphi\,, \qquad \varphi(y) \in \Gamma(y, z)\,,$$

and moreover $y \neq \varphi(y)$. Hence, by the transitivity of E_φ, we can infer that $(x, z) \in E_\varphi$. Moreover, since $\varphi(x) = \varphi(y) \in \Gamma(y, z)$ and $\varphi(x) = \varphi(y) \in \{y\}^c$, by using property 9.1 (3) we can also infer that $\varphi(x) \in \Gamma(x, z)$. Therefore, $(x, z) \in Q_{\varphi\,\Gamma}$ also holds.

Corollary 11.4. *We have*

(1) $Q_{\varphi\,\Gamma} = \Delta_X \cup \big(Q_{\varphi\Gamma} \setminus \Delta_X \big)$;

(2) $Q_{\varphi\,\Gamma} \setminus \Delta_X = \big\{ (x, y) \in E_\varphi \setminus \Delta_X : \ \ \varphi(x) \in \Gamma(x, y) \big\}$.

Proof. By Theorem 11.3, we have $\Delta_X \subset Q_{\varphi\,\Gamma}$, and thus

$$Q_{\varphi\,\Gamma} = \Delta_X \cup \big(Q_{\varphi\Gamma} \setminus \Delta_X \big) .$$

Moreover, by the corresponding definitions, it is clear that

$$\begin{aligned} Q_{\varphi\,\Gamma} \setminus \Delta_X &= \big\{ (x, y) \in E_\varphi : \ \ \varphi(x) \in \Gamma(x, y) \big\} \setminus \Delta_X \\ &= \big\{ (x, y) \in E_\varphi \setminus \Delta_X : \ \ \varphi(x) \in \Gamma(x, y) \big\} . \end{aligned}$$

Corollary 11.5. *If φ is an injection (involution), then $Q_{\varphi\,\Gamma} = \Delta_X$.*

Proof. By Notation 11.1 and Theorem 2.4 (Remark 2.8), we have $Q_{\varphi\,\Gamma} \subset E_\varphi = \Delta_X$. Therefore, $Q_{\varphi\,\Gamma} \setminus \Delta_X = \emptyset$, and thus by Corollary 11.4 the required equality is also true.

Example 11.6. If in particular Γ is as in Example 9.9, then $Q_{\varphi\,\Gamma} = E_\varphi$. Namely, by the corresponding definitions, we have

$$Q_{\varphi\,\Gamma} = \big\{ (x, y) \in E_\varphi : \ \ \varphi(x) \in \Gamma(x, y) \big\} = \big\{ (x, y) \in E_\varphi : \ \ \varphi(x) \in X \big\} .$$

Example 11.7. If in particular Γ is as in Example 9.10, then

$$Q_{\varphi\,\Gamma} = \Delta_X \cup \big\{ (x, y) \in E_\varphi \setminus \Delta_X : \ \ \varphi(x) \neq 0 \big\} .$$

Namely, by the corresponding definitions, we have

$$\begin{aligned} Q_{\varphi\Gamma} \setminus \Delta_X &= \big\{ (x, y) \in E_\varphi \setminus \Delta_X : \ \ \varphi(x) \in \Gamma(x, y) \big\} \\ &= \big\{ (x, y) \in E_\varphi \setminus \Delta_X : \ \ \varphi(x) \in \{0\}^c \big\} . \end{aligned}$$

Therefore, by Corollary 11.4, the required equality is also true.

Example 11.8. If in particular Γ is as in Example 10.1, then

$$Q_{\varphi\,\Gamma} = \Delta_X \cup \big\{ (x, y) \in E_\varphi \setminus \Delta_X : \ \ \varphi(x) = x \ \text{ or } \ \varphi(x) = y \big\} .$$

Namely, by the corresponding definitions, we have

$$\begin{aligned} Q_{\varphi\,\Gamma} \setminus \Delta_X &= \big\{ (x, y) \in E_\varphi : \ \ \varphi(x) \in \Gamma(x, y) \big\} \\ &= \big\{ (x, y) \in E_\varphi \setminus \Delta_X : \ \ \varphi(x) \in \{x, y\} \big\} . \end{aligned}$$

Therefore, by Corollary 11.4, the required equality is also true.

