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SET THEORETIC OPERATIONS ON BOX AND TOTALIZATION RELATIONS

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Abstract. In this paper, we shall only study the most simple set theoretic operations on box and totalization relations

$$\Gamma_{(A\,,\,B\,)} = A \times B \qquad \text{and} \qquad \tilde{F} = F \cup \Gamma_{(\,D_F^{\,c}\,,\,Y\,)}\,,$$

where $A \subset X$, $B \subset Y$ and F is a relation on X to Y with domain D_F. The relation theoretic operations, and several algebraic and topological properties of these relations, will be studied elsewhere.

This line of investigations is mainly motivated by the fact that the relations

$$\tilde{\Gamma}_{(A\,,\,B\,)} = \widetilde{\Gamma_{(A\,,\,B\,)}} \qquad \text{and} \qquad \tilde{\Gamma}_A = \tilde{\Gamma}_{(A\,,\,A\,)}$$

play an important role in the uniformization of various topological structures such as proximities, closures and topologies, for instance. Moreover, the relations $\tilde{F}$ can be used to prove a useful reduction theorem for the intersection convolution of relations. The latter operation allows of a natural treatment of the Hahn-Banach type extension theorems.

1. A few basic facts on relations

A subset F of a product set $X{\times}Y$ is called a relation on X to Y. If in particular F is a relation on X to itself, then we may simply say that F is a relation on X. Thus, a relation F on X to Y is also a relation on $X \cup Y$.

If F is a relation on X to Y, then for any $x \in X$ and $A \subset X$ the sets $F(x) = \{\, y \in Y : \ (x,\,y) \in F \}$ and $F\,[\,A\,] = \bigcup_{a \in A} F(a)$ are called the images of x and A under F, respectively.

Moreover, the sets $D_F = \{\, x \in X : \ F(x) \neq \emptyset \,\}$ and $R_F = F\,[\,X\,]$ are called the domain and range of F, respectively. If in particular $X = D_F$, then we say that F is a relation of X to Y, or that F is a total relation on X to Y. While, if $Y = R_F$, then we say that F is a relation on X onto Y.

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If F is a relation on X to Y and $U \subset D_F$, then the relation $F \,|\, U = F \cap (U \times Y)$ is called the restriction of F to U. Moreover, if F and G are relations on X to Y such that $D_F \subset D_G$ and $F = G \,|\, D_F$, then G is called an extension of F.

In particular, a relation f on X to Y is called a function if for each $x \in D_f$ there exists $y \in Y$ such that $f(x) = \{y\}$. In this case, by identifying singletons with their elements, we may simply write $f(x) = y$ instead of $f(x) = \{y\}$.

Concerning relations, we shall only quote here the following basic theorems from [23].

Theorem 1.1. *If F is a relation on X to Y, then*

$$F = \bigcup_{x \in X} \{x\} \times F(x) = \bigcup_{x \in D_F} \{x\} \times F(x) .$$

Remark 1.2. By this theorem, a relation F on X to Y can be naturally defined by specifying $F(x)$ for all $x \in X$, or by specifying D_F and $F(x)$ for all $x \in D_F$.

Corollary 1.3. *If F and G are relations on X to Y, then the following assertions are equivalent:*

(1) $F \subset G$;

(2) $F(x) \subset G(x)$ *for all* $x \in X$; *(3)* $F(x) \subset G(x)$ *for all* $x \in D_F$.

Corollary 1.4. *If F and G are relations on X to Y, then the following assertions are equivalent:*

(1) $F = G$;

(2) $F(x) = G(x)$ *for all* $x \in X$;

(3) $D_F = D_G$ *and* $F(x) = G(x)$ *for all* $x \in D_F$.

Theorem 1.5. *If F is a relation on X to Y, then for any $A, B \subset X$ we have*

(1) $F[A \cap B] \subset F[A] \cap F[B]$; *(2)* $F[A \cup B] = F[A] \cup F[B]$.

Theorem 1.6. *If F is a relation on X to Y, then for any $A, B \subset X$ we have*

$$F[A] \setminus F[B] \subset F[A \setminus B] .$$

Corollary 1.7. *If F is a relation on X onto Y, then for any $A \subset X$ we have*

$$F[A]^c \subset F[A^c] .$$

Remark 1.8. If in particular the inverse $F^{-1} = \big\{ (y, x) : \ (x, y) \in F \big\}$ of F is a function, then the equality also holds in Theorems 1.5 and 1.6 and Corollary 1.7.

Theorem 1.9. *If F and G are relations on X to Y, then for any $x \in X$ we have*

(1) $\big(F \cap G\big)(x) = F(x) \cap G(x)$; *(2)* $\big(F \cup G\big)(x) = F(x) \cup G(x)$.

Theorem 1.10. *If F and G are relations on X to Y, then for any $A \subset X$ we have*

(1) $\big(F \cap G\big)[A] \subset F[A] \cap G[A]$; (2) $\big(F \cup G\big)[A] = F[A] \cup G[A]$.

Remark 1.11. Theorems 1.5, 1.9 and 1.10 can be naturally extended to arbitrary families of sets and relations.

Theorem 1.12. *If F and G are relations on X to Y, then for any $x \in X$ and $A \subset X$ we have*

(1) $(F \setminus G)(x) = F(x) \setminus G(x)$; (2) $F[A] \setminus G[A] \subset (F \setminus G)[A]$.

Corollary 1.13. *If F is a relation on X to Y, then for any $x \in X$ and $A \subset X$, with $A \neq \emptyset$, we have*

(1) $F^c(x) = F(x)^c$; (2) $F[A]^c \subset F^c[A]$.

Theorem 1.14. *If F is a relation on X to Y, then for any $A \subset X$ we have*

$$F^c[A]^c = \bigcap_{x \in A} F(x).$$

2. Box and totalization relations

Definition 2.1. For any $A \subset X$ and $B \subset Y$, we define

$$\Gamma_{(A,\,B)} = A \times B.$$

Remark 2.2. In particular, we shall also write

$$\Gamma_A = \Gamma_{(A,\,A)} \qquad \text{and} \qquad \Gamma_{(a,\,B)} = \Gamma_{(\{a\},\,B)}$$

for any $a \in X$.

Concerning box relations, the following theorems have been proved in [23].

