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AN INSTRUCTIVE TREATMENT AND SOME NATURAL EXTENSIONS OF A SET-VALUED FUNCTION OF ZSOLT PÁLES

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Abstract. In this paper, we offer an instructive treatment and some natural extensions of a subadditive set-valued function of Zs. Páles.

This function shows that the boundedness condition in a set-valued generalization of Hyers's stability theorem, proved by Z. Gajda and R. Ger, is essential.

Here, instead of set-valued functions, we shall use relations. Thus, the results will also illustrate the appropriateness of the relational methods of the present author.

Introduction

Hyers [24] in 1941, giving a partial answer to a general problem formulated by S. M. Ulam during a talk at the University of Wisconsin in 1940, proved a slightly weaker Banach space particular case the following stability theorem.

Theorem 1. *If f is an ε–approximately additive function of a commutative semigroup X to a Banach space Y, for some $\varepsilon > 0$, in the sense that*

$$\| f(x+y) - f(x) - f(y) \| \leq \varepsilon$$

for all $x, y \in X$, then there exists an additive function g of X to Y such that g is ε–near to f in the sense that

$$\| f(x) - g(x) \| \leq \varepsilon$$

for all $x \in X$.

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Remark 1. Hence, by using the $\mathbb{N}$–homogeneity of g, one can infer that

$$g(x) = \lim_{n\to\infty} n^{-1} f(nx)$$

for all $x \in X$. Therefore, the additive function g is uniquely determined.

By Ger [17], M. Laczkovich announced at a conference that a strict inequality form of the $X = \mathbb{N}$, $Y = \mathbb{R}$ and $\varepsilon = 1$ particular case of the above theorem was already proved by Pólya and Szegő [37, Aufgabe 99, pp. 17, 171] in 1925. Moreover, this particular case is actually equivalent to the original theorem. (For some ideas in this respect, see [63].)

However, it is now more important to note that Hyers's theorem was already transformed into set-valued settings by W. Smajdor [47] and Gajda and Ger [14], in 1986 and 1987, respectively, by making use the following observations.

If f and g are as in Theorem 1 and $B = \big\{ y \in Y : \ \ \| y \| \leq \varepsilon \big\}$, then

$$g(x) - f(x) \in B \qquad \text{and} \qquad f(x+y) - f(x) - f(y) \in B,$$

and hence

$$g(x) \in f(x) + B \qquad \text{and} \qquad f(x+y) \in f(x) + f(y) + B$$

for all $x, y \in X$.

Therefore, by defining

$$F(x) = f(x) + B$$

for all $x \in X$, we can get a set-valued function F of X to Y such that g is a selection of F and F is subadditive. That is,

$$g(x) \in F(x) \qquad \text{and} \qquad F(x+y) \subset F(x) + F(y)$$

for all $x, y \in X$.

Thus, the essence of Hyers's theorem is nothing but the statement of the existence of an additive selection function of a certain subadditive set-valued function. A similar observation, in connection with the Hahn–Banach extension theorems, was already announced by Rodríguez-Salinas and Bou [43] in 1974 and Gajda, A. Smajdor and W. Smajdor [15] in 1992. (See also [49], [53] and [20] for some further developments.)

In particular, in 1987 Gajda and Ger [14] proved the following generalization of Theorem 1. (See also Gajda [13, Theorem 4.2] for a further generalization.)

Theorem 2. *If F is a subadditive set-valued function of a commutative semigroup X to a Banach space Y such that the values of F are nonempty, closed and convex, and moreover*

$$\sup \big\{ \operatorname{diam}\big(F(x)\big) : \quad x \in X \big\} < +\infty,$$

then F has an additive selection function f.

Remark 2. Hence, by using the $\mathbb{N}$–homogeneity of f and the above boundedness condition on F, one can infer that

$$\{f(x)\} = \bigcap_{n=1}^{\infty} n^{-1} F(nx)$$

for all $x \in X$. Therefore, the additive selection function f of F is uniquely determined.

At the same time, Gajda and Ger [14] also proved an extension of this theorem to a separated, sequentially complete topological vector space Y. (See also Gajda [13, Theorem 4.3] for a further generalization.)

The importance of the observations of W. Smajdor, Gajda and Ger was soon recognized by Hyers and Rassias [25], Rassias [40], Hyers, Isac and Rassias [26, pp. 204–231], and Czerwik [9, pp. 301–329].

Moreover, the results of Gajda and Ger [14] have been generalized and improved by Popa [38, 39], Badora [3], Badora, Ger and Páles [4], Piao [36], Lu and Park [30], and the present author [58, 60].

However, it is now more important to note that, by finding the following counterexample, Zs. Páles showed at a conference that the boundedness condition on the function F is essential for the proof of Theorem 2.

This counterexample, which also clarifies the importance of the infimality condition of [58], was not originally published by Páles. However, it was cited by Gajda and Ger [14] in 1987, Hyers and Rassias [25] in 1992, Rassias [40], and Hyers, Isac and Rassias [26, p. 210] in 1998. (Moreover, A. Smajdor [46] in 1990 considered a superadditive counterpart of it.)

Example. Define $\mathbb{R}_+ = [0, +\infty[$ and

$$F(x) = [x^2, +\infty[$$

for all $x \in \mathbb{R}_+$. Then, F is a subadditive set-valued function of the semigroup $\mathbb{R}_+$ to the Banach space $\mathbb{R}$ such that values of F are nonempty, closed and convex, but F still does not have any additive selection function.

To prove the latter fact, following [14], assume on the contrary that f is an additive selection function of F. Then, by [1, Theorem 2.1], f can be extended to an additive function g of $\mathbb{R}$ to itself. Moreover, we can note that

$$g(x) = f(x) \in F(x) = [x^2, +\infty[,$$

and thus $x^2 \le g(x)$ for all $x \in \mathbb{R}_+$. Therefore, g is bounded below by 0 on $\mathbb{R}_+$. Thus, by [1, Corollary 2.5], there exists a number $c \in \mathbb{R}$ such that $g(x) = cx$ for all $x \in \mathbb{R}$. Hence, we can already infer that $x^2 \le g(x) = cx$, and thus $x \le c$ for all $x \in \mathbb{R}$ with $x > 0$. This contradiction proves the the required assertion.

Unfortunately, the set-valued function F of Páles is defined only on a semigroup. Therefore, in view of the counterexamples of Á. Száz and G. Száz [64], Godini [21], Sablik [45], Paganoni [35], Forti and Schwaiger [11], Forti [10], Gajda [12], Rassias and Šemrl [41], Găvruţă [16], Kazhdan [27], and Špakula and Zlatoš [50], it seems to be of some interest to find some reasonable extensions of the function F to $\mathbb{R}$.

This problem and some of its immediate generalizations, motivated by the results of Aczél et al. [2], were posed by the present author at some special courses for students and several talks with colleagues. However, no answers have been obtained. Therefore, it seems reasonable to present here some possible solutions. These will also well illustrate the appropriateness of our relational methods offered in [63], where we have only considered a natural totalization of F.

1. Relations and functions

A subset F of a product set $X \times Y$ is called a relation on X to Y. If in particular $F \subset X^2$, then we may simply say that F is a relation on X. Thus, a relation on X to Y is also a relation on $X \cup Y$.

If F is a relation on X to Y, then for any $x \in X$ and $A \subset X$ the sets $F(x) = \{ y \in Y : \ (x, y) \in F \}$ and $F[A] = \bigcup_{a \in A} F(a)$ are called the images of x and A under F, respectively.

Instead of $y \in F(x)$ sometimes we shall also write $x F y$. Moreover, the sets $D_F = \{ x \in X : \ F(x) \neq \emptyset \}$ and $R_F = F[X] = F[D_F]$ will be called the domain and range of F, respectively.

If in particular $D_F = X$, then we say that F is a relation of X to Y, or that F is a total relation on X to Y. While, if $R_F = Y$, then we say that F is a relation on X onto Y.

If F is a relation on X to Y, then $F = \bigcup_{x \in X} \{x\} \times F(x) = \bigcup_{x \in D_F} \{x\} \times F(x)$. Therefore, a relation F on X to Y can be naturally defined by by specifying $F(x)$ for all $x \in X$, or by specifying D_F and $F(x)$ for all $x \in D_F$.

For instance, if F is a relation on X to Y, then the inverse relation F^{-1} of F can be naturally defined such that $F^{-1}(y) = \{ x \in X : \ y \in F(x) \}$ for all $y \in Y$. Thus, we also have $F^{-1} = \{ (y, x) : \ (x, y) \in F \}$.

Moreover, if in addition G is a relation on Y to Z, then the composition relation $G \circ F$ of G and F can be naturally defined such that $(G \circ F)(x) = G[F(x)]$ for all $x \in X$. Thus, we also have $(G \circ F)[A] = G[F[A]]$ for all $A \subset X$.

In particular, a relation f on X to Y is called a function if for each $x \in D_f$ there exists $y \in Y$ such that $f(x) = \{y\}$. In this case, by identifying singletons with their elements, we may simply write $f(x) = y$ in place of $f(x) = \{y\}$.

A relation F on X to Y can be naturally identified with the set-valued function $\mathfrak{F}$ defined by $\mathfrak{F}(x) = F(x)$ for all $x \in X$. However, thus in contrast to $F \subset X \times Y$ we have $\mathfrak{F} \subset X \times \mathcal{P}(Y)$. Therefore, F is a more convenient tool than $\mathfrak{F}$.