Example 11.9. If in particular Γ is as in Example 10.3, then

$$Q_{\varphi\,\Gamma} = \Delta_X \cup \big\{ (x, y) \in E_\varphi \setminus \Delta_X : \ \ \exists \ \lambda \in K : \ \ \varphi(x) = \lambda\, x + (1 - \lambda)\, y \big\} .$$

Namely, by the corresponding definitions, we have

$$\begin{aligned} Q_{\varphi\,\Gamma} \setminus \Delta_X &= \big\{ (x, y) \in E_\varphi \setminus \Delta_X : \ \ \varphi(x) \in \Gamma(x, y) \big\} \\ &= \big\{ (x, y) \in E_\varphi \setminus \Delta_X : \ \ \exists \ \lambda \in K : \ \ \varphi(x) = \lambda\, x + (1 - \lambda)\, y \big\} . \end{aligned}$$

Therefore, by Corollary 11.4, the required equality is also true.

Remark 11.10. Note that in the statements of above examples, we may simply write E_φ in place $E_\varphi \setminus \Delta_X$.

12. Metrics derived from d and ρ by $Q_{\varphi\Gamma}$

Notation 12.1. *Now, according to Theorem 11.3 and Notation 7.1, we define* $d_{\rho\varphi\Gamma} = d_{\rho\varphi Q_{\varphi\Gamma}}$.

Remark 12.2. Thus, for any $x, y \in X$, we have

$$d_{\rho\varphi\Gamma}(x, y) = \begin{cases} d(x, y) & \text{if} \quad (x, y) \in Q_{\varphi\Gamma}, \\ d_{\rho\varphi}(x, y) & \text{if} \quad (x, y) \notin Q_{\varphi\Gamma}. \end{cases}$$

Moreover, in particular, by Theorem 11.3 and Remark 7.2, we have $d_{\rho\Delta_X\Gamma} = \rho$.

Furthermore, as an immediate consequence of Theorems 11.3 and 7.3, we can at once state the following

Theorem 12.3. *The function* $d_{\rho\varphi\Gamma}$ *is a metric on* X *such that*

(1) $d_{\rho\varphi\Gamma}(x, y) = \rho(x, y)$ *for all* $x, y \in A_\varphi$;

(2) $d(x, y) \le d_{\rho\varphi\Gamma}(x, y)$ *for all* $x, y \in X$ *whenever* $d(u, v) \le \rho(u, v)$ *for all* $u, v \in \varphi[X]$.

Now, analogously to Corollaries 7.4 and 7.5, we can also state

Corollary 12.4. *If in particular* φ *is a projection, then* $d_{\rho\varphi\Gamma}(x, y) = \rho(x, y)$ *for all* $x, y \in \varphi[X]$.

Corollary 12.5. *If in particular* φ *is an injection (involution), then* $d_{\rho\varphi\Gamma}(x, y) = \rho(x, y)$ *for all* $x, y \in B_\varphi$.

Notation 12.6. *In the sequel, analogously to Notation 7.6, we shall simply write* $d_{\varphi\Gamma}$ *in place of* $d_{\rho\varphi\Gamma}$ *whenever* ρ *is the restriction of* d *to* $\varphi[X]^2$.

Remark 12.7. Thus, for any $x, y \in X$, we have

$$d_{\varphi\Gamma}(x, y) = \begin{cases} d(x, y) & \text{if} \quad (x, y) \in Q_{\varphi\Gamma}, \\ d_{\varphi}(x, y) & \text{if} \quad (x, y) \notin Q_{\varphi\Gamma}. \end{cases}$$

Moreover, for instance, Theorem 12.3 can be specialized in the following form.

Theorem 12.8. *The function* $d_{\varphi\Gamma}$ *is a metric on* X *such that*

(1) $d_{\varphi\Gamma}(x, y) = \rho(x, y)$ *for all* $x, y \in A_\varphi$;

(2) $d(x, y) \le d_{\varphi\Gamma}(x, y)$ *for all* $x, y \in X$.