Theorem 2.3. *If $A \subset X$ and $B \subset Y$, then for any $x \in X$ we have*

$$\Gamma_{(A,\,B)}(x) = \begin{cases} B & \textit{if} \quad x \in A, \\ \emptyset & \textit{if} \quad x \notin A. \end{cases}$$

Remark 2.4. Thus, in particular if $A \subset X$, then for any $x \in X$ we have

$$\Gamma_A(x) = \begin{cases} A & \text{if} \quad x \in A, \\ \emptyset & \text{if} \quad x \notin A. \end{cases}$$

Theorem 2.5. *If* $A \subset X$ *and* $B \subset Y$, *then for any* $U \subset X$ *we have*

$$\Gamma_{(A,\,B)}\,[\,U\,] = \begin{cases} \emptyset & \textit{if} \quad U \subset A^c, \\ B & \textit{if} \quad U \not\subset A^c. \end{cases}$$

Remark 2.6. Thus, in particular if $A \subset X$, then for any $U \subset X$ we have

$$\Gamma_A\,[\,U\,] = \begin{cases} \emptyset & \text{if} \quad U \subset A^c, \\ A & \text{if} \quad U \not\subset A^c. \end{cases}$$

Definition 2.7. For any relation F on one set X to another Y, we define

$$\tilde{F} = F \cup \Gamma_{(D_F^c,\,Y)}\,.$$

Remark 2.8. If $Y \neq \emptyset$, then the relation $\tilde{F}$ may be called the natural totalization of F. Its usefulness will be cleared up by the forthcoming results.

In particular, for any $A, B \subset X$, the totalizations

$$\tilde{\Gamma}_A = \widetilde{\Gamma_A} \qquad \text{and} \qquad \tilde{\Gamma}_{(A,\,B)} = \widetilde{\Gamma_{(A,\,B)}}$$

may be called the Davis–Pervin and the Hunsaker–Lindgren relations on X, respectively.

The latter relations play an important role in the generalized uniformization of various topological structures such as proximities, closures, topologies, and filters, for instance. (See [2], [14], [21] and [1, pp. 42, 193], [6], [16].)

While, the relations $\tilde{F}$ can be used to prove a useful reduction theorem for the intersection convolution of relations [22]. The latter operation allows of a natural treatment of the Hahn-Banach type extension theorems. (See [17] and [5].)

Concerning totalization relations, the following theorems have been proved in [23].

Theorem 2.9. *If* F *is a relation on* X *to* Y, *then for any* $U \subset X$ *we have*

$$\tilde{F}\,[\,U\,] = \begin{cases} F\,[\,U\,] & \textit{if} \quad U \subset D_F, \\ Y & \textit{if} \quad U \not\subset D_F. \end{cases}$$

Corollary 2.10. *If* F *is a relation on* X *to* Y, *then for some* $U \subset X$ *we have* $F[\,U\,] = \tilde{F}\,[\,U\,]$ *if and only if either* $U \subset D_F$ *or* $F\,[\,U\,] = Y$.

Corollary 2.11. *If* F *is a relation on* X *to* Y, *then for some* $x \in X$ *we have* $F(x) = \tilde{F}(x)$ *if and only if either* $x \in D_F$ *or* $F(x) = Y$.

Theorem 2.12. *If* F *is a relation on* X *to* Y, *then for any* $x \in X$ *we have*

$$\tilde{F}(\,x) = \begin{cases} F(\,x) & \textit{if} \quad x \in D_F, \\ Y & \textit{if} \quad x \notin D_F. \end{cases}$$

Corollary 2.13. *If F is a relation on X to Y, then $\tilde{F}$ is an extension of F such that $F = \tilde{F}$ if and only if $F(x) = Y$ for all $x \in D_F^c$.*

Corollary 2.14. *If F is a relation on X to Y and $Y \neq \emptyset$, then $F = \tilde{F}$ if and only if F is total.*

Theorem 2.15. *If $A \subset X$ and $B \subset Y$, then*

$$\tilde{\Gamma}_{(A,\,B)} = \begin{cases} \Gamma_{(X,\,Y)} & \text{if} \quad B = \emptyset\,, \\ \Gamma_{(A,\,B)} \cup \Gamma_{(A^c,\,Y)} & \text{if} \quad B \neq \emptyset\,. \end{cases}$$

Remark 2.16. Thus, in particular if A is a nonvoid subset of X, then

$$\tilde{\Gamma}_A = \Gamma_A \cup \Gamma_{(A^c,\,X)}\,.$$

Moreover, we can at once see that the latter equality is also true for $A = \emptyset$.

Theorem 2.17. *If $A \subset X$ and $B \subset Y$ such that $B \neq \emptyset$, then for any $U \subset X$, with $U \neq \emptyset$, we have*

$$\tilde{\Gamma}_{(A,\,B)}\,[\,U\,] = \begin{cases} B & \text{if} \quad U \subset A\,, \\ Y & \text{if} \quad U \not\subset A\,. \end{cases}$$

Remark 2.18. Thus, in particular if A and U are nonvoid subsets of X, then

$$\tilde{\Gamma}_A\,[\,U\,] = \begin{cases} A & \text{if} \quad U \subset A\,, \\ X & \text{if} \quad U \not\subset A\,. \end{cases}$$

Moreover, we can easily see that the latter equality is also true for $A = \emptyset$.

Corollary 2.19. *If $A \subset X$ and $B \subset Y$ such that $B \neq \emptyset$, then for any $x \in X$ we have*

$$\tilde{\Gamma}_{(A,\,B)}(x) = \begin{cases} B & \text{if} \quad x \in A\,, \\ Y & \text{if} \quad x \notin A\,. \end{cases}$$

Remark 2.20. Thus, in particular if A is a nonvoid subset of X and $x \in X$, then

$$\tilde{\Gamma}_A(x) = \begin{cases} A & \text{if} \quad x \in A\,, \\ X & \text{if} \quad x \notin A\,. \end{cases}$$

Moreover, we can easily see that the latter equality is also true for $A = \emptyset$.

Remark 2.21. Note that if $A \subset X$ and $\emptyset$ is considered as a subset of Y, then by Theorems 2.15 and 2.3 we have $\tilde{\Gamma}_{(A,\,\emptyset)}(x) = \Gamma_{(X,\,Y)}(x) = Y$ for all $x \in X$. Therefore, the assumption $B \neq \emptyset$ is indispensable in Corollary 2.19 and Theorem 2.17.

3. Intersections and unions of box relations and their totalizations

Theorem 3.1. *If* $A, C \subset X$ *and* $B, D \subset Y$*, then*

$$\Gamma_{(A,B)} \cap \Gamma_{(C,D)} = \Gamma_{(A \cap C,\ B \cap D)}.$$

Proof. By the corresponding definitions, for any $x \in X$ and $y \in Y$, we have

$$\begin{aligned}
&(x, y) \in \Gamma_{(A,B)} \cap \Gamma_{(C,D)} \iff (x, y) \in (A \times B) \cap (C \times D) \\
&\iff (x, y) \in A \times B, \quad (x, y) \in C \times D \iff x \in A, \ y \in B, \quad x \in C, \ y \in D \\
&\iff x \in A \cap C, \quad y \in B \cap D \iff (x, y) \in (A \cap C) \times (B \cap D) \\
&\iff (x, y) \in \Gamma_{(A \cap C,\ B \cap D)}.
\end{aligned}$$

Therefore, the required equality is true.

Remark 3.2. Thus, in particular if $A, B \subset X$, then

$$\Gamma_A \cap \Gamma_B = \Gamma_{A \cap B}.$$

Concerning unions, we can only prove a less convenient theorem.