If F is a relation on X to Y, then a subset Φ of F is called a partial selection relation of F. Thus, we also have $D_\Phi \subset D_F$. Therefore, a partial selection relation Φ of F may be called total if $D_\Phi = D_F$.

In the literature, the total selection functions of a relation F are usually called the selections of F. Thus, in particular, the Axiom of Choice can be briefly expressed by saying that every relation F has a selection.

If F is a relation on X to Y and $U \subset X$, then the relation $F | U = F \cap (U \times Y)$ is called the restriction of F to U. Moreover, F and G are relations on X to Y such that $D_F \subset D_G$ and $F = G | D_F$, then G is called an extension of F.

2. Computations with sets

A function $\star$ of a set X to itself is called an unary operation on X. Moreover, a function $*$ of X^2 to X is called a binary operation in X. In these cases, for any $x, y \in X$, we usually write $x^\star$ and $x * y$ in place of $\star(x)$ and $*\big((x, y)\big)$, respectively.

A set X, equipped with a binary operation $+$, is called a groupoid. Instead of groupoids, it is usually sufficient to consider only semigroups (associative groupoids) or even monoids (semigroups with zero).

However, several definitions on semigroups can be naturally extended to groupoids. For instance, if X is a groupoid, then for any $x \in X$ and $n \in \mathbb{N}$, with $n \neq 1$, we may naturally define $n\,x = (n-1)\,x + x$ with the convention that $1\,x = x$.

Moreover, for any $n \in \mathbb{N}$ and $A \subset X$ we may also naturally define $n\,A = \{\,n\,a : \ a \in A\,\}$. And, for any $A,\,B \subset X$, we may naturally define $A + B = \{\,a+b : \ a \in A,\ b \in B\,\}$. Thus, for instance, $2\,A$ can be easily confused with the possibly larger set $A + A$.

If in particular X is a group, then for any $k \in \mathbb{Z}$ and $A \subset X$ we may also define $k\,A = \{\,k\,a : \ a \in A\,\}$. And, for any $A,\,B \subset X$, we may also write $-A = (-1)\,A$ and $A - B = A + (-B)$ despite that the family $\mathcal{P}(X)$ is, in general, only a monoid with involution.

If more specially X is a vector space over K, then for any $\lambda \in K$ and $A \subset X$ we may also define $\lambda\,A = \{\,\lambda\,a : \ a \in A\,\}$. Thus, only two axioms of a vector space may fail to hold for $\mathcal{P}(X)$. Namely, only the one point subsets of X can have additive inverses. Moreover, in general we only have $(\lambda + \mu)\,A \subset \lambda\,A + \mu\,A$.

A subset A of a groupoid X is called additive, subadditive and superadditive if $A = A + A$, $A \subset A + A$ and $A + A \subset A$, respectively. Moreover, for some $n \in \mathbb{N}$, the set A is called n–homogeneous, n–subhomogeneous and n–superhomogeneous if $A = n\,A$, $A \subset n\,A$ and $n\,A \subset A$, respectively.

In particular, a subset A of a group X is called symmetric if $A = -A$. Moreover, for some $\lambda \in K$, a subset A of a vector space X over K is called λ–affine, λ–subaffine and λ–superaffine if $A = \lambda\,A + (1-\lambda)\,A$, $A \subset \lambda\,A + (1-\lambda)\,A$ and $\lambda\,A + (1-\lambda)\,A \subset A$, respectively.

Thus, a subset A of a vector space X over $\mathbb{R}$ may be called convex if A is $[0,\,1]$–superaffine in the sense that A is λ–superaffine for all $\lambda \in [0,\,1]$. Note that the inclusions $0\,A + (1-0)\,A \subset A$ and $1\,A + (1-1) \subset A$ always hold. Therefore, we may take here $]\,0,\,1\,[$ in place of $[\,0\,,\,1\,]$.

3. Computations with intervals

In the set $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty,\ +\infty\}$ of the extended real numbers, beside the usual ordering, we shall only consider some restricted addition and multiplication. Thus, in contrast some recent trends, expressions like $0\,(+\infty)$ and $-\infty + (+\infty)$ will not be defined.

Moreover, for any $a\,,\,b \in \overline{\mathbb{R}}$, with $a \leq b$, we shall write $[\,a,\,b\,] = \{x \in \overline{\mathbb{R}} : \ a \leq x \leq b\,\}$,

$$[\,a,\,b\,[\, = \{x \in \overline{\mathbb{R}} : \ a \leq x < b\} \qquad \text{and} \qquad]\,a,\,b\,] = \{x \in \overline{\mathbb{R}} : \ a < x \leq b\,\}.$$

Thus, we have $[\,a,\,a\,] = \{a\}$ and $[\,a,\,a\,[\, = \,]\,a,\,a] = \emptyset$. Therefore, we shall usually assume that $a < b$.

Concerning half-open intervals, in the sequel we shall only need some particular cases the following obvious facts.

Theorem 3.1. *If $a,\,b \in \mathbb{R}$ and $c \in \mathbb{R} \cup \{+\infty\}$ such that $b < c$, then*

$$a + [\,b,\,c\,[\, = [\,a + b,\,a + c\,[\,.$$

Corollary 3.2. *If* $a \in \mathbb{R}$ *and* $b \in \mathbb{R} \cup \{+\infty\}$ *such that* $a < b$, *then*

$$[\, a,\, b\, [\, = a + \, [\, 0, -a + b\, [\, .$$

Theorem 3.3. *If* $a \in \mathbb{R}$, $b \in \mathbb{R} \cup \{+\infty\}$ *such that* $a < b$, *then for any* $\lambda \in \mathbb{R}$ *we have*

$$\lambda\, [\, a,\, b\, [\, = \begin{cases} \{0\} & \text{if} \quad \lambda = 0\,; \\ [\, \lambda\, a,\ \lambda\, b\, [& \text{if} \quad \lambda > 0\,; \\ ]\, \lambda\, b,\ \lambda\, a\,] & \text{if} \quad \lambda < 0\,. \end{cases}$$

Lemma 3.4. *If* $a, b \in \mathbb{R} \cup \{+\infty\}$ *such that* $0 < a,\, b$, *then*

$$[\, 0,\, a\, [\, + [\, 0,\, b\, [\, = \, [\, 0\,, a + b\, [\, .$$

Hint. If $x \in \, [\, 0\,, a + b\, [$ and $a\,,\, b \neq +\infty$, then by taking $y = a\, (\, a + b\,)^{-1} x$ and $z = b\, (\, a + b\,)^{-1} x$ we can easily see that $y \in \, [0,\, a\, [$ and $z \in \, [0,\, b\, [$ such that $x = y + z$. Therefore, $[\, 0\,, a + b\, [\, \subset [\, 0,\, a\, [\, + [\, 0,\, b\, [$.

While, if for instance $a = +\infty$, then we also have $a + b = +\infty$, and thus $[\, 0\,,\ a + b\, [\, = [\, 0,\, a\, [$. Hence, since $0 \in [\, 0,\, b\, [$, and thus $[\, 0,\, a\, [\, \subset [\, 0,\, a\, [\, + [\, 0,\, b\, [$, it is already clear that the required inclusion is again true.

Theorem 3.5. *If* $a,\, c \in \mathbb{R}$ *and* $b,\, d \in \mathbb{R} \cup \{+\infty\}$ *such that* $a < b$ *and* $c < d$, *then*

$$[\, a,\, b\, [\, + [\, c,\, d\, [\, = \, [\, a + c,\ b + d\, [\, .$$

Proof. By Corollary 3.2 and Lemma 3.4, we have

$$[\, a,\, b\, [\, + [\, c,\, d\, [\, = a + [\, 0, -a + b\, [\, + c + [\, 0, -c + d\, [\, = a + c + \, [\, 0, -a + b\, [\, + [\, 0, -c + d\, [$$
$$= a + c + \, [\, 0,\ -a + b - c + d\, [\, = a + c + \, [\, 0,\ -(a + c) + b + d\, [\, = \, [\, a + c,\ b + d\, [\, .$$

Remark 3.6. In the last section of the paper, we shall also need some similar results for the infimuma and suprema of subsets of $\mathbb{R}$.

For instance, if A and B are nonvoid subsets of $\mathbb{R}$, then it can be easily shown that

$$\inf(\, A + B\,) = \inf(A) + \inf(B) \qquad \text{and} \qquad \sup(\, A + B\,) = \sup(A) + \sup(B)\, .$$

4. Additive and homogeneous relations

Analogously to the definitions of additive, subadditive and superadditive functions, studied in [1, 44, 23, 28], we may also naturally have the following

Definition 4.1. A relation F on one groupoid X to another Y is called

(1) additive if $F\, (\, x + y\,) = F(x) + F(y)$,

(2) subadditive if $F\, (\, x + y\,) \subset F(x) + F(y)$,

(3) superadditive if $F(x) + F(y) \subset F\, (\, x + y\,)$

for all $x,\, y \in X$.

Remark 4.2. Moreover, the relation F may, for instance, be naturally called semi-additive (left-quasi-additive) if the equality $F(x+y) = F(x) + F(y)$ is required to hold only for all $x, y \in D_F$ $(x \in D_F$ and $y \in X)$.