Notation 12.9. *In the forthcoming illustrating examples, by specializing our former notation, we shall assume that* $X = \mathbb{C}$ *and* d *is the Euclidean metric on* X.

Example 12.10. If $\varphi(x) = 0$ for all $x \in X$, then it is clear that φ is a projection,

$$A_\varphi = \{0\}, \qquad B_\varphi = X \qquad \text{and} \qquad D_\varphi = \Delta_{\{0\}}, \qquad E_\varphi = X^2.$$

Moreover, if $x, y \in X$, then according to Notation 4.1 we can also easily see that

$$d_{\rho\varphi}(x, y) = d(x, 0) + \rho(0, 0) + d(0, y) = |x| + |y|.$$

Note that, by Theorem 4.3, $d_{\rho\varphi}$ is a φ–metric on X. Thus, according to property 3.1(4), we have $d_{\rho\varphi}(x, y) = 0$ if and only if $(x, y) \in \Delta_{\{0\}}$, i. e., $x = y = 0$. But, in contrast to property 3.1(3), the corresponding equality is also true.

Furthermore, if Γ is as in Example 9.10, then by Example 11.7, Remark 11.10 and the definition of φ we have

$$Q_{\varphi\Gamma} = \Delta_X \cup \big\{ (x, y) \in X^2 : \quad 0 \neq 0 \big\} = \Delta_X \cup \emptyset = \Delta_X.$$

Now, if $x, y \in X$, then by the above observations we can see that, according to Notation 12.1, we simply have

$$d_{\rho\varphi\Gamma}(x, y) = \begin{cases} 0 & \text{if} \quad x = y, \\ |x| + |y| & \text{if} \quad x \neq y. \end{cases}$$

Therefore, in the present particular case, $d_{\rho\varphi\Gamma}$ is just the postman metric on X mentioned in the Introduction.

Remark 12.11. Note that if Γ is as in Example 9.9, then by Example 11.6 we have $Q_{\varphi\Gamma} = E_\varphi = X^2$. Therefore, by Notation 12.1, we have $d_{\rho\varphi\Gamma}(x, y) = d(x, y)$ for all $x, y \in X$, and thus $d_{\rho\varphi\Gamma} = d$.

Remark 12.12. While, if Γ is as in Example 10.1, then by Example 11.8, Remark 11.10 and the definition of φ we have

$$Q_{\varphi\Gamma} = \Delta_X \cup \big\{ (x, y) \in X^2 : \quad 0 = x \quad \text{or} \quad 0 = y \big\} = \Delta_X \cup \{0\} \times \mathbb{R} \cup \mathbb{R} \times \{0\}.$$

Therefore, by Notation 12.1, for any $x, y \in X$ we have

$$d_{\rho\varphi\Gamma}(x, y) = \begin{cases} |x - y| & \text{if} \quad x = y \quad \text{or} \quad x\,y = 0, \\ |x| + |y| & \text{if} \quad x \neq y \quad \text{and} \quad x\,y \neq 0. \end{cases}$$

Therefore, in the present particular case, $d_{\rho\varphi\Gamma}$ is again the postman metric on X.

13. A similar derivation of the radial metric

Analogously to the above derivations of the postman metric, the identically zero function can also be used to derive the radial metric.

Example 13.1. Suppose now that φ is as in Example 12.10, but Γ is as in Example 10.3 with $K = \mathbb{R}$. Then, in addition to the corresponding statements of Example 12.10, by Example 11.9, Remark 11.10 and the definition of φ we have

$$Q_{\varphi\,\Gamma} = \Delta_X \cup \big\{ (x,\, y) \in X^2 : \quad \exists\ \lambda \in \mathbb{R} : \quad 0 = \lambda\, x + (1-\lambda)\, y \big\}\,.$$

Next, we show that, in the present particular case, we simply have

$$Q_{\varphi\,\Gamma} = \big\{ (x,\, y) \in X^2 : \quad (x\,\bar{y})_2 = 0 \big\}\,.$$