Theorem 3.3. *If* $A, C \subset X$ *and* $B, D \subset Y$*, then*

(1) $\Gamma_{(A,B)} \cup \Gamma_{(A,D)} = \Gamma_{(A,\ B \cup D)}$;

(2) $\Gamma_{(A,B)} \cup \Gamma_{(C,B)} = \Gamma_{(A \cup C,\ B)}$.

Proof. By the corresponding definitions, for any $x \in X$ and $y \in Y$, we have

$$\begin{aligned}
&(x, y) \in \Gamma_{(A,B)} \cup \Gamma_{(A,D)} \iff (x, y) \in (A \times B) \cup (A \times D) \iff \\
&(x, y) \in A \times B \quad \text{or} \quad (x, y) \in C \times D \iff (x \in A, \ y \in B) \quad \text{or} \quad (x \in A, \ y \in D) \\
&\iff x \in A, \ (y \in B \ \text{ or } \ y \in D) \iff x \in A, \ y \in B \cup D \\
&\iff (x, y) \in A \times (B \cup D) \iff (x, y) \in \Gamma_{(A,\ B \cup D)}.
\end{aligned}$$

Therefore, equality (1) is true. The proof of (2) is quite similar.

Theorem 3.4. *If* $A, C \subset X$ *and* $B, D \subset Y$ *such that* $D \neq \emptyset$*, then*

$$\Gamma_{(A,B)} \cap \tilde{\Gamma}_{(C,D)} = \Gamma_{(A \cap C,\ B \cap D)} \cup \Gamma_{(A \setminus C,\ B)}.$$

Proof. By Theorems 2.15 and 3.1, we have

$$\begin{aligned}
\Gamma_{(A,B)} \cap \tilde{\Gamma}_{(C,D)} &= \Gamma_{(A,B)} \cap \big(\Gamma_{(C,D)} \cup \Gamma_{(C^c,Y)}\big) \\
&= \big(\Gamma_{(A,B)} \cap \Gamma_{(C,D)}\big) \cup \big(\Gamma_{(A,B)} \cap \Gamma_{(C^c,Y)}\big) = \Gamma_{(A \cap C,\ B \cap D)} \cup \Gamma_{(A \cap C^c,\ B)}.
\end{aligned}$$

Thus, since $A \cap C^c = A \setminus C$, the required equality is also true.

Remark 3.5. Thus, in particular if A, $B \subset X$ such that $B \neq \emptyset$, then

$$\Gamma_A \cap \tilde{\Gamma}_B = \Gamma_{A \cap B} \cup \Gamma_{(A \setminus B,\, A)}\,.$$

Moreover, we can easily see that the latter equality is also true for $B = \emptyset$.

Theorem 3.6. *If A, $C \subset X$ and B, $D \subset Y$ such that $B \neq \emptyset$ and $D \neq \emptyset$, then*

$$\tilde{\Gamma}_{(A,\,B)} \cap \tilde{\Gamma}_{(C,\,D)} = \Gamma_{(A \cap C,\ B \cap D)} \cup \Gamma_{(A \setminus C,\ B)} \cup \Gamma_{(C \setminus A,\ D)} \cup \Gamma_{((A \cup C)^c,\ Y)}\,.$$

Proof. By using Theorems 2.15 and 3.4, we can see that

$$\begin{aligned}
\tilde{\Gamma}_{(A,\,B)} \cap \tilde{\Gamma}_{(C,\,D)} &= \big(\Gamma_{(A,\,B)} \cup \Gamma_{(A^c,\,Y)}\big) \cap \tilde{\Gamma}_{(C,\,D)} \\
&= \big(\Gamma_{(A,\,B)} \cap \tilde{\Gamma}_{(C,\,D)}\big) \cup \big(\Gamma_{(A^c,\,Y)} \cap \tilde{\Gamma}_{(C,\,D)}\big) \\
&= \Gamma_{(A \cap C,\ B \cap D)} \cup \Gamma_{(A \setminus C,\ B)} \cup \Gamma_{(A^c \cap C,\ D)} \cup \Gamma_{(A^c \setminus C,\ Y)}\,.
\end{aligned}$$

Hence, since $A^c \cap C = C \setminus A$ and $A^c \setminus C = A^c \cap C^c = (A \cup C)^c$, the required equality is also true.

Remark 3.7. Thus, in particular if A and B are nonvoid subsets of X, then

$$\tilde{\Gamma}_A \cap \tilde{\Gamma}_B = \Gamma_{A \cap B} \cup \Gamma_{(A \setminus B,\, A)} \cup \Gamma_{(B \setminus A,\, B)} \cup \Gamma_{((A \cup B)^c,\, X)}\,.$$

Moreover, we can easily see that this equality is also true even if either $A = \emptyset$ or $B = \emptyset$.

Remark 3.8. Note that if F is a relation on X to Y, and moreover $C \subset X$ and $\emptyset$ is considered as a subset of Y, then by Theorem 2.15 we have

$$F \cap \tilde{\Gamma}_{(C,\,\emptyset)} = F \cap \Gamma_{(X,\,Y)} = F \cap (X \times Y) = F.$$

Therefore, the nonvoidness conditions are indispensable in Theorems 3.4 and 3.6.

Theorem 3.9. *If $A \subset X$ and B, $D \subset Y$, then*

(1) $\Gamma_{(A,\,B)} \cup \tilde{\Gamma}_{(A,\,D)} = \tilde{\Gamma}_{(A,\,B \cup D)}$ *if* $D \neq \emptyset$*;*

(2) $\tilde{\Gamma}_{(A,\,B)} \cup \tilde{\Gamma}_{(A,\,D)} = \tilde{\Gamma}_{(A,\,B \cup D)}$ *if* $B \neq \emptyset$ *and* $D \neq \emptyset$.

Proof. If $D \neq \emptyset$, then by Theorems 2.15 and 3.3, we have

$$\Gamma_{(A,\,B)} \cup \tilde{\Gamma}_{(A,\,D)} = \Gamma_{(A,\,B)} \cup \Gamma_{(A,\,D)} \cup \Gamma_{(A^c,\,Y)} = \Gamma_{(A,\ B \cup D)} \cup \Gamma_{(A^c,\,Y)} = \tilde{\Gamma}_{(A,\,B \cup D)}\,.$$

While, if $B \neq \emptyset$ and $D \neq \emptyset$, then by the above mentioned theorems we have

$$\begin{aligned}
&\tilde{\Gamma}_{(A,\,B)} \cup \tilde{\Gamma}_{(A,\,D)} = \Gamma_{(A,\,B)} \cup \Gamma_{(A^c,\,Y)} \cup \Gamma_{(A,\,D)} \cup \Gamma_{(A^c,\,Y)} \\
&\quad = \Gamma_{(A,\,B)} \cup \Gamma_{(A,\,D)} \cup \Gamma_{(A^c,\,Y)} \cup \Gamma_{(A^c,\,Y)} = \Gamma_{(A,\ B \cup D)} \cup \Gamma_{(A^c,\,Y)} = \tilde{\Gamma}_{(A,\,B \cup D)}\,.
\end{aligned}$$

Theorem 3.10. *If* $A, C \subset X$ *and* $B \subset Y$ *such that* $B \neq \emptyset$, *then*

(1) $\Gamma_{(A,B)} \cup \tilde{\Gamma}_{(C,B)} = \Gamma_{(A \cup C,\, B)} \cup \Gamma_{(C^c, Y)}$.