Furthermore, if in particular X has a zero element (X is a group), then the relation F may, for instance, be naturally called left-zero-additive (inversion-additive) if $F(x) = F(0) + F(x)$ $(F(0) = F(x) + F(-x))$ for all $x \in X$.

Definition 4.3. For some $n \in \mathbb{N}$, a relation F on one groupoid X to another Y is called

(1) n–homogeneous if $F(nx) = nF(x)$,

(2) n–subhomogeneous if $F(nx) \subset nF(x)$,

(3) n–superhomogeneous if $nF(x) \subset F(nx)$

for all $x \in X$.

Remark 4.4. Moreover, the relation F may, for instance, be naturally called n–semihomogeneous if the equality $F(nx) = nF(x)$ is required to hold only for all $x \in D_F$.

Furthermore, for some $A \subset \mathbb{N}$, the relation F may for instance, be naturally called A–homogeneous if it is n–homogeneous for all $n \in A$.

The following two simple theorems reveal some intimate connections between additivity and homogeneity properties.

Theorem 4.5. *If F is a superadditive relation on one groupoid X to another Y, then F is $\mathbb{N}$–superhomogeneous.*

Corollary 4.6. *If f is an additive function of one groupoid X to another Y, then f is $\mathbb{N}$–homogeneous.*

Theorem 4.7. *If F is a subadditive relation on a groupoid X to a vector space Y over $\mathbb{Q}$ such that the value $F(x)$ is n^{-1}–superaffine for all $x \in X$ and $n \in \mathbb{N}$, then F is $\mathbb{N}$–subhomogeneous.*

Proof. If $x \in X$ and $n \in \mathbb{N}$ such that $F(nx) \subset nF(x)$, then we also have

$$\begin{aligned} F\big((n+1)x\big) &= F(nx+x) \subset F(nx) + F(x) \subset nF(x) + F(x) = F(x) + nF(x) \\ &= (n+1)\Big((n+1)^{-1}F(x) + \big(1-(n+1)^{-1}\big)F(x)\Big) \subset (n+1)F(x). \end{aligned}$$

Definition 4.8. A relation F on one group X to another Y is called odd if $F(-x) = -F(x)$ for all $x \in X$.

Remark 4.9. Quite similarly, the relation F may be naturally called even if $F(-x) = F(x)$ for all $x \in X$.

Moreover, the relation F may, for instance, be naturally called semi-subodd if $F(-x) \subset -F(x)$ for all $x \in D_F$.

However, by the following obvious theorem, some further similar weakenings of Definition 4.8 need not be introduced.

Theorem 4.10. *If F is a relation on one group X to another Y, then the following assertions are equivalent:*

(1) F is odd;

(2) $F(-x) \subset -F(x)$ for all $x \in X$;

(3) $-F(x) \subset F(-x)$ for all $x \in D_F$.

The fact that odd relations are more important than the even ones is apparent from the following two basic theorems.

Theorem 4.11. *If f is an additive function of one group X to another Y, then f is odd.*

Theorem 4.12. *If F is a nonvoid, odd and superadditive relation on one group X to another Y, then $0 \in F(0)$ and F is quasi-additive.*

Remark 4.13. This theorem can be improved by assuming only that Y is a monoid and F is quasi-odd in the sense that $0 \in F(x) + F(-x)$ for all $x \in D_F$.

To see the importance of odd relations, it is also worth mentioning that, by using an obvious analogue of Definition 4.3, we can also easily prove the following

Theorem 4.14. *If F is an odd, n–subhomogeneous (n–superhomogeneous) relation on one group X to another Y, for some $n \in \mathbb{N}$, then F is $-n$–subhomogeneous ($-n$–superhomogeneous).*

Hence, by Corollary 4.6 and Theorem 4.11, it is clear that in particular we also have

Corollary 4.15. *If f is an additive function of one group X to another Y, then $\mathbb{Z}$–homogeneous.*

By using an obvious analogue of Definition 4.3, we can also easily prove the following

Theorem 4.16. *If F is a λ–subhomogeneous (λ–superhomogeneous) relation on one vector space X over K to another Y, for some $\lambda \in K \setminus \{0\}$, then F is λ^{-1}–superhomogeneous (λ^{-1}–subhomogeneous).*

Proof. Namely, if F λ–subhomogeneous, then we also have

$$\lambda^{-1} F(x) = \lambda^{-1} F\big(\lambda\, \lambda^{-1} x\big) \subset \lambda^{-1} \lambda\, F\big(\lambda^{-1} x\big) = F\big(\lambda^{-1} x\big)$$

for all $x \in X$. Therefore, F is λ^{-1}–superhomogeneous.

Remark 4.17. In the sequel, a relation F on one vector space X over $\mathbb{R}$ to another Y will be called convex-valued if $F(x)$ is a convex subset of Y for all $x \in X$.

Moreover, the relation F will be called convex if it is λ–superaffine for all $\lambda \in [0, 1]$ in the sense that $\lambda F(x) + (1-\lambda) F(y) \subset F\big(\lambda x + (1-\lambda) y\big)$ for all $x, y \in X$.

Note that thus a convex relation is always convex-valued, but the converse statement need not be true. Moreover, the relation F is convex if and only if it is a convex subset of the product space $X \times Y$.

However, it is now more important to note that a subset A of Y is convex if and only if the relation $X \times A$ is convex. Therefore, the definition and properties of convex sets can also be derived from those of convex relations.

5. The global negative of a relation

Definition 5.1. For any relation F on one group X to another Y, we define a relation $F^{\wedge}$ on X to Y such that

$$F^{\wedge}(x) = -F(-x)$$

for all X. Moreover, we also define $F^{\vartriangle} = F \cap F^{\wedge}$.

Remark 5.2. Thus, we have $D_{F^{\wedge}} = -D_F$ and

$$F^{\wedge} = \big\{ (-x, -y) : \quad (x, y) \in F \big\} .$$

Therefore, the relation $F^{\wedge}$ will be called the global negative of F. (See [18].)

Remark 5.3. The partial negatives $-F$ and $F^{\vee}$ of F can defined such that $-F(x) = -F(x)$ and $F^{\vee}(x) = F(-x)$ for all $x \in X$. Note that any one of the above three negatives can be expressed in terms of the other two. Moreover, $-F$ can easily be confused with $F^{\wedge}$.

Concerning the operations $\wedge$ and $\vartriangle$, the following simple theorems have been proved in [63].

Theorem 5.4. *For any relation F on one group X to another Y, we have*

(1) $F^{\wedge} = F^{\wedge\wedge}$; *(2)* $F^{\vartriangle} = F^{\vartriangle\wedge} = F^{\wedge\vartriangle}$; *(2)* $F^{\vartriangle} = F^{\vartriangle\vartriangle}$.

Remark 5.5. Thus, $\wedge$ is an involution and $\vartriangle$ is an idempotent operation on the family $\mathcal{P}(X \times Y)$ of all relations on X to Y. Moreover, $\wedge$ and $\vartriangle$ commute, and $F^{\vartriangle}$ is $\wedge$–invariant. That is, $F^{\vartriangle}$ is a fixed point of $\wedge$.

Theorem 5.6. *For any relation F on one group X to another Y, the following assertions are equivalent:*

(1) F *is odd;* *(2)* $F^{\wedge}$ *is odd;*

(3) $F = F^{\wedge}$; *(4)* $F = F^{\vartriangle}$; *(5)* $F^{\wedge} = F^{\vartriangle}$.

Remark 5.7. In this respect, it is also worth mentioning that F is quasi-odd if and only if $D_F = D_{F^{\vartriangle}}$. Moreover, $F^{\vartriangle}$ is total if and only if F is total and quasi-odd.

From Theorem 5.7, by Theorem 4.11, it is clear that in particular we have

Corollary 5.8. *If f is an additive function of one group X to another Y, then $f = f^{\wedge} = f^{\vartriangle}$.*

Theorem 5.9. *For any relation F on one group X to another Y,*

(1) $F^{\vartriangle}$ *is the largest odd partial selection relation of* F;

(2) $F^{\vartriangle}$ *is the largest odd partial selection relation of* $F^{\wedge}$.

Proof. By definition, we have $F^{\vartriangle} = F \cap \hat{F} \subset F$. Therefore, $F^{\vartriangle}$ is a partial selection relation of F. Moreover, by Theorem 5.4, we have $F^{\vartriangle\wedge} = F^{\vartriangle}$. Therefore, by Theorem 5.6, $F^{\vartriangle}$ is always odd.

On the other hand, if Φ is an odd partial selection relation of F, then by Theorem 5.6 we have $\Phi = \Phi^{\vartriangle}$. Moreover, by the corresponding definitions, we have $\Phi \subset F$, and hence $\Phi^{\vartriangle} \subset F^{\vartriangle}$. Therefore, $\Phi \subset F^{\vartriangle}$ also holds.

Hence, it is clear that (1) is true. Moreover, by Theorem 5.4, we have $F^{\vartriangle} = F^{\wedge\vartriangle}$. Therefore, (2) can be immediately derived from (1). by writing $F^{\wedge}$ in place F.

Corollary 5.10. *If F is a relation on one group X to another Y and Φ is an odd partial selection relation of F, then $\Phi \subset F^{\triangle}$.*

Remar 5.11. In contrast to $\wedge$, the operation $\triangle$ is not compatible with most of the set and relation theoretic operations. Moreover, the relation $F^{\triangle}$ fails to inherit several basic properties of F.