For this, note that, if $(x,\, y) \in Q_{\varphi\,\Gamma}$, such that $x \neq y$, then there exists $\lambda \in \mathbb{R}$ such that $0 = \lambda\, x + (1-\lambda-1)\, y$, and thus

$$\lambda\, x = (\lambda - 1)\, y\,.$$

Hence, if $y \neq 0$, then we can infer that $\lambda \neq 0$, and thus

$$x = \big(1 - 1/\lambda\big)\, y.$$

Now, by taking $\mu = 1 - 1/\lambda$, we can also see that

$$x\,\bar{y} = \mu\, y\,\bar{y} = \mu\,|\,y\,|^2\,,$$

and thus

$$(x\,\bar{y})_2 = 0 \qquad \text{and} \qquad (x\,\bar{y})_1 = \mu\,|\,y\,|^2\,.$$

Thus, in particular $(x,\, y) \in Q_{\varphi\,\Gamma}$ implies $(x\,\bar{y})_2 = 0$ whenever $x \neq y$ and $y \neq 0$. Moreover, we can also note that

$$(x\,\bar{x})_2 = \big(\,|\,x\,|^2\big)_2 = 0 \qquad \text{and} \qquad (x\,\bar{0})_2 = (x\,0)_2 = 0_2 = 0$$

for all $x \in X$. Therefore,

$$Q_{\varphi\,\Gamma} \subset \big\{ (x,\, y) \in X^2 : \quad (x\,\bar{y})_2 = 0 \big\}\,.$$

To prove the converse inclusion, suppose now that $x,\, y \in X$ such that $(x\,\bar{y})_2 = 0$. Now, if $y \neq 0$, and thus $|\,y\,| \neq 0$, then by defining

$$\mu = (x\,\bar{y})_1 / |\,y\,|^2\,,$$

we can see that

$$x\,\bar{y} = (x\,\bar{y})_1 = \mu\,|\,y\,|^2 = \mu\, y\,\bar{y}\,.$$

Hence, since $\bar{y} \neq 0$, we can infer that

$$x = \mu\, y\,.$$

Now, if $x \neq y$, and thus $\mu \neq 1$, then we can also see that

$$\frac{1}{1-\mu} x + \left(1 - \frac{1}{1-\mu}\right) y = \frac{1}{1-\mu}\, \mu\, y + \frac{-\mu}{1-\mu}\, y = 0\,.$$

Hence, we can already see that $(x, y) \in Q_{\varphi\Gamma}$. Thus, in particular $(x\bar{y})_2 = 0$ implies $(x, y) \in Q_{\varphi\Gamma}$ whenever $x \neq y$ and $y \neq 0$. Moreover, by the reflexivity of $Q_{\varphi\Gamma}$ and the equality $0x = (0-1)0$, we can also note that

$$(x, x) \in Q_{\varphi\Gamma} \qquad \text{and} \qquad (x, 0) \in Q_{\varphi\Gamma}$$

for all $x \in X$. Therefore,

$$\big\{ (x, y) \in X^2 : \quad (x\bar{y})_2 = 0 \big\} \subset Q_{\varphi\Gamma},$$

and thus the required equality is also true.

Now, if $x, y \in X$, then by the above observations we can see that, according to Notation 12.1, we simply have

$$d_{\rho\varphi\Gamma}(x, y) = \begin{cases} |x-y| & \text{if} \quad (x\bar{y})_2 = 0, \\ |x| + |y| & \text{if} \quad (x\bar{y})_2 \neq 0. \end{cases}$$

Hence, by noticing that

$$(x\bar{y})_2 = 0 \iff x_2 y_1 - x_1 y_2 = 0 \iff x_1 y_2 = x_2 y_1,$$

we can see that, in the present particular case, $d_{\rho\varphi\Gamma}$ is just the radial metric on X mentioned in the Introduction.

Remark 13.2. Here, it is also worth mentioning that if $X = R^2$ with an integral domain R and

$$Q = \big\{ (x, y) \in X^2 : \quad x_1 y_2 = x_2 y_1 \big\},$$

then Q is a φ–equivalence relation on X with $\varphi = X \times \{0\}$.