(2) $\tilde{\Gamma}_{(A,B)} \cup \tilde{\Gamma}_{(C,B)} = \Gamma_{(A \cup C,\, B)} \cup \Gamma_{((A \cap C)^c,\, Y)}$.

Proof. By Theorems 2.15 and 3.3, we have

$$\Gamma_{(A,B)} \cup \tilde{\Gamma}_{(C,B)} = \Gamma_{(A,B)} \cup \Gamma_{(C,B)} \cup \Gamma_{(C^c,Y)} = \Gamma_{(A \cup C,\, B)} \cup \Gamma_{(C^c,Y)}.$$

Moreover, by the above mentioned theorems, we also have

$$\begin{aligned} \tilde{\Gamma}_{(A,B)} \cup \tilde{\Gamma}_{(C,B)} &= \Gamma_{(A,B)} \cup \Gamma_{(A^c,Y)} \cup \Gamma_{(C,B)} \cup \Gamma_{(C^c,Y)} \\ &= \Gamma_{(A,B)} \cup \Gamma_{(C,B)} \cup \Gamma_{(A^c,Y)} \cup \Gamma_{(C^c,Y)} = \Gamma_{(A \cup C,\, B)} \cup \Gamma_{(C^c \cup A^c,\, Y)}. \end{aligned}$$

Hence, since $C^c \cup A^c = (A \cap B^c$, equality (2) is also true.

Remark 3.11. Note that if F is a relation on X to Y, and moreover $C \subset X$ and $\emptyset$ is considered as a subset of Y, then by Theorem 2.15 we have

$$F \cup \tilde{\Gamma}_{(C,\emptyset)} = F \cup \Gamma_{(X,Y)} = F \cup (X \times Y) = X \times Y = \Gamma_{(X,Y)}.$$

Therefore, the nonvoidness conditions are indispensable in Theorems 3.9 and 3.10.

4. Complements of box and totalization relations

Theorem 4.1. *If* $A \subset X$ *and* $B \subset Y$, *then*

$$\Gamma^c_{(A,B)} = \Gamma_{(A^c,Y)} \cup \Gamma_{(X,B^c)}$$

Proof. By the corresponding definitions, for any $x \in X$ and $y \in Y$, we have

$$\begin{aligned} (x, y) \in \Gamma^c_{(A,B)} &\iff (x, y) \notin \Gamma_{(A,B)} \iff (x, y) \notin A \times B \\ &\iff x \notin A \;\text{ or }\; y \notin B \iff (x, y) \in A^c \times Y \;\text{ or }\; (x, y) \in X \times B^c \\ &\iff (x, y) \in \Gamma_{(A^c,Y)} \;\text{ or }\; (x, y) \in \Gamma_{(X,B^c)} \iff (x, y) \in \Gamma_{(A^c,Y)} \cup \Gamma_{(X,B^c)}. \end{aligned}$$

Therefore, the required equality is true.

Remark 4.2. Thus, in particular if $A \subset X$, then

$$\Gamma^c_A = \Gamma_{(A^c,X)} \cup \Gamma_{(X,A^c)}.$$

Corollary 4.3. *If* $A \subset X$ *and* $B \subset Y$, *then for any* $U \subset X$, *with* $U \neq \emptyset$, *we have*

$$\Gamma^c_{(A,B)}[U] = \begin{cases} B^c & \text{if } \; U \subset A, \\ Y & \text{if } \; U \not\subset A. \end{cases}$$

Proof. By Theorems 4.1, 1.10 and 2.5, we have

$$\begin{aligned} \Gamma^c_{(A,B)}[U] &= \big(\Gamma_{(X,B^c)} \cup \Gamma_{(A^c,Y)}\big)[U] \\ &= \Gamma_{(X,B^c)}[U] \cup \Gamma_{(A^c,Y)}[U] = B^c \cup \begin{cases} \emptyset & \text{if } \; U \subset A, \\ Y & \text{if } \; U \not\subset A. \end{cases} \end{aligned}$$

Therefore, the required equality is also true

Remark 4.4. Thus, in particular if $A \subset X$, then for any $U \subset X$, with $U \neq \emptyset$, we have

$$\Gamma_A^c[U] = \begin{cases} A^c & \text{if} \quad U \subset A\,, \\ X & \text{if} \quad U \not\subset A\,. \end{cases}$$

Corollary 4.5. *If $A \subset X$ and $B \subset Y$, then for any $x \in X$ we have*

$$\Gamma_{(A,\,B)}^c(x) = \begin{cases} B^c & \textit{if} \quad x \in A\,, \\ Y & \textit{if} \quad x \notin A\,. \end{cases}$$

Remark 4.6. Thus, in particular if $A \subset X$, then for any $x \in X$ we have

$$\Gamma_A^c(x) = \begin{cases} A^c & \text{if} \quad x \in A\,, \\ X & \text{if} \quad x \notin A\,. \end{cases}$$

In addition to Theorem 4.1, it is also worth proving the following

Theorem 4.7. *If $A \subset X$ and $B \subset Y$, then*

(1) $\Gamma_{(A,\,B)}^c = \Gamma_{(A,\,B^c)} \cup \Gamma_{(A^c,\,Y)}$*;*

(2) $\Gamma_{(A,\,B)}^c = \Gamma_{(A,\,B^c)} \cup \Gamma_{(A^c,\,B)} \cup \Gamma_{(A^c,\,B^c)}$.

Proof. By Theorems 4.1 and 3.3, we have

$$\Gamma_{(A,\,B)}^c = \Gamma_{(X,\,B^c)} \cup \Gamma_{(A^c,\;Y)} = \Gamma_{(A,\,B^c)} \cup \Gamma_{(A^c,\;B^c)} \cup \Gamma_{(A^c,\;Y)}\,.$$

Hence, since $\Gamma_{(A^c,\,B^c)} \subset \Gamma_{(A^c,\,Y)}$, it is clear that (1) is true.

Moreover, by (1) and Theorem 3.3, we can see that

$$\Gamma_{(A,\,B)}^c = \Gamma_{(A,\,B^c)} \cup \Gamma_{(A^c,\;Y)} = \Gamma_{(A,\,B^c)} \cup \Gamma_{(A^c,\;B)} \cup \Gamma_{(A^c,\;B^c)}$$

also holds.