6. The Hyers transform of a relation

Definition 6.1. If F is a relation on a group X to a vector space Y over $\mathbb{Q}$, then for any $k \in \mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$ we define a relation F_k on X to Y such that

$$F_k(x) = k^{-1} F(kx)$$

for all $x \in X$.

Remark 6.2. Thus, we have $D_{F_k} = \{ x \in X : \ kx \in D_F \}$ and

$$F_k = \{ (x, y) \in X \times Y : \quad (kx, \ ky) \in F \} .$$

The relation F_k or the family $(F_k)_{k \in \mathbb{Z}^*}$ will be called the Hyers transform of F. Though, in contrast to Pólya and Szegő [37, pp. 17, 171], Hyers [24] originally used the functional case of the subfamily $(F_{2^n})_{n \in \mathbb{N}}$.

The set-valued case has been first studied by W. Smajdor [47] and Gajda and Ger [14]. For some further developments, see Popa [38], Nikodem and Popa [34], Lu and Park [30] and the present author [58, 60].

Remark 6.3. Note that if in particular X is also a vector space over $\mathbb{Q}$, then we may also naturally define $F_\lambda(x) = \lambda^{-1} F(\lambda x)$ for all $x \in X$ and $\lambda \in \mathbb{Q} \setminus \{0\}$.

Concerning the relations F_k, the following simple theorems have also been proved in [63].

Theorem 6.4. *If F is a relation on a group X to a vector space Y over $\mathbb{Q}$, then for any $k \in \mathbb{Z}^*$*

(1) F is k–subhomogeneous if and only if $F_k \subset F$;

(2) F is k–superhomogeneous if and only if $F \subset F_k$.

Corollary 6.5. *If F is as in Theorem 6.4, then for any $k \in \mathbb{Z}^*$ the relation F is k–homogeneous if and only if $F = F_k$.*

Hence, by Corollary 4.15, it is clear that in particular we also have

Corollary 6.6. *If f is an additive function of a group X to a vector space Y over $\mathbb{Q}$, then $f = f_k$ for all $k \in \mathbb{Z}^*$.*

Moreover, as an immediate consequence of Theorem 6.4, we can also state

Corollary 6.7. *If F is a relation on one group X to another Y and Φ is a k–superhomogeneous partial selection relation of F, for some $k \in \mathbb{Z}^*$, then $\Phi \subset F_k$.*

Theorem 6.8. *If F is a relation on a group X to a vector space Y over $\mathbb{Q}$, then for any $k, l \in \mathbb{Z}^*$ we have*

(1) $(F_k)_l = (F_l)_k = F_{kl}$;

(2) $(F_k)^\wedge = (F^\wedge)_k = F_{-k}$; (3) $(F_k)^\Delta = (F^\Delta)_k = F_k \cap F_{-k}$.

Hence, it is clear that in particular we also have

Corollary 6.9. *If F is as in Theorem 6.8, then*

(1) $F^\wedge = F_{-1}$; (2) $F^\Delta = F \cap F_{-1}$.

Moreover, from Theorem 6.8, by using Theorem 5.6, we can immediately get

Corollary 6.10. *If F is an odd relation on one group X to another Y, then F_k is also odd for all $k \in \mathbb{Z}^*$.*

7. Two superhomogenizations of a relation

Definition 7.1. For any relation F on a group X to a vector space Y over $\mathbb{Q}$, we also define

$$F^\star = \bigcap_{n \in \mathbb{N}} F_n \qquad \text{and} \qquad F^* = \bigcap_{k \in \mathbb{Z}^*} F_k .$$

Remark 7.2. Thus, we have

$$F^* \subset F^\star \subset F_1 = F \qquad \text{and} \qquad F^* \subset F_1 \cap F_{-1} = F \cap F^\wedge = F^\Delta .$$

Concerning operations $\star$ and $*$, the following simple theorems have also been proved in [63].

Theorem 7.3. *If F is a relation on a group X to a vector space Y over $\mathbb{Q}$, then the following assertions are equivalent:*

(1) F *is* $\mathbb{N}$*–superhomogeneous;* (2) $F \subset F^\star$; (3) $F = F^\star$.

Theorem 7.4. *If F is a relation on a group X to a vector space Y over $\mathbb{Q}$, then the following assertions are equivalent:*

(1) F *is* $\mathbb{Z}^*$*–superhomogeneous;* (2) $F \subset F^*$; (3) $F = F^*$.

Hence, by Corollary 4.15, it is clear that in particular we have

Corollary 7.5. *If f is an additive function on a group X to a vector space Y over $\mathbb{Q}$, then $f = f^\star = f^*$.*

Theorem 7.6. *If F is a relation on a group X to a vector space Y over $\mathbb{Q}$, then*

(1) $F^{\wedge\star} = F^{\star\wedge}$; (2) $F^{\Delta\star} = F^{\star\Delta} = F^*$;

(3) $\big(F_k\big)^\star = \big(F^\star\big)_k$ *for all* $k \in \mathbb{Z}^*$; (4) $F^{\star\star} = F^\star$.

Remark 7.7. In this respect, it is worth noticing that

$$F^{\star\wedge} = \bigcap_{n\in\mathbb{N}} F_{-n} \qquad \text{and} \qquad \big(F^{\star}\big)_k = \bigcap_{n\in\mathbb{N}} F_{nk}$$

for all $k \in \mathbb{Z}^*$.

Theorem 7.8. *If F is a relation on a group X to a vector space Y over $\mathbb{Q}$, then*

(1) $F^{\wedge\,*} = F^{*\wedge} = F^*$; *(2)* $F^{\Delta*} = F^{*\Delta} = F^*$;

(3) $\big(F_k\big)^* = \big(F^*\big)_k$ *for all* $k \in \mathbb{Z}^*$; *(4)* $F^{\star*} = F^{*\star} = F^{**} = F^*$.

From Theorems 7.6 and 7.8, by using Theorem 5.6, we can immediately get

Corollary 7.9. *If F is an odd relation on a group X to a vector space Y, then $F^{\star}$ and F^* are also odd.*

Theorem 7.10. *If F is a relation on a group X to a vector space Y over $\mathbb{Q}$, then*

(1) $F^{\star}$ *is the largest* $\mathbb{N}$*–superhomogeneous relation contained in* F ;

(2) F^* *is the largest* $\mathbb{Z}^*$*–superhomogeneous relation contained in* F .

Corollary 7.11. *If F is a relation on a group X to a vector space Y over $\mathbb{Q}$ and Φ is an $\mathbb{N}$–superhomogeneous ($\mathbb{Z}^*$–superhomogeneous) partial selection relation of F, then $\Phi \subset F^{\star}$ ($\Phi \subset F^*$).*

Remar 7.12. In contrast to $F \mapsto F_k$, the operations $\star$ and $*$ are not compatible with most of the set and relation theoretic operations. Moreover, the relations $F^{\star}$ and F^* fail to inherit several basic properties of F.

8. The relational equivalent of a set-valued function of Zsolt Páles

Definition 8.1. Define

$$\mathbb{R}_+ = [\,0\,, +\infty\,[\qquad \text{and} \qquad \varphi(x) = x^2$$

for all $x \in \mathbb{R}_+$.

Then, the function φ and the set $\mathbb{R}_+$ can easily be seen to have the following useful properties.

Theorem 8.2.

(1) φ *is increasing and convex ;*

(2) φ *is superadditive and* $\varphi(0) = 0$;

(3) φ *is* $[\,0,\,1\,]$*–subhomogeneous and* $[\,1,\,+\infty\,[$*–superhomogeneous.*

Proof. By two well-known theorems in calculus [51 , (4.18) and (4.47)], assertion (1) is immediate from the facts that $\varphi'(x) = 2\,x \geq 0$ and $\varphi''(x) = 2 \geq 0$ for all $x \in \mathbb{R}_+$.

Moreover, if $x \in \mathbb{R}_+$ and $0 \leq \lambda \leq 1$, then by using the second parts of (1) and (2), we can easily see that

$$\varphi(\lambda x) = \varphi\big(\lambda x + (1-\lambda)\,0\big) \leq \lambda\,\varphi(x) + (1-\lambda)\,\varphi(0) = \lambda\,\varphi(x)\,.$$

Therefore, the first part of (3) is true. Hence, the second part of (3) can be immediately derived by using an functional analogue of Theorem 4.16.

Finally, to complete the proof, we can note that if $x, y \in \mathbb{R}_+$, then $0 \leq 2\,x\,y$, and thus

$$\varphi(x) + \varphi(y) = x^2 + y^2 \leq x^2 + y^2 + 2\,x\,y = (x+y)^2 = \varphi(x+y)\,.$$

Therefore, the first part of (2) is also true.

Remark 8.3. Because of the above non-direct proof of (1), it is worth noticing that by an improvement of Rathore's [42, Theorem 1] the superadditivity of φ on $]\,0\,, +\infty\,[$ can also be immediately derived from the fact that $x^{-1}\,\varphi(x) = x \leq 2\,x = \varphi'(x)$ for all $x \in\,]\,0\,, +\infty\,[$. Moreover, by the second part of (2), the function φ is zero-additive.

In this respect, it is also worth mentioning that if f is a superadditive function of $\mathbb{R}_+$ to itself, then f is increasing and $f(0) = 0$. Moreover, by Matkowski [31, Lemma 2], f is differentiable at 0 and $f\,'(0) = \inf_{x>0} x^{-1}\,f(x)$.