Moreover, we have

$$Q\big((0, 0)\big) = X, \qquad Q\big((s, 0)\big) = R \times \{0\},$$

and

$$Q\big((s, t)\big) = \big\{ (u, v) \in X : \quad sv = tu \big\} = t/s$$

for all $s, t \in R$ with $s \neq 0$.

Thus, in particular $Q\big((s, t)\big)$, with $s, t \in R$ and $s \neq 0$, is a maximal partial multiplier, and so also a maximal partial homomorphism on R to itself. Moreover, the family $Q[\{0\}^c \times R]$ is the classical quotient field of R. (For some generalizations of the above ideas, see [18], [19], and the references therein.)

14. A similar derivation of the river metrics

Now, in contrast to the derivations the postman and radial metrics, the identically zero function has to be replaced by a non-constant one to derive the river metric.

Example 14.1. If $\varphi(x) = x_1$ for all $x \in X$, then it is clear that φ is a projection,

$$A_\varphi = \mathbb{R}\,, \quad B_\varphi = X \qquad \text{and} \qquad D_\varphi = \Delta_{\mathbb{R}}\,, \quad E_\varphi = \big\{ (x,\, y) \in X^2 : \ \ x_1 = y_1 \big\}\,.$$

Moreover, if $x,\, y \in X$, then by Remark 4.7, we can also easily see that

$$d_\varphi(x,\, y) = d(x,\, x_1\,) + d(x_1\,,\, y_1) + d(y_1\,,\, y) = |\, x_2 \,| \,+\, |\, x_1 - y_1 \,| \,+\, |\, y_2 \,|\,.$$

Note that, by the corresponding particular case of Theorem 4.3, d_φ is a φ–metric on X. Thus, according to property 3.1(4), we have $d_\varphi(x,\, y) = 0$ if and only if $(x,\, y) \in \Delta_{\mathbb{R}}$, i. e., $x = y \in \mathbb{R}$.

Furthermore, if Γ is as in Example 10.3, then by Example 11.9, Remark 11.10 and the definition of φ we have

$$Q_{\varphi\,\Gamma} = \Delta_X \cup \big\{ \,(x,\, y) \in X^2 : \quad x_1 = y_1\,, \quad \exists\ \lambda \in \mathbb{R} : \ \ x_1 = \lambda\, x + (\,1 - \lambda\,)\, y \,\big\}\,.$$

Next, we show that, in the present particular case, we simply have

$$Q_{\varphi\,\Gamma} = E_\varphi = \big\{ \,(x,\, y) \in X^2 : \quad x_1 = y_1 \big\}\,.$$

For this, note that, by Notation 11.1, $Q_{\varphi\,\Gamma} \subset E_\varphi$ automatically holds. Moreover, if $(\,x,\, y\,) \in E_\varphi$ such that $x \neq y$, then

$$x_1 = y_1 \qquad \text{and} \qquad x_2 \neq y_2\,.$$

Hence, by taking

$$\lambda = \frac{y_2}{y_2 - x_2}\,,$$

we have not only

$$x_1 = \lambda\, x_1 + (\,1 - \lambda\,)\, y_1\,, \qquad \text{but also} \qquad 0 = \lambda\, x_2 + (\,1 - \lambda\,)\, y_2\,.$$

Therefore,

$$x_1 = \lambda\, x + (\,1 - \lambda\,)\, y\,,$$

and thus $(\,x,\, y\,) \in Q_{\varphi\Gamma}$ also holds. Hence, by the reflexivity of $Q_{\varphi\,\Gamma}$, it is clear that $E_\varphi \subset Q_{\varphi\,\Gamma}$, and thus the required equality is also true.

Now, if $x,\, y \in X$, then by the above observations we can see that, according to Notation 12.6, we simply have

$$d_{\varphi\,\Gamma}(x,\, y) = \begin{cases} \quad |\, x_2 - y_2 \,| & \text{if} \quad x_1 = y_1\,, \\ |\, x_2 \,| \,+\, |\, x_1 - y_1 \,| \,+\, |\, y_2 \,| & \text{if} \quad x_1 \neq y_1\,. \end{cases}$$

Therefore, in the present particular case, $d_{\varphi\Gamma}$ is just the river metric on X mentioned in the Introduction.