Remark 4.8. Thus, in particular if $A \subset X$, then

$$\Gamma_A^c = \Gamma_{(A,\,A^c)} \cup \Gamma_{(A^c,\,X)} = \Gamma_{A^c} \cup \Gamma_{(A,\,A^c)} \cup \Gamma_{(A^c,\;A)}\,.$$

Now, as an immediate consequence of Theorems 4.7 and 2.15, we can also state

Theorem 4.9. *If $A \subset X$ and $B \subset Y$ such that $B \neq Y$, then*

$$\Gamma_{(A,\,B)}^c = \tilde{\Gamma}_{(A,\,B^c)}\,.$$

Remark 4.10. Thus, in particular if A is a proper subset of X, then

$$\Gamma_A^c = \tilde{\Gamma}_{(A,\,A^c)}\,.$$

Remark 4.11. Note that, by the corresponding definitions and Theorem 2.15, we have

$$\Gamma_X^c = \big(X^2 \big)^c = \emptyset \qquad \text{and} \qquad \tilde{\Gamma}_{(X\,,\,X^c)} = \tilde{\Gamma}_{(X\,,\,\emptyset)} = \Gamma_X\,.$$

Therefore, the assumptions $A \neq X$ and $B \neq Y$ are indispensable in Remark 4.10 and Theorem 4.9, respectively.

By using Theorem 4.1, we can also easily prove the following

Theorem 4.12. *If F is a relation on X to Y, then*

$$\tilde{F}^c = F^c \cap \Gamma_{(D_F,\ Y)}\,.$$

Proof. By Definition 2.7, DeMorgan's law and Theorem 4.1, we have

$$\begin{aligned} \tilde{F}^c = \big(F \cup \Gamma_{(D_F^c,\ Y)} \big)^c = F^c \cap \Gamma^c_{(D_F^c,\ Y)} \\ = F^c \cap \big(\Gamma_{(D_F,\ Y)} \cup \Gamma_{(X,\emptyset)} \big) = F^c \cap \Gamma_{(D_F,\ Y)}\,. \end{aligned}$$

In principle the following theorem can be naturally derived from Theorem 4.12. However, it can now be more easily proved with the help of Theorem 2.12.

Theorem 4.13. *If F is a relation on X to Y, then for any $x \in X$ we have*

$$\tilde{F}^c(x) = \begin{cases} F(x)^c & \textit{if} \quad x \in D_F\,, \\ \emptyset & \textit{if} \quad x \notin D_F\,. \end{cases}$$

Proof. By Corollary 1.13 and Theorem 2.12, we have

$$\tilde{F}^c(x) = \tilde{F}\,(x)^c = \begin{cases} F(x)^c & \text{if} \quad x \in D_F\,, \\ \emptyset & \text{if} \quad x \notin D_F\,. \end{cases}$$

Theorem 4.14. *If F is a relation on X to Y, then for any $U \subset X$ we have*

$$\tilde{F}^c\,[\,U\,] = F^c\,[\,U \cap D_F\,]\,.$$

Proof. By Theorems 1.5, 4.13 and Corollary 1.13, we have

$$\begin{aligned} \tilde{F}^c\,[\,U\,] &= \tilde{F}^c\,[\,(\,U \cap D_F\,) \cup (\,U \setminus D_F\,)\,] \\ &= \tilde{F}^c\,[\,U \cap D_F\,] \cup \tilde{F}^c\,[\,U \setminus D_F\,] = \Big(\bigcup_{x \in U \cap D_F} \tilde{F}^c(x) \Big) \cup \Big(\bigcup_{x \in U \setminus D_F} \tilde{F}^c(x) \Big) \\ &= \bigcup_{x \in U \cap D_F} F(x)^c = \bigcup_{x \in U \cap D_F} F^c(x) = F^c\,[\,U \cap D_F\,]\,. \end{aligned}$$

In principle the following theorem can be naturally derived from Theorem 4.12. However, it can now be more easily proved with the help of Theorem 4.9.

Theorem 4.15. *If $A \subset X$ and $B \subset Y$ such that $B \neq \emptyset$, then*

$$\tilde{\Gamma}^{c}_{(A,\,B)} = \Gamma_{(A,\,B^c)}\,.$$

Proof. From Theorem 4.9, we can can see that

$$\tilde{\Gamma}_{(A,\,B)} = \Gamma^{c}_{(A,\,B^c)}.$$

Hence, it is clear that the required equality is also true.

Remark 4.16. Thus, in particular if A is a nonvoid subset of X, then

$$\tilde{\Gamma}^{c}_{A} = \Gamma_{(A,\,A^c)}\,.$$

Moreover, we can easily see that the latter equality is also true for $A = \emptyset$.

Corollary 4.17. *Under the conditions of Theorem 4.15, for any $U \subset X$, we have*

$$\tilde{\Gamma}^{c}_{(A,\,B)}\,[\,U\,] = \begin{cases} \emptyset & \text{if} \quad U \subset A^c\,, \\ B^c & \text{if} \quad U \not\subset A^c\,. \end{cases}$$

Proof. By Theorems 4.15 and 2.5, we have

$$\tilde{\Gamma}^{c}_{(A,\,B)}[\,U\,] = \Gamma_{(A,\,B^c)}[\,U\,] = \begin{cases} \emptyset & \text{if} \quad U \subset A^c\,, \\ B^c & \text{if} \quad U \not\subset A^c\,. \end{cases}$$

Remark 4.18. Thus, in particular if A is a nonvoid subset of X, then for any $U \subset X$ we have

$$\tilde{\Gamma}^{c}_{A}\,[\,U\,] = \begin{cases} \emptyset & \text{if} \quad U \subset A^c\,, \\ A^c & \text{if} \quad U \not\subset A^c\,. \end{cases}$$

Moreover, we can easily see that the latter equality is also true for $A = \emptyset$.

Corollary 4.19. *Under the conditions of Theorem 4.15, for any $x \in X$, we have*

$$\tilde{\Gamma}^{c}_{(A,\,B)}(x) = \begin{cases} B^c & \text{if} \quad x \in A\,, \\ \emptyset & \text{if} \quad x \notin A\,. \end{cases}$$

Remark 4.20. Thus, in particular if A is a nonvoid subset of X, then for any $x \in X$ we have

$$\tilde{\Gamma}^{c}_{A}(x) = \begin{cases} B^c & \text{if} \quad x \in A\,, \\ \emptyset & \text{if} \quad x \notin A\,. \end{cases}$$

Moreover, we can easily see that the latter equality is also true for $A = \emptyset$.

Remark 4.21. Note that if $A \subset X$ and $\emptyset$ is considered as a subset of Y, then by Theorem 2.15 we have

$$\tilde{\Gamma}^{c}_{(A,\,\emptyset)} = \Gamma^{c}_{(X,\,Y)} = \big(\,X \times Y\,\big)^{c} = \emptyset\,,$$

and hence $\tilde{\Gamma}^{c}_{(A,\,\emptyset)}(x) = \emptyset$ for all $x \in X$. Therefore, the assumption $B \neq \emptyset$ is indispensable in Corollaries 4.19 and 4.17 and Theorem 4.15.