In connection with Theorem 8.2, it is also worth mentioning that by Rosenbaum [44, Theorem 1.4.6] a finite-valued convex function is subadditive if and only if it is $[1, +\infty\,[$–subhomogeneous. Moreover, by Burai and Száz [8, Corollary 4.5], a 2–homogeneous real-valued function is subadditive if and only if it is 2^{-1}–subaffine.

Theorem 8.4.

(1) $\mathbb{R}_+$ *is a closed and convex;*

(2) $\mathbb{R}_+ = \mathbb{R}_+ + \mathbb{R}_+$ *and* $\mathbb{R} = \mathbb{R}_+ - \mathbb{R}_+$*;*

(3) $\mathbb{R}_+ = \lambda\,\mathbb{R}_+$ *if* $\lambda > 0$ *and* $-\mathbb{R}_+ = \lambda\,\mathbb{R}_+$ *if* $\lambda < 0$.

Proof. By the corresponding definitions, it is clear that (1) is true. Moreover, we can note that (3) and the first part of (2) are particular cases of the corresponding statements of Theorem 3.3 and Lemma 3.4. While, to prove the second part of (2) it is enough to note that $0 \in \mathbb{R}_+$, and thus $-\mathbb{R}_+ \subset \mathbb{R}_+ - \mathbb{R}_+$ and $\mathbb{R}_+ \subset \mathbb{R}_+ - \mathbb{R}_+$. Therefore, $\mathbb{R} = \mathbb{R}_+ \cup (-\mathbb{R}_+) \subset \mathbb{R}_+ - \mathbb{R}_+$ also holds.

Theorem 8.5. *For any* $x, y \in \mathbb{R}$, *the following assertions are equivalent:*

(1) $x \leq y$*;* *(2)* $-x + y \in \mathbb{R}_+$*;* *(3)* $y \in x + \mathbb{R}_+$.

Proof. Namely, we evidently have

$$x \leq y \iff 0 \leq -x + y \iff -x + y \in \mathbb{R}_+ \iff y \in x + \mathbb{R}_+\,.$$

Remark 8.6. Therefore, by defining a relation Θ on $\mathbb{R}$ such that

$$\Theta(t) = t + \mathbb{R}_+$$

for all $t \in \mathbb{R}$, we can state that Θ is the usual inequality relation on $\mathbb{R}$. Moreover, it is also worth noticing that by [54 , Theorem 3.2], Θ is the unique translation relation on $\mathbb{R}$ such that $\Theta(0) = \mathbb{R}_+$.

The importance of translation relations lies mainly in the fact that each vector topology can be derived from a family of translation relations by [54]. Moreover, the multiplicative forms of translation functions can be used to extended various algebraic structures by [52] and [56] and the references therein.

Now, in addition to Definition 8.1, we may also naturally introduce the following

Definition 8.7. Define a relation Φ on $\mathbb{R}_+$ to $\mathbb{R}$ such that

$$\Phi(x) = \varphi(x) + \mathbb{R}_+$$

for all $x \in \mathbb{R}_+$.

Remark 8.8. Thus, for all $x \in \mathbb{R}_+$, we also have

$$\Phi(x) = x^2 + [\,0\,,\, +\infty\,[= [\,x^2,\, +\infty\,[\,.$$

Therefore, Φ corresponds to the set-valued function of Zs. Páles mentioned earlier.

Moreover, it also worth noticing that

$$\Phi(x) = \varphi(x) + \mathbb{R}_+ = \Theta\big(\varphi(x)\big) = \big(\Theta \circ \varphi\big)(x)$$

for all $x \in \mathbb{R}_+$. Therefore, $\Phi = \Theta \circ \varphi$.

The relation Φ can also be easily seen to have the following useful properties.

Theorem 8.9.

(1) Φ *is decreasing and convex ;*

(2) Φ *is closed and convex valued ;*

(3) Φ *is zero-additive and subadditive ;*

(4) $\mathbb{R} = \Phi(x) - \Phi(y)$ *for all* $x, y \in \mathbb{R}_+$ *;*

(5) Φ *is* $[\,0,\, 1\,]$*–superhomogeneous and* $[\,1,\, +\infty\,[$*–subhomogeneous .*

Hint. If $x, y \in \mathbb{R}_+$ such that $x \leq y$, then by (1) in Theorem 8.2 we have $\varphi(x) \leq \varphi(y)$. Hence, by Theorem 8.5, it follows that $\varphi(y) \in \varphi(x) + \mathbb{R}_+$. Now, by using Definition 8.7 and (2) in Theorem 8.4, we can see that

$$\Phi(y) = \varphi(y) + \mathbb{R}_+ \subset \varphi(x) + \mathbb{R}_+ + \mathbb{R}_+ = \varphi(x) + \mathbb{R}_+ = \Phi(x)\,.$$

Therefore, Φ is decreasing.

If $x, y \in \mathbb{R}_+$, then by (2) in Theorem 8.2 we have $\varphi(x) + \varphi(y) \le \varphi(x + y)$. Hence, by Theorem 8.5, it follows that $\varphi(x + y) \in \varphi(x) + \varphi(y) + \mathbb{R}_+$. Now, by using Definition 8.7, we can see that

$$\begin{aligned}\Phi(x + y) = \varphi(x + y) + \mathbb{R}_+ \subset \varphi(x) + \varphi(y) + \mathbb{R}_+ + \mathbb{R}_+ \\ = \varphi(x) + \mathbb{R}_+ + \varphi(y) + \mathbb{R}_+ = \Phi(x) + \Phi(y).\end{aligned}$$

Therefore, Φ is subadditive.

If $x, y \in \mathbb{R}_+$ and $0 < \lambda < 1$, then by (1) in Theorem 8.2 we have

$$\varphi\big(\lambda x + (1 - \lambda) y\big) \le \lambda \varphi(x) + (1 - \lambda) \varphi(y).$$

Hence, by Theorem 8.5, it follows that

$$\lambda \varphi(x) + (1 - \lambda) \varphi(y) \in \varphi\big(\lambda x + (1 - \lambda) y\big) + \mathbb{R}_+.$$

Now, by using Definition 8.7 and (3) and (2) in Theorem 8.4, we can see that

$$\begin{aligned}\lambda \Phi(x) + (1 - \lambda) \Phi(y) = \lambda \big(\varphi(x) + \mathbb{R}_+\big) + (1 - \lambda) \big(\varphi(y) + \mathbb{R}_+\big) = \\ \lambda \varphi(x) + \lambda \mathbb{R}_+ + (1 - \lambda)\varphi(y) + (1 - \lambda) \mathbb{R}_+ = \lambda \varphi(x) + \mathbb{R}_+ + (1 - \lambda)\varphi(y) + \mathbb{R}_+ \\ = \lambda \varphi(x) + (1 - \lambda)\varphi(y) + \mathbb{R}_+ + \mathbb{R}_+ \subset \varphi\big(\lambda x + (1 - \lambda) y\big) + \mathbb{R}_+ + \mathbb{R}_+ + \mathbb{R}_+ \\ = \varphi\big(\lambda x + (1 - \lambda) y\big) + \mathbb{R}_+ = \Phi\big(\lambda x + (1 - \lambda) y\big).\end{aligned}$$

Therefore, Φ is convex.

Remark 8.10. The above theorem can be proved more directly by using the results of Section 3 instead of Theorems 8.4 and 8.5.

However, the above arguments can also be well used in the case when $\mathbb{R}_+$ and φ are replaced by some more general objects.

9. Odd and superhomogeneous partial selection relations of Φ

Remark 9.1. In the sequel, to apply the transformations $\triangle$ and $\star$ to Φ, we shall consider Φ as a relation on $\mathbb{R}$.

Thus, by the corresponding definitions, for any $x \in \mathbb{R}$ we have

$$\Phi(x) = \begin{cases} \emptyset & \text{if} \quad x < 0, \\ [\, x^2, +\infty\, [& \text{if} \quad x \ge 0. \end{cases}$$

Theorem 9.2. *We have*

$$\Phi^{\triangle} = \{(0, 0)\}.$$

Proof. By Remark 9.1, for any $x \in \mathbb{R}$, we have

$$\Phi(-x) = \begin{cases} \emptyset & \text{if} \quad x > 0, \\ [\, x^2, +\infty\, [& \text{if} \quad x \le 0. \end{cases}$$

Hence, by Theorem 3.3, it is clear that

$$\Phi^{\wedge}(x) = -\Phi(-x) = \begin{cases} \emptyset & \text{if} \quad x > 0, \\ ]-\infty, -x^2] & \text{if} \quad x \le 0. \end{cases}$$

Now, by the corresponding definitions, we can also easily see that

$$\Phi^{\triangle}(x) = \big(\Phi \cap \Phi^{\wedge}\big)(x) = \Phi(x) \cap \Phi^{\wedge}(x) = \begin{cases} \emptyset & \text{if} \quad x \ne 0, \\ \{0\} & \text{if} \quad x = 0. \end{cases}$$

Therefore, the required equality is also true.