Remark 14.2. Note that if Γ is as in Example 9.9, then by Example 11.6 we also have $Q_{\varphi\,\Gamma} = E_\varphi$. Therefore, by the above observations, $d_{\varphi\,\Gamma}$ is again the river metric on X.

Remark 14.3. While, if Γ is as in Example 9.10, then by Example 11.7, Remark 11.10 and the definition of φ we have

$$Q_{\varphi\Gamma} = \Delta_X \cup \big\{ (x, y) \in X^2 : \quad x_1 = y_1 , \quad x_1 \neq 0 \big\} .$$

Therefore, by Notation 12.6, for any $x, y \in X$ we have

$$d_{\varphi\Gamma}(x, y) = \begin{cases} 0 & \text{if} \quad x = y , \\ |x_2 - y_2| & \text{if} \quad x_1 = y_1 , \quad x_1 \neq 0 , \\ |x_2| + |x_1 - y_1| + |y_2| & \text{if} \quad x_1 \neq y_1 \ \text{ or } \ x_1 = 0 , \ \ x \neq y . \end{cases}$$

Remark 14.4. Moreover, if Γ is as in Example 10.1, then by Example 11.8, Remark 11.10 and the definition of φ we have

$$Q_{\varphi\Gamma} = \Delta_X \cup \big\{ (x, y) \in X^2 : \quad x_1 = y_1 , \quad x \in \mathbb{R} \quad \text{or} \quad y \in \mathbb{R} \big\} .$$

Therefore, by Notation 12.6, for any $x, y \in X$ we have

$$d_{\varphi\Gamma}(x, y) = \begin{cases} 0 & \text{if} \quad x = y , \\ |x_2 - y_2| & \text{if} \quad x_1 = y_1 , \ x \in \mathbb{R} \ \text{ or } \ y \in \mathbb{R} , \\ |x_2| + |x_1 - y_1| + |y_2| & \text{if} \quad x_1 \neq y_1 \ \text{ or } \ x, y \notin \mathbb{R} , \ \ x \neq y . \end{cases}$$

15. Some further illustrating examples

Example 15.1. If $\varphi(x) = \bar{x}$ for all $x \in X$, then it is clear that φ is an involution,

$$A_\varphi = B_\varphi = \mathbb{R} \qquad \text{and} \qquad D_\varphi = \Delta_{\mathbb{R}} , \qquad E_\varphi = \Delta_X .$$

Moreover, if $x, y \in X$, then according to Remark 4.7 we can also easily see that

$$d_\varphi(x, y) = |x - \bar{x}| + |\bar{x} - \bar{y}| + |\bar{y} - y| = 2|x_2| + |x - y| + 2|y_2| .$$

Note that, by the corresponding particular case of Theorem 4.3, d_φ is a φ–metric on X. Thus, according to property 3.1(4), we have $d_\varphi(x, y) = 0$ if and only if $(x, y) \in \Delta_{\mathbb{R}}$, i. e., $x = y \in \mathbb{R}$.

Furthermore, by Corollary 11.5, we can now at once see that $Q_{\varphi\Gamma} = \Delta_X$. Therefore, if $x, y \in X$, then by the above observations we can see that, according to Notation 12.6, we simply have

$$d_{\varphi\Gamma}(x, y) = \begin{cases} 0 & \text{if} \quad x = y , \\ |x - y| + 2|x_2| + 2|y_2| & \text{if} \quad x \neq y . \end{cases}$$

Example 15.2. If for all $x \in X$ we have

$$\varphi(x) = \begin{cases} 0 & \text{if} \quad x = 0\,, \\ 1/x & \text{if} \quad x \neq 0\,, \end{cases}$$

then it is clear that φ is an involution,

$$A_\varphi = B_\varphi = \{-1,\,0,\,1\} \qquad \text{and} \qquad D_\varphi = \Delta_{\{-1,\,0,\,1\}}\,, \qquad E_\varphi = \Delta_X\,.$$