5. Differences of box relations and their totalizations

Theorem 5.1. *If $A, C \subset X$ and $B, D \subset Y$, then*

(1) $\Gamma_{(A,B)} \setminus \Gamma_{(C,D)} = \Gamma_{(A,\ B\setminus D)} \cup \Gamma_{(A\setminus C,\ B)}$ *;*

(2) $\Gamma_{(A,B)} \setminus \Gamma_{(C,D)} = \Gamma_{(A\cap C,\ B\setminus D)} \cup \Gamma_{(A\setminus C,\ B)}$.

Proof. By using Theorems 4.1 and 3.1, we can see that

$$\begin{aligned}\Gamma_{(A,B)} \setminus \Gamma_{(C,D)} &= \Gamma_{(A,B)} \cap \Gamma^{c}_{(C,D)} = \\ \Gamma_{(A,B)} \cap \big(\Gamma_{(X,D^c)} \cup \Gamma_{(C^c,Y)}\big) &= \big(\Gamma_{(A,B)} \cap \Gamma_{(X,D^c)}\big) \cup \big(\Gamma_{(A,B)} \cap \Gamma_{(C^c,Y)}\big) \\ &= \Gamma_{(A,\ B\cap D^c)} \cup \Gamma_{(A\cap C^c,\ B)} = \Gamma_{(A,\ B\setminus D)} \cup \Gamma_{(A\setminus C,\ B)} .\end{aligned}$$

Moreover, by using Theorem 3.3, we can see that

$$\Gamma_{(A,\ B\setminus D)} = \Gamma_{((A\cap C)\cup(A\setminus C),\ B\setminus D)} = \Gamma_{(A\cap C,\ B\setminus D)} \cup \Gamma_{(A\setminus C,\ B\setminus D)} .$$

Hence, by using that $\Gamma_{(A\setminus C,\ B\setminus D)} \subset \Gamma_{(A\setminus C,\ B)}$, we can already see that

$$\begin{aligned}\Gamma_{(A,B)} \setminus \Gamma_{(C,D)} &= \Gamma_{(A,\ B\setminus D)} \cup \Gamma_{(A\setminus C,\ B)} \\ &= \Gamma_{(A\cap C,\ B\setminus D)} \cup \Gamma_{(A\setminus C,\ B\setminus D)} \cup \Gamma_{(A\setminus C,\ B)} = \Gamma_{(A\cap C,\ B\setminus D)} \cup \Gamma_{(A\setminus C,\ B)}\end{aligned}$$

is also true.

Remark 5.2. Thus, in particular if $A, B \subset X$, then

$$\Gamma_A \setminus \Gamma_B = \Gamma_{(A,\ A\setminus B)} \cup \Gamma_{(A\setminus B,\ A)} = \Gamma_{(A\cap B,\ A\setminus B)} \cup \Gamma_{(A\setminus B,\ A)}$$

Theorem 5.3. *If $A, C \subset X$ and $B, D \subset Y$ such that $B \neq \emptyset$, then*

$$\tilde{\Gamma}_{(A,B)} \setminus \Gamma_{(C,D)} = \Gamma_{(A\cap C,\ B\setminus D)} \cup \Gamma_{(A\setminus C,\ B)} \cup \Gamma_{(C\setminus A,\ D^c)} \cup \Gamma_{((A\cup C)^c,\ Y)} .$$

Proof. By Theorems 2.15 and 5.1, we have

$$\begin{aligned}\tilde{\Gamma}_{(A,B)} \setminus \Gamma_{(C,D)} &= \big(\Gamma_{(A,B)} \cup \Gamma_{(A^c,Y)}\big) \setminus \Gamma_{(C,D)} \\ &= \big(\Gamma_{(A,B)} \setminus \Gamma_{(C,D)}\big) \cup \big(\Gamma_{(A^c,Y)} \setminus \Gamma_{(C,D)}\big) \\ &= \Gamma_{(A\cap C,\ B\setminus D)} \cup \Gamma_{(A\setminus C,\ B)} \cup \Gamma_{(A^c\cap C,\ D^c)} \cup \Gamma_{(A^c\setminus C,\ Y)} .\end{aligned}$$

Thus, since $A^c \cap C = C \setminus A$ and $A^c \setminus C = A^c \cap C^c = (A \cup C)^c$, the required equality is also true.

Remark 5.4. Thus, in particular if $A, B \subset X$ such that $A \neq \emptyset$, then

$$\tilde{\Gamma}_A \setminus \Gamma_B = \Gamma_{(A\cap B,\ A\setminus B)} \cup \Gamma_{(A\setminus B,\ A)} \cup \Gamma_{(B\setminus A,\ B^c)} \cup \Gamma_{((A\cup B)^c,\ X)} .$$

Moreover, by considering $\emptyset$ as a subset of X and using Remarks 2.16 and 4.8, we can see that

$$\tilde{\Gamma}_\emptyset \setminus \Gamma_B = X^2 \setminus \Gamma_B = \Gamma_B^c = \Gamma_{(B,B^c)} \cup \Gamma_{(B^c,\ X)} .$$

Therefore, since $\Gamma_\emptyset = \emptyset$, the former equality is true even if $A = \emptyset$.

Remark 5.5. On the other hand, if $A\,,\,C\subset X$ and $D\subset Y$, and $\emptyset$ is considered as a subset Y, then by using Theorems 2.15 and 4.7, we can see that

$$\tilde{\Gamma}_{(A,\,\emptyset)}\setminus\Gamma_{(C,\,D)}=(\,X\times Y\,)\setminus\Gamma_{(C,\,D)}=\Gamma^{c}_{(C,\,D)}=\Gamma_{(C,\,D^{c})}\cup\Gamma_{(C^{c},\;Y)}\,.$$

Therefore, the assumption $B\neq\emptyset$ is indispensable in Theorem 5.3.

Theorem 5.6. *If* $A\,,\,C\subset X$ *and* $B\,,\,D\subset Y$ *such that* $D\neq\emptyset$*, then*

$$\Gamma_{(A,\,B)}\setminus\tilde{\Gamma}_{(C,\,D)}=\Gamma_{(A\cap C,\;B\setminus D)}\,.$$

Proof. By Theorems 4.15 and 3.1, we have

$$\Gamma_{(A,\,B)}\setminus\tilde{\Gamma}_{(C,\,D)}=\Gamma_{(A,\,B)}\cap\tilde{\Gamma}^{c}_{(C,\,D)}=\Gamma_{(A,\,B)}\cap\Gamma_{(C,\,D^{c})}=\Gamma_{(A\cap C,\;B\cap D^{c})}\,.$$

Thus, since $B\cap D^{c}=B\setminus D$, the required equality is also true.