Theorem 9.3. *For a relation Ω on $\mathbb{R}$, the following assertions are equivalent:*

(1) $\Omega = \emptyset$ *or* $\Omega = \{(0,\,0)\}$ *;*

(2) Ω *is an odd partial selection relation of* Φ *.*

Hint. If (2) holds, then by Corollary 5.10 and Theorem 9.2 we have

$$\Omega \subset \Phi^{\Delta} = \{(0,\,0)\}\,.$$

Therefore, either $\Omega = \emptyset$ or $\Omega = \{(0,\,0)\}$. Thus, (1) also holds.

Theorem 9.4. *We have*

$$\Phi^{\star} = \{0\} \times \mathbb{R}_+\,.$$

Proof. If $x \in \mathbb{R}$, then by Remark 9.1, for any $n \in \mathbb{N}$, we have

$$\Phi(\,n\,x\,) = \begin{cases} \emptyset & \text{if} \quad x < 0, \\ [\,n^2x^2,\ +\infty\,[& \text{if} \quad x \geq 0\,. \end{cases}$$

Hence, by Theorem 3.3, it is clear that

$$\Phi_n(x) = n^{-1}\,\Phi(\,n\,x\,) = \begin{cases} \emptyset & \text{if} \quad x < 0, \\ [\,n\,x^2,\ +\infty\,[& \text{if} \quad x \geq 0\,. \end{cases}$$

Now, by the corresponding definitions, we can also easily see that

$$(\Phi^{\star})(x) = \Big(\bigcap_{n=1}^{\infty} \Phi_n\Big)(x) = \bigcap_{n=1}^{\infty} \Phi_n(x) = \begin{cases} \emptyset & \text{if} \quad x \neq 0, \\ [\,0,\,+\infty\,[& \text{if} \quad x = 0\,. \end{cases}$$

Therefore, the required equality is also true.

Theorem 9.5. *For a relation Ω on $\mathbb{R}$, the following assertions are equivalent:*

(1) Ω *is an* $\mathbb{N}$*–superhomogeneous partial selection relation of* Φ *;*

(2) $\Omega = \{0\} \times A$ *for some* $\mathbb{N}$*–superhomogeneous subset* A *of* $\mathbb{R}_+$ *.*

Hint. If (1) holds, then by Corollary 7.11 and Theorem 9.4 we have

$$\Omega \subset \Phi^{\star} = \{0\} \times \mathbb{R}_+\,.$$

Therefore, $\Omega = \{0\} \times A$ with $A = \Omega(0) \subset \mathbb{R}_+$. Moreover, we can also see that

$$n\,A = n\,\Omega(0) \subset \Omega\,(n\,0) = \Omega(0) = A$$

for all $n \in \mathbb{N}$. Therefore, A is $\mathbb{N}$–superhomogeneous. Thus, (2) also holds.

Theorem 9.6. *For a relation Ω on $\mathbb{R}$, the following assertions are equivalent:*

(2) Ω *is a superadditive partial selection relation of* Φ *;*

(1) $\Omega = \{0\} \times A$ *for some superadditive subset* A *of* $\mathbb{R}_+$ *.*

Hint. If (2) holds, then by Theorem 4.5 Ω is $\mathbb{N}$–superhomogeneous. Thus, by Theorem 9.5, we have $\Omega = \{0\} \times A$ for some subset A of $\mathbb{R}_+$. Moreover, we can also see that

$$A + A = \Omega(0) + \Omega(0) \subset \Omega(0) = A\,.$$

Therefore, A is superadditive. Thus, (1) also holds.

Remark 9.7. If $x \in \mathbb{R}_+$, then by the corresponding definitions we also have

$$\varphi_n(x) = n^{-1}\,\varphi(\,n\,x\,) = n\,x^2$$

for all $n \in \mathbb{N}$, and thus

$$\lim_{n\to\infty} \varphi_n(x) = \begin{cases} 0 & \text{if} \quad x = 0, \\ +\infty & \text{if} \quad x > 0\,. \end{cases}$$

Therefore, φ is also rather irregular in the sense of $[\,22\,,$ Definition 3.1$]$.

10. Some basic properties of the global negative $\Phi^{\wedge}$ of Φ

Remark 10.1. By defining $\mathbb{R}_- =]-\infty\,,\,0\,]$, for any $x \in \mathbb{R}_-$, we have

$$\Phi^{\wedge}(x) = -\Phi\,(-x\,) = -\big(\,\varphi(-x\,) + \mathbb{R}_+\big) = -\varphi(-x\,) - \mathbb{R}_+ = \varphi^{\wedge}(x) + \mathbb{R}_-\,.$$

However, the basic properties of $\Phi^{\wedge}$ can be more easily derived from those of Φ.

For instance, by considering $\Phi^{\wedge}$ as a relation on $\mathbb{R}_-$, from Theorem 8.9 we can immediately get the following

Theorem 10.2.

(1) $\Phi^{\wedge}$ *is increasing and convex;*

(2) $\Phi^{\wedge}$ *is closed and convex valued;*

(3) $\Phi^{\wedge}$ *is zero-additive and subadditive;*

(4) $\mathbb{R} = \Phi^{\wedge}(x) - \Phi^{\wedge}(y)$ *for all* $x,\, y \in \mathbb{R}_-$*;*

(5) $\Phi^{\wedge}$ *is* $[\,0,\,1\,]$*–superhomogeneous and* $[\,1,\,+\infty\,[$*–subhomogeneous.*

Hint. If $x,\, y \in \mathbb{R}_-$ such that $x \leq y$, then $-x, -y \in \mathbb{R}_+$ such that $-y \leq -x$. Therefore, by (1) in Theorem 8.9, we have $\Phi(-x\,) \subset \Phi(-y\,)$. Hence, it is clear $\Phi^{\wedge}(x) = -\Phi(-x\,) \subset -\Phi(-y\,) = \Phi^{\wedge}(y)$. Therefore, $\Phi^{\wedge}$ is increasing.

If $x,\, y \in \mathbb{R}_-$ and $0 \leq \lambda \leq 1$, then again by (1) in Theorem 8.9 we have

$$\lambda\,\Phi\,(-x\,) + (\,1-\lambda\,)\Phi\,(-y\,) \subset \Phi\big(\,\lambda(-x\,) + (\,1-\lambda\,)(-y\,)\big) = \Phi\,\big(-\big(\,\lambda\,x + (\,1-\lambda\,)\,y\,\big)\big)\,.$$

Hence, it is clear that

$$\begin{aligned} \lambda\,\Phi^{\wedge}(x) + (\,1-\lambda\,)\Phi^{\wedge}(y) &= \lambda\,\big(-\Phi\,(-x\,)\big) + (\,1-\lambda\,)\big(\,-\Phi\,(-y\,)\big) \\ &= -\big(\,\lambda\,\Phi\,(-x\,) + (\,1-\lambda\,)\,\Phi\,(-y\,)\big) \subset -\Phi\,\big(-\big(\,\lambda\,x + (\,1-\lambda\,)\,y\,\big)\big) \\ &= \Phi^{\wedge}\big(\,\lambda\,x + (\,1-\lambda\,)\,y\,\big)\,. \end{aligned}$$

Therefore, $\Phi^{\wedge}$ is also convex.

In the following theorem, we shall again consider $\Phi^{\wedge}$ as a relation on $\mathbb{R}$.

Theorem 10.3. *We have*

(1) $\Phi^{\wedge \Delta} = \{(0, 0)\}$ *;* *(2)* $\Phi^{\wedge \star} = \{0\} \times \mathbb{R}_-$ *.*

Proof. By Theorems 5.4 and 9.2, we have $\Phi^{\wedge \Delta} = \Phi^{\wedge} = \{(0, 0)\}$. Moreover, by Theorems 7.6 and 9.4, we have

$$\Phi^{\wedge \star} = \Phi^{\star \wedge} = \big(\{0\} \times \mathbb{R}_+ \big)^{\wedge} = \{0\} \times (-\mathbb{R}_+) = \{0\} \times \mathbb{R}_- \,.$$

Now, analogously to Theorems 9.3, 9.5 and 9.6, we can also easily establish the following theorems.

Theorem 10.4. *For a relation* Ω *on* $\mathbb{R}$ *, the following assertions are equivalent :*

(1) $\Omega = \emptyset$ *or* $\Omega = \{(0, 0)\}$ *;*

(2) Ω *is an odd partial selection relation of* $\Phi^{\wedge}$ *.*

Theorem 10.5. *For a relation* Ω *on* $\mathbb{R}$ *, the following assertions are equivalent :*

(1) Ω *is an* $\mathbb{N}$*–superhomogeneous partial selection relation of* $\Phi^{\wedge}$ *;*

(2) $\Omega = \{0\} \times A$ *for some* $\mathbb{N}$*–superhomogeneous subset* A *of* $\mathbb{R}_-$ *.*

Theorem 10.6. *For a relation* Ω *on* $\mathbb{R}$ *, the following assertions are equivalent :*

(1) Ω *is a superadditive partial selection relation of* $\Phi^{\wedge}$ *;*

(2) $\Omega = \{0\} \times A$ *for some superadditive subset* A *of* $\mathbb{R}_-$ *.*

Remark 10.7. The latter three theorems can also be easily derived from Theorems 9.3, 9.5 and 9.6.

For instance, if (1) in Theorem 10.6 holds, then $\Omega^{\wedge} \subset \Phi^{\wedge\wedge} = \Phi$. Moreover, $\Omega^{\wedge}$ is also superadditive. Thus, by Theorem 9.6, $\Omega^{\wedge} = \{0\} \times B$ for some superadditive subset B of $\mathbb{R}_+$. Hence, by noticing that $\Omega = \Omega^{\wedge\wedge} = \big(\{0\} \times B \big)^{\wedge} = \{0\} \times (-B)$ and $-B$ is a superadditive subset of $\mathbb{R}_-$, we can see that (2) in Theorem 10.6 also holds.