Moreover, according to Remark 4.7, we can also easily see that

$$\begin{aligned} d_\varphi(0,\,0) &= |\,0-0\,| + |\,0-0\,| + |\,0-0\,| = 0\,, \\ d_\varphi(x,\,0) &= \Big|\,x - \frac{1}{x}\,\Big| + \Big|\,\frac{1}{x} - 0\Big| + |\,0-0\,| = \frac{|\,x^2-1\,|}{|\,x\,|} + \frac{1}{|\,x\,|}\,, \\ d_\varphi(0,\,y) &= |\,0-0\,| + \Big|\,0 - \frac{1}{y}\,\Big| + \Big|\,\frac{1}{y} - y\,\Big| = \frac{1}{|\,y\,|} + \frac{|\,y^2-1\,|}{|\,y\,|} \end{aligned}$$

and

$$\begin{aligned} d_\varphi(x,\,y) &= \Big|\,x - \frac{1}{x}\,\Big| + \Big|\,\frac{1}{x} - \frac{1}{y}\,\Big| + \Big|\,\frac{1}{y} - y\,\Big|\,, \\ &= \frac{|\,x^2-1\,|}{|\,x\,|} + \frac{|\,x-y\,|}{|\,x\,y\,|} + \frac{|\,y^2-1\,|}{|\,y\,|}\,, \end{aligned}$$

for all $x,\,y \in X$ with $x \neq 0$ and $y \neq 0$.

Note that, by the corresponding particular case of Theorem 4.3, d_φ is a φ–metric on X. Thus, according to property 3.1(4), we have $d_\varphi(x,\,y) = 0$ if and only if $(x,\,y) \in \Delta_{\{-1,\,0,\,1\}}$, i. e., $x = y \in \{-1,\,0,\,1\}$.

Furthermore, by Corollary 11.5, we can now at once see that $Q_{\varphi\Gamma} = \Delta_X$. Therefore, if $x,\,y \in X$, then by the above observations we can see that, according to Notation 12.6, we have

$$d_{\varphi\Gamma}(x,\,y) = \begin{cases} 0 & \text{if} \quad x = y\,, \\ \frac{|\,x^2-1\,|+1}{|\,x\,|}\,, & \text{if} \quad x \neq 0\,, \quad y = 0\,, \\ \frac{|\,y^2-1\,|+1}{|\,y\,|} & \text{if} \quad x = 0\,, \quad y \neq 0\,, \\ \frac{|\,x^2y-y\,|+|\,xy^2-x\,|+|\,x-y\,|}{|\,x\,y\,|} & \text{if} \quad x \neq 0\,, \quad y \neq 0\,, \quad x \neq y\,. \end{cases}$$

Example 15.3. If for all $x \in X$ we have $\varphi(x) = \operatorname{sgn}(x)$, i. e.,

$$\varphi(x) = \begin{cases} 0 & \text{if} \quad x = 0\,, \\ x/|\,x\,| & \text{if} \quad x \neq 0\,, \end{cases}$$

then it is clear that φ is a projection,

$$A_\varphi = \{0\} \cup S\,, \qquad B_\varphi = X \qquad \text{and} \qquad D_\varphi = \Delta_{\{0\} \cup S}\,,$$

where $S = \{ x \in X : \ |x| = 1 \}$. Moreover, we can also easily see that

$$E_\varphi = \Delta_{\{0\}} \cup \big\{ (x, r\,x) : \quad x \in \{0\}^c, \quad r > 0 \big\}.$$

Namely, for any $x, y \in X$, we have

$$\begin{aligned} (x, y) \in E_\varphi \iff & \varphi(x) = \varphi(y) \\ \iff & \big(x = 0, \ \ y = 0\big) \ \text{ or } \ \big(x \neq 0, \ \ y \neq 0, \ \ x/|x| = y/|y|\big) \\ \iff & \big(x = 0, \ \ y = 0\big) \ \text{ or } \ \big(x \neq 0, \ \ y \neq 0, \ \ y = r\,x \ \text{ with } \ r = |y|/|x|\big). \\ \iff & \big(x = 0, \ \ y = 0\big) \ \text{ or } \ \big(x \neq 0, \ \ y = r\,x \ \text{ with } \ r > 0\big). \end{aligned}$$