Remark 5.7. Thus, in particular if $A\,,\,B\subset X$ such that $B\neq\emptyset$, then

$$\Gamma_{A}\setminus\tilde{\Gamma}_{B}=\Gamma_{(A\cap B,\;A\setminus B)}\,.$$

Moreover, by using Remark 2.16, we can easily see that the latter equality is true even if $B=\emptyset$.

Theorem 5.8. *If* $A\,,\,C\subset X$ *and* $B\,,\,D\subset Y$ *such that* $B\neq\emptyset$ *and* $D\neq\emptyset$*, then*

$$\tilde{\Gamma}_{(A,\,B)}\setminus\tilde{\Gamma}_{(C,\,D)}=\Gamma_{(A\cap C,\;B\setminus D)}\cup\Gamma_{(C\setminus A,\;D^{c})}.$$

Proof. By Theorems 4.15 and 3.4, we have

$$\begin{aligned}\tilde{\Gamma}_{(A,\,B)}\setminus\tilde{\Gamma}_{(C,\,D)}&=\tilde{\Gamma}_{(A,\,B)}\cap\tilde{\Gamma}^{c}_{(C,\,D)}=\tilde{\Gamma}_{(A,\,B)}\cap\Gamma_{(C,\,D^{c})}=\\&=\Gamma_{(C,\;D^{c})}\cap\tilde{\Gamma}_{(A,\,B)}=\Gamma_{(C\cap A,\;D^{c}\cap B)}\cup\Gamma_{(C\setminus A,\;D^{c})}\,.\end{aligned}$$

Thus, since $D^{c}\cap B=B\setminus D$, the required equality is also true.

Remark 5.9. Thus, in particular if A and B are nonvoid subsets of X, then

$$\tilde{\Gamma}_{A}\setminus\tilde{\Gamma}_{B}=\Gamma_{(A\cap B,\;A\setminus B)}\cup\Gamma_{(B\setminus A,\;B^{c})}.$$

Moreover, by using Remarks 2.16 and 4.16, we can easily see that the latter equality is true even if either $A=\emptyset$ or $B=\emptyset$.

Remark 5.10. Note that if F is a relation on X to Y, and moreover $C\subset X$ and $\emptyset$ is considered as a subset of Y, then by Theorem 2.15, we have

$$F\setminus\tilde{\Gamma}_{(C,\,\emptyset)}=F\setminus\Gamma_{(X,\,Y)}=F\setminus X\times Y=\emptyset\,.$$

Therefore, the nonvoidness conditions are indispensable in Theorems 5.6 and 5.8.

6. A few basic facts on relators

A family $\mathcal{R}$ on relations on one set X to another Y is called a relator on X to Y. Moreover, the ordered pair $(X, Y)(\mathcal{R}) = \big((X, Y), \mathcal{R}\big)$ is called a relator space. For the origins of this notion, see [15] and the references therein.

If in particular $\mathcal{R}$ is a relator on X to itself, then we may simply say that $\mathcal{R}$ is a relator on X. Moreover, by identifying singletons with their elements, we may naturally write $X(\mathcal{R})$ in place of $(X, X)(\mathcal{R})$.

Quite similarly, if R is a relation on X to Y, then we may simply write $(X, Y)(R)$ in place of $(X, Y)(\{R\})$. More generally, the same convention can also be applied when $\mathfrak{F}$ is a function of relators on X to Y.

Relator spaces of the simpler type $X(R)$ and $X(\mathcal{R})$ are substantial generalizations of ordered sets and uniform spaces [3]. However, they are insufficient to include the theory of context spaces [4], and to naturally express continuity properties of relations [18].

If $\mathcal{R}$ is a relator on X to Y, then for any $A \subset X$ and $B \subset Y$, we write:

(1) $A \in \operatorname{Int}_{\mathcal{R}}(B)$ if $R[A] \subset B$ for some $R \in \mathcal{R}$;

(2) $A \in \operatorname{Lb}_{\mathcal{R}}(B)$ if $B \subset R^c[A]^c$ for some $R \in \mathcal{R}$.

To see the appropriateness of the latter apparently very strange definition, recall that, by the corresponding definition and Theorem 1.14, we have

$$R[A] = \bigcup_{a \in A} R(a) \qquad \text{and} \qquad R^c[A]^c = \bigcap_{a \in A} R(a).$$

Thus, in particular $B \subset R^c[A]^c$ if and only if $B \subset R(a)$ for all $a \in A$. That is, $b \in R(a)$, i.e., $a\,R\,b$ for all $a \in A$ and $b \in B$. Therefore, A is a lower bound of B with respect to R.

In this respect, it is also worth noticing that $B \subset R^c[A]^c$ if and only if $R^c[A] \subset B^c$. Therefore,

$$\operatorname{Lb}_{\mathcal{R}}(B) = \operatorname{Int}_{\mathcal{R}^c}(B^c) \qquad \text{and} \qquad \operatorname{Int}_{\mathcal{R}}(B) = \operatorname{Lb}_{\mathcal{R}^c}(B^c),$$

where $\mathcal{R}^c = \{R^c : R \in \mathcal{R}\}$. Thus, in contrast to a common belief, the basic topological and order theoretic notions can be expressed in terms of each other. This fact, and the use of the notation Lb, was first put forward in [19].

Now, if $\mathcal{R}$ is a relator on X to Y, then for any $a \in X$ and $B \subset Y$, we may simply write:

(3) $a \in \operatorname{int}_{\mathcal{R}}(B)$ if $\{a\} \in \operatorname{Int}_{R}(B)$; (4) $a \in \operatorname{lb}_{\mathcal{R}}(B)$ if $\{a\} \in \operatorname{Lb}_{R}(B)$;

(5) $B \in \mathcal{E}_{\mathcal{R}}$ if $\operatorname{int}_{\mathcal{R}}(B) \neq \emptyset$; (6) $B \in \mathfrak{L}_{\mathcal{R}}$ if $\operatorname{lb}_{\mathcal{R}}(B) \neq \emptyset$.

Moreover, if in particular $\mathcal{R}$ is a relator on X, then for any $A \subset X$ we may also write:

(7) $A \in \tau_{\mathcal{R}}$ if $A \in \operatorname{Int}_{\mathcal{R}}(A)$; (8) $A \in \mathcal{T}_{\mathcal{R}}$ if $A \subset \operatorname{int}_{\mathcal{R}}(A)$;

(9) $A \in l_{\mathcal{R}}$ if $A \in \operatorname{Lb}_{\mathcal{R}}(A)$; (10) $A \in \mathcal{L}_{\mathcal{R}}$ if $A \subset \operatorname{lb}_{\mathcal{R}}(A)$.

The relations $\operatorname{Int}_{\mathcal{R}}$ and $\operatorname{int}_{\mathcal{R}}$ are called the proximal and topological interiors on Y to X induced by $\mathcal{R}$, respectively. While, the members of the families $\tau_{\mathcal{R}}$, $\mathcal{T}_{\mathcal{R}}$, and $\mathcal{E}_{\mathcal{R}}$ are called the proximally open, topologically open and fat subsets of $X(\mathcal{R})$, respectively.