11. An almost odd extension of Φ to $\mathbb{R}$

Definition 11.1. Define $\mathbb{R}_-^* = \mathbb{R}_- \setminus \{0\}$, and

$$\Psi = \Phi^{\wedge} \,|\, \mathbb{R}_-^* \qquad \text{and} \qquad F = \Phi \cup \Psi \,.$$

Remark 11.2. Thus, for any $x \in \mathbb{R}$, we have

$$F(x) = \big(\Phi \cup \Psi \big)(x) = \Phi(x) \cup \Psi(x) = \begin{cases} \Phi(x) & \text{if} \quad x \geq 0 \,, \\ \Psi(x) & \text{if} \quad x < 0 \,. \end{cases}$$

Therefore, F is an extension of both Φ and Ψ. Moreover, by the corresponding definitions, we also have

$$F(x) = \begin{cases} [\, x^2, +\infty \,[& \text{if} \quad x \geq 0 \,, \\ ]-\infty, -x^2] & \text{if} \quad x < 0 \,. \end{cases}$$

By using the corresponding properties of Φ and $\Phi^{\wedge}$, we can also easily prove the following

Theorem 11.3.

(1) F *is subadditive,*

(2) F *is closed and convex valued;*

(3) $F(-x) = -F(x)$ *for all* $x \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}$*;*

(4) F *is* $[0, 1]$*–superhomogeneous and* $[1, +\infty[$*–subhomogeneous;*

Hint. To prove (1), note that if for instance $x, y \in \mathbb{R}$ such that $x, y < 0$, then by Remark 11.2 and (3) in Theorem 10.2 we have

$$F(x+y) = \Phi^\wedge(x+y) \subset \Phi^\wedge(x) + \Phi^\wedge(y) = F(x) + F(y).$$

While, if for instance $x, y \in \mathbb{R}$ such that $x \geq 0$ and $y < 0$, then by Remark 11.2 and (4) in Theorem 8.9 we have

$$F(x) + F(y) = \Phi(x) + \Phi^\wedge(y) = \Phi(x) - \Phi(-y) = \mathbb{R}.$$

Therefore, $F(x+y) \subset F(x) + F(y)$ trivially holds.

Concerning the relation F, we can also easily prove the following

Theorem 11.4. *We have*

$$F^\Delta = \{(0, 0)\} \cup \big(F \,|\, \mathbb{R}^*\big).$$

Proof. By (3) in Theorem 11.3, we have

$$F^\wedge(x) = -F(-x) = F(x)$$

for all $x \in \mathbb{R}$ with $x \neq 0$. Moreover, since F is an extension of Φ, we have

$$F^\wedge(0) = -F(0) = -\Phi(0) = -\mathbb{R}_+ = \mathbb{R}_-.$$

Hence, by the corresponding definitions, it is clear that

$$F^\Delta(x) = \big(F \cap F^\wedge\big)(x) = F(x) \cap F^\wedge(x) = \begin{cases} \{0\} & \text{if} \quad x = 0, \\ F(x) & \text{if} \quad x \neq 0. \end{cases}$$

Therefore, the required equality is also true.

Now, in contrast to Theorem 9.3, we can only prove the following

Theorem 11.5. *For a relation* Ω *on* $\mathbb{R}$*, the following assertions are equivalent:*

(1) Ω *is an odd partial selection relation of* F*;*

(2) $\Omega = \Lambda \cup \Lambda^\wedge$ *for some partial selection relation* Λ *of* $\{(0, 0)\} \cup \big(\Phi \,|\, \mathbb{R}^*\big)$.

Hint. If (1) holds, then by Corollary 5.10 and Theorem 11.4, we have

$$\Omega \subset F^\Delta = \{(0, 0)\} \cup \big(F \,|\, \mathbb{R}^*\big).$$

Hence, since $\Phi = F \,|\, \mathbb{R}_+$, it is clear that

$$\Lambda = \Omega \,|\, \mathbb{R}_+ \subset \{(0, 0)\} \cup \big(F \,|\, \mathbb{R}_+^*\big) = \{(0, 0)\} \cup \big(\Phi \,|\, \mathbb{R}_+^*\big).$$

Thus, Λ is a partial selection relation of $\{(0, 0)\} \cup \big(\Phi \,|\, \mathbb{R}^*\big)$. Moreover, if $x \in \mathbb{R}_-$, then since $-x \in \mathbb{R}_+$ and Ω is odd we can easily see that

$$\Lambda^\wedge(x) = -\Lambda(-x) = -\big(\Omega \,|\, \mathbb{R}_+\big)(-x) = -\Omega(-x) = \Omega(x) = \big(\Omega \,|\, \mathbb{R}_-\big)(x).$$

Hence, it is clear that

$$\Omega = \big(\Omega \,|\, \mathbb{R}_+\big) \cup \big(\Omega \,|\, \mathbb{R}_-\big) = \Lambda \cup \Lambda^\wedge,$$

and thus (2) also holds.

In addition to Theorem 11.4, we can also easily prove the following

Theorem 11.6. *We have*

$$F^\star = \{0\} \times \mathbb{R}_+ \, .$$

Proof. If $x \in \mathbb{R}$, then by Remark 11.2, for any $n \in \mathbb{N}$, we have

$$F(n\,x) = \begin{cases} [\,n^2 x^2\,,\ +\infty\,[& \text{if} \quad x \geq 0\,, \\ ]-\infty\,,\ -n^2 x^2] & \text{if} \quad x < 0\,. \end{cases}$$

Hence, by Theorem 3.3, it is clear that

$$F_n(x) = n^{-1} F(n\,x) = \begin{cases} [\,n\,x^2\,,\ +\infty\,[& \text{if} \quad x \geq 0\,, \\ ]-\infty\,,\ -n\,x^2] & \text{if} \quad x < 0\,. \end{cases}$$

Now, by the corresponding definitions, we can also easily see that

$$F^\star(x) = \Big(\bigcap_{n=1}^{\infty} F_n \Big)(x) = \bigcap_{n=1}^{\infty} F_n(x) = \begin{cases} \emptyset & \text{if} \quad x \neq 0, \\ [\,0,\ +\infty\,[& \text{if} \quad x = 0\,. \end{cases}$$

Therefore, the required equality is also true.

Now, analogously to Theorems 9.5 and 9.6, we can also easily establish the following two theorems.

Theorem 11.7. *For a relation Ω on $\mathbb{R}$, the following assertions are equivalent:*

(1) Ω *is an $\mathbb{N}$–superhomogeneous partial selection relation of F;*

(2) $\Omega = \{0\} \times A$ *for some $\mathbb{N}$–superhomogeneous subset A of $\mathbb{R}_+$.*

Theorem 11.8. *For a relation Ω on $\mathbb{R}$, the following assertions are equivalent:*

(1) Ω *is a superadditive partial selection relation of F;*

(2) $\Omega = \{0\} \times A$ *for some superadditive subset A of $\mathbb{R}_+$.*

12. Another natural extension of Φ to $\mathbb{R}$

Because of the results of [61], we may also naturally introduce the following

Definition 12.1. Define

$$\Gamma = \mathbb{R}^*_- \times \mathbb{R} \qquad \text{and} \qquad G = \Phi \cup \Gamma\,.$$

Remark 12.2. Thus, for any $x \in \mathbb{R}$, we have

$$G(x) = \big(\,\Phi \cup \Gamma\,\big)(x) = \Phi(x) \cup \Gamma(x) = \begin{cases} \Phi(x) & \text{if} \quad x \geq 0\,, \\ \Gamma(x) & \text{if} \quad x < 0\,. \end{cases}$$

Therefore, G is an extension of both Φ and Γ. Moreover, by the corresponding definitions, we also have

$$G(x) = \begin{cases} \mathbb{R} & \text{if} \quad x < 0\,, \\ [\,x^2,\ +\infty\,[& \text{if} \quad x \geq 0\,. \end{cases}$$

Now, analogously to the the results of Section 11, we can also easily prove the following theorems.

Theorem 12.3.

(1) G *is closed and convex valued;*

(2) G *is subadditive and zero-additive;*

(3) G *is* $[\,0,\,1\,]$*–superhomogeneous and* $[\,1,\,+\infty\,[$*–subhomogeneous;*

Hint. To prove (2), note that if for instance $x,\,y\in\mathbb{R}$ such that $x<0$, then by Remark 12.2 we have

$$G(x)+G(y)=\mathbb{R}+G(y)=\mathbb{R}\,.$$

Therefore, $G\,(\,x+y\,)\subset G(x)+G(y)$ trivially holds. Moreover, we also have $G(x)+G(0)=\mathbb{R}=G(x)$.

Remark 12.4. Note that convexity of G on $\mathbb{R}^*_-$ is an immediate consequence of the $\mathbb{R}^*_+$–linearity of G on $\mathbb{R}^*_-$.