Moreover, according to Remark 4.7, we can also easily see that

$$\begin{aligned} d_\varphi(0, 0) &= |0 - 0| + |0 - 0| + |0 - 0| = 0, \\ d_\varphi(x, 0) &= \Big| x - \frac{x}{|x|} \Big| + \Big| \frac{x}{|x|} - 0 \Big| + \big| 0 - 0 \big| = \big| |x| - 1 \big| + 1, \\ d_\varphi(0, y) &= \big| 0 - 0 \big| + \Big| 0 - \frac{y}{|y|} \Big| + \Big| \frac{y}{|y|} - y \Big| = 1 + \big| |y| - 1 \big|, \end{aligned}$$

and

$$\begin{aligned} d_\varphi(x, y) &= \Big| x - \frac{x}{|x|} \Big| + \Big| \frac{x}{|x|} - \frac{y}{|y|} \Big| + \Big| \frac{y}{|y|} - y \Big| \\ &= \big| |x| - 1 \big| + \frac{\big| x |y| - |x| y \big|}{|x||y|} + \big| |y| - 1 \big|, \end{aligned}$$

for all $x, y \in X$ with $x \neq 0$ and $y \neq 0$.

Note that, by the corresponding particular case of Theorem 4.3, d_φ is a φ–metric on X. Thus, according to property 3.1(4), we have $d_\varphi(x, y) = 0$ if and only if $(x, y) \in \Delta_{\{0\} \cup S}$, i.e., $x = y$ and either $y = 0$ or $|y| = 1$.

Furthermore, if Γ is as in Example 9.10, then by Example 11.7, Remark 11.10, and the definition of φ we can see that

$$\begin{aligned} Q_{\varphi\Gamma} &= \Delta_X \cup \big\{ (x, y) \in E_\varphi : \quad \varphi(x) \neq 0 \big\} \\ &= \Delta_X \cup \big\{ (x, r\,x) : \quad x \in \{0\}^c, \ \ r > 0 \big\} \\ &= \Delta_{\{0\}} \cup \big\{ (x, r\,x) : \quad x \in \{0\}^c, \ \ r > 0 \big\} = E_\varphi. \end{aligned}$$

Namely, for some $x \in X$, we have $\varphi(x) \neq 0 \iff x \neq 0 \iff x \in \{0\}^c$. Moreover, for some $x \in \{0\}^c$ and $y \in X$, we have $(x, y) \in E_\varphi$ if and only if $y = r\,x$ for some $r > 0$.

Now, if $x, y \in X$, then by the above observations we can see that, according to Notation 12.6, we have $d_{\varphi\Gamma}(x, y)$

$$= \begin{cases} 0 & \text{if} \quad x = 0, \ \ y = 0, \\ |x - y| & \text{if} \quad x \neq 0, \ \ y/x > 0, \\ \big| |x| - 1 \big| + 1 & \text{if} \quad x \neq 0, \ \ y = 0, \\ \big| |y| - 1 \big| + 1 & \text{if} \quad x = 0, \ \ y \neq 0, \\ \big| |x| - 1 \big| + \frac{| x |y| - |x| y |}{|x||y|} + \big| |y| - 1 \big| & \text{if} \quad x\,y \neq 0, \ \ y/x \not> 0. \end{cases}$$

Remark 15.4. Note that if Γ is as in Example 9.9, then by Example 11.6 we also have $Q_{\varphi\Gamma} = E_{\varphi}$. Therefore, $d_{\varphi\Gamma}$ is again as above. The case when Γ is as in Example 10.1 or 10.3 is more difficult.

Remark 15.5. In addition to Examples 13.1, 14.1 and 15.3, it would be useful to consider the case when φ is the natural projection of the unit ball or square in X and Γ is one of the relations given in Examples 9.9, 9.10, 10.1 and 10.3.

Moreover, it would also be useful to find some further collinearity or pre-collinearity relations for X. And to establish some additional axioms to the ones given in Section 9 in order that we could get a relational characterization of the natural collinearity relation given in Example 10.3.

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