Theorem 12.5. *We have*

$$G^{\triangle}=\{(0,\,0)\}\cup\big(F\,|\,\mathbb{R}^*\big)\,.$$

Proof. By Remark 12.2, for any $x\in\mathbb{R}$, we have

$$G(-x\,)=\begin{cases}\quad\mathbb{R} & \text{if}\quad x>0\,,\\ [\,x^2,\,+\infty\,[& \text{if}\quad x\le 0\,.\end{cases}$$

Hence, by Theorem 3.3, it is clear that

$$G^{\wedge}(x)=-G(-x\,)=\begin{cases}\quad\mathbb{R} & \text{if}\quad x>0\,,\\ ]-\infty\,,\,-x^2] & \text{if}\quad x\le 0\,.\end{cases}$$

Now, by the corresponding definitions, we can also easily see that

$$G^{\triangle}(x)=\big(\,G\cap G^{\wedge}\big)(x)=G(x)\cap G^{\wedge}(x)=\begin{cases}\quad\{0\} & \text{if}\quad x=0\,,\\ [\,x^2,\;+\infty\,[& \text{if}\quad x>0\,,\\ ]-\infty\,,\;-x^2\,] & \text{if}\quad x<0\,.\end{cases}$$

Hence, by Remark 11.2, it is clear that the required equality is also true

Theorem 12.6. *For a relation* Ω *on* $\mathbb{R}$*, the following assertions are equivalent:*

(1) Ω *is an odd partial selection relation of* G*;*

(2) $\Omega=\Lambda\cup\Lambda^{\wedge}$ *for some partial selection relation* Λ *of* $\{(0,\,0)\}\cup\big(\,\Phi\,|\,\mathbb{R}^*\big)$.

Theorem 12.7. *We have*

$$G^\star = \Gamma \cup \big(\{0\} \times \mathbb{R}_+\big)\,.$$

Proof. If $x \in \mathbb{R}\,,$ then by Remark 12.2 for any $n \in \mathbb{N}\,,$ we have

$$G(\,n\,x\,) = \begin{cases} \quad\mathbb{R} & \text{if} \quad x < 0\,, \\ [\,n^2 x^2\,,\ +\infty\,[& \text{if} \quad x \geq 0\,. \end{cases}$$

Hence, by Theorem 3.3, it is clear that

$$G_n(x) = n^{-1} G(\,n\,x\,) = \begin{cases} \quad\mathbb{R} & \text{if} \quad x < 0\,, \\ [\,n\,x^2\,,\ +\infty\,[& \text{if} \quad x \geq 0\,. \end{cases}$$

Now, by the corresponding definitions, we can also easily see that

$$G^\star(x) = \Big(\bigcap_{n=1}^{\infty} G_n\Big)(x) = \bigcap_{n=1}^{\infty} G_n(x) = \begin{cases} \emptyset & \text{if} \quad x > 0\,, \\ \mathbb{R} & \text{if} \quad x < 0\,, \\ \mathbb{R}_+ & \text{if} \quad x = 0\,. \end{cases}$$

Therefore, by the corresponding definitions, the required equality is also true.

Now, we can also easily prove the following two theorems.

Theorem 12.8. *For a relation Ω on $\mathbb{R}\,,$ the following assertions are equivalent:*

(1) Ω is an $\mathbb{N}$–superhomogeneous partial selection relation of $G\,;$

(2) Ω is an $\mathbb{N}$–superhomogeneous relation on $\mathbb{R}_-$ to $\mathbb{R}$ such that $\Omega(0) \subset \mathbb{R}_+\,.$

Theorem 12.9. *For a relation Ω on $\mathbb{R}\,,$ the following assertions are equivalent:*

(1) Ω is a superadditive partial selection relation of $G\,;$

(2) Ω is a superadditive relation on $\mathbb{R}_-$ to $\mathbb{R}$ such that $G(0) \subset \mathbb{R}_+\,.$

13. Further natural extensions of Φ to $\mathbb{R}$

Suppose now that H is a subadditive relation of $\mathbb{R}$ to itself such that H is an extension of $\Phi\,.$ Moreover, define

$$p(x) = \inf\big(H\,(x)\big) \qquad \text{and} \qquad q(x) = \sup\big(H\,(x)\big)$$

for all $x \in \mathbb{R}\,.$

Then, since $H\,(x) \neq \emptyset$ for all $x \in \mathbb{R}\,,$ it is clear that p and q are functions of $\mathbb{R}$ to $\mathbb{R} \cup \{-\infty\}$ and $\mathbb{R} \cup \{+\infty\}\,,$ respectively. Moreover, we evidently have

$$H\,(x) \subset \,[\,p(x)\,,\ q(x)\,]$$

for all $x \in \mathbb{R}$.

Furthermore, since $H(u) = \Phi(u) = [\,\varphi(u),\, +\infty\,[$ for all $u \in \mathbb{R}_+$, we can also at once state that

$$p(u) = \varphi(u) \qquad \text{and} \qquad q(u) = +\infty$$

for all $u \in \mathbb{R}_+$.

On the other hand, by using that $H(x+y) \subset H(x) + H(y)$ for all $x, y \in \mathbb{R}$, we can also easily see that

$$\begin{aligned} p(x) + p(y) &= \inf\big(H(x)\big) + \inf\big(H(y)\big) \\ &= \inf\big(H(x) + H(y)\big) \leq \inf\big(H(x+y)\big) = p(x+y) \end{aligned}$$

and

$$\begin{aligned} q(x+y) &= \sup\big(H(x+y)\big) \leq \sup\big(H(x) + F(y)\big) \\ &= \sup\big(H(x)\big) + \sup\big(H(y)\big) = q(x) + q(y) \end{aligned}$$

for all $x, y \in \mathbb{R}$. Therefore, p is superadditive and q is subadditive.

Now, if $u, v \in \mathbb{R}_+$, then we can already see that

$$\varphi(u+v) + p(-v) = p(u+v) + p(-v) \leq p(u) = \varphi(u),$$

and thus

$$p(-v) \leq -\big(\varphi(u+v) - \varphi(u)\big) = -\big((u+v)^2 - u^2\big) = -(2u+v)\,v.$$

Hence, if $v \neq 0$, then by letting $u \to +\infty$ we can already infer that

$$p(-v) \leq -\infty, \qquad \text{and thus} \qquad p(-v) = -\infty.$$

Therefore, for any $x \in \mathbb{R}$, we have

$$p(x) = \begin{cases} x^2 & \text{if} \quad x \geq 0, \\ -\infty & \text{if} \quad x < 0. \end{cases}$$

It can also be easily seen that

$$p(-v) = \inf_{u \in \mathbb{R}_+} \big(\varphi(u) - \varphi(u+v)\big)$$

for all $v \in \mathbb{R}_+^*$. Therefore, according to Barton and Laatsch [6], p is just the maximal superadditive extension of φ to $\mathbb{R}$.

Unfortunately, concerning the function q we cannot prove a similar statement. Namely, if ψ is a subadditive function of $\mathbb{R}_-^*$ to $\mathbb{R} \cup \{+\infty\}$ and

$$\rho(x) = \begin{cases} +\infty & \text{if} \quad x \geq 0, \\ \psi(x) & \text{if} \quad x < 0, \end{cases}$$

then it can be easily seen that ρ is a subadditive.

However, if in addition to the subadditivity of H, we assume that H is closed and convex valued, then we can note that

$$H(x) = \begin{cases} \mathbb{R} & \text{if} \quad x < 0 \quad \text{and} \quad q(x) = +\infty, \\ ]-\infty,\, q(x)] & \text{if} \quad x < 0 \quad \text{and} \quad q(x) \neq +\infty. \end{cases}$$

Hence, it is clear that the implication (1) $\Longrightarrow$ (2) is true in the following

Theorem 13.1. *For any relation H of $\mathbb{R}$ to itself such that H is an extension of Φ, the following assertions are equivalent:*

(1) H is subadditive and closed and convex valued;

(2) there exists a subadditive function ψ of $\mathbb{R}^*_-$ to $\mathbb{R} \cup \{+\infty\}$ such that

$$H(x) = \begin{cases} \mathbb{R} & \text{if} \quad x < 0 \quad \text{and} \quad \psi(x) = +\infty, \\ ]-\infty, \ \psi(x)] & \text{if} \quad x < 0 \quad \text{and} \quad \psi(x) \neq +\infty. \end{cases}$$

Hint. To check the subadditivity of H, note that if for instance $x, y \in \mathbb{R}$ such that $x, y < 0$ and $\psi(x)$, $\psi(y) < +\infty$, then $\psi(x+y) \le \psi(x) + \psi(y) < +\infty$. Therefore, by a dual of Theorem 3.5, we have

$$\begin{aligned} H(x+y) =]-\infty, \ \psi(x+y)] \subset]-\infty, \ \psi(x) + \psi(y)] \\ =]-\infty, \ \psi(x)] +]-\infty, \ \psi(y)] = H(x) + H(y). \end{aligned}$$

On the other hand, if for instance $x, y \in \mathbb{R}$ such $x < 0$, $0 \le y$ and $\psi(x) < +\infty$, then

$$H(x) + H(y) =]-\infty, \ \psi(x)] + [\varphi(y), \ +\infty[= \mathbb{R}.$$

Moreover, if for instance $x \in \mathbb{R}$ such that $x < 0$ and $\psi(x) = +\infty$, then for any $y \in \mathbb{R}$ we have

$$H(x) + H(y) = \mathbb{R} + H(y) = \mathbb{R}.$$

Therefore, the inclusion $H(x+y) \subset H(x) + H(y)$ trivially holds.

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