The use of notation Int instead of $\Subset$ was first suggested in [15]. While, the fact that the fat sets are usually more important tools than the open ones was first stressed in [16], and at the Seventh Topological Symposium in Prague in 1991.

Now, by Remark 2.2 and Definition 2.1, we may naturally introduce the following generated relators.

Definition 6.1. For any family $\mathcal{A} \subset \mathcal{P}(X)$, we define

$$\mathcal{R}_{\mathcal{A}} = \left\{ \Gamma_A : \quad A \in \mathcal{A} \right\}.$$

Moreover, for any relations $\mathfrak{f}$ and $\mathfrak{F}$ on $\mathcal{P}(Y)$ to X and $\mathcal{P}(X)$, respectively, we define

$$\mathcal{R}_{\mathfrak{f}} = \left\{ \Gamma_{(a,\,B)} : \quad a \in \mathfrak{f}(B) \right\} \qquad \text{and} \qquad \mathcal{R}_{\mathfrak{F}} = \left\{ \Gamma_{(A,\,B)} : \quad A \in \mathfrak{F}(B) \right\}.$$

Remark 6.2. Note that if in particular $\mathfrak{F}_{\mathcal{A}}$ is the identity function of $\mathcal{A}$, then $\mathcal{R}_{\mathcal{A}} = \mathcal{R}_{\mathfrak{F}_{\mathcal{A}}}$.

While, if in particular $\mathfrak{F}_{\mathfrak{f}}(B) = \left\{ \{a\} : \quad a \in \mathfrak{f}(B) \right\}$ for all $B \subset Y$, then $\mathcal{R}_{\mathfrak{f}} = \mathcal{R}_{\mathfrak{F}_{\mathfrak{f}}}$.

Moreover, by Definition 2.7, we may also also naturally introduce the following totalization relator.

Definition 6.3. For any relator $\mathcal{R}$ on X to Y, we define

$$\tilde{\mathcal{R}} = \left\{ \tilde{R} : \quad R \in \mathcal{R} \right\}.$$

Remark 6.4. Thus, for any family $\mathcal{A}$ of subsets of X and relation $\mathfrak{F}$ on X, the totalizations $\tilde{\mathcal{R}}_{\mathcal{A}}$ and $\tilde{\mathcal{R}}_{\mathfrak{F}}$ may be called the Davis–Pervin and the Hunsaker–Lindgren relators on X generated by $\mathcal{A}$ and $\mathfrak{F}$.

However, in the sequel we shall only interested in some characteristic properties of the induced order theoretic basic tools. For this, it will be enough to investigate only the generated relators $\mathcal{R}_{\mathfrak{F}}$, $\mathcal{R}_{\mathfrak{f}}$ and $\mathcal{R}_{\mathcal{A}}$ considered in Definition 6.1.

7. Some applications to relator spaces

In addition to our former results on the complements of box relations, we shall also need the following auxiliary theorem and its consequences.

Theorem 7.1. *If $A \subset X$ and $B \subset Y$, then for any $U \subset X$ and $V \subset Y$, with $U \neq \emptyset$ and $V \neq Y$, the following assertions are equivalent:*

(1) $\Gamma^c_{(A,\,B)}[U] \subset V$; *(2)* $U \subset A$ *and* $V^c \subset B$.

Proof. By Corollary 4.3, we have

$$\Gamma^c_{(A,\,B)}[U] = \begin{cases} B^c & \text{if} \quad U \subset A, \\ Y & \text{if} \quad U \not\subset A. \end{cases}$$

Hence, since $V \neq Y$, we can already see that (1) is equivalent to the requirements that $U \subset A$ and $B^c \subset V$, i.e., $V^c \subset B$. Therefore, (1) and (2) are also equivalent.

Remark 7.2. Thus, in particular if $A \subset X$, then for any $U\,,\,V \subset X$, with $U \neq \emptyset$ and $V \neq X$, we have $\Gamma_A^c\,[\,U\,] \subset V$ if and only if $U \cup V^c \subset A$.

Corollary 7.3. *If $A \subset X$ and $B \subset Y$, then for any $u \in X$ and $V \subset Y$, with $V \neq Y$, the following assertions are equivalent:*

(1) $\Gamma_{(A,\,B)}^c(u) \subset V$; *(2)* $u \in A$ *and* $V^c \subset B$.

Remark 7.4. Thus, in particular if $A \subset X$, then for any $u \in X$ and $V \subset X$, with $V \neq X$, we have $\Gamma_A^c(u) \subset V$ if and only if $u \in A$ and $V^c \subset A$.

Corollary 7.5. *If $a \in X$ and $B \subset Y$, then for any $u \in X$ and $V \subset Y$, with $V \neq Y$, the following assertions are equivalent:*

(1) $\Gamma_{(a,\,B)}^c(u) \subset V$; *(2)* $u = a$ *and* $V^c \subset B$.

Now, by using the corresponding definitions and Theorem 7.1, we can also easily prove the following

Theorem 7.6. *If* Lb *is a relation on $\mathcal{P}(Y)$ to $\mathcal{P}(X)$, then for any $U \subset X$ and $V \subset Y$, with $U \neq \emptyset$ and $V \neq \emptyset$, the following assertions are equivalent:*

(1) $U \in \mathrm{Lb}_{\mathcal{R}_{\mathrm{Lb}}}(V)$;

(2) $A \in \mathrm{Lb}(B)$ *for some* $A \subset X$ *and* $B \subset Y$ *with* $U \subset A$ *and* $V \subset B$.

Proof. By the corresponding definitions, we have (1) if and only if there exist $A \subset X$ and $B \subset Y$, with $A \in \mathrm{Lb}(B)$, such that $V \subset \Gamma_{(A,\,B)}^c\,[\,U\,]^c$, i.e., $\Gamma_{(A,\,B)}^c\,[\,U\,] \subset V^c$. Moreover, by Theorem 7.1, the latter inclusion is equivalent to the requirements that $U \subset A$ and $V \subset B$. Therefore, (1) and (2) are also equivalent.

In principle, the following theorem can be derived from Theorem 7.6 by using Remark 6.2. However, it can be more easily proved with the help of Corollary 7.5.

Theorem 7.7. *If* lb *is a relation on $\mathcal{P}(Y)$ to X, then for any $u \in X$ and $V \subset Y$, with $V \neq \emptyset$, the following assertions are equivalent:*

(1) $u \in \mathrm{lb}_{\mathcal{R}_{\mathrm{lb}}}(V)$;

(2) $u \in \mathrm{lb}\,(B)$ *for some* $B \subset Y$ *with* $V \subset B$.

Proof. By the corresponding definition, we have (1) if and only if $\{u\} \in \mathrm{Lb}_{\mathcal{R}_{\mathrm{lb}}}(V)$. This means that there exist $a \in X$ and $B \subset Y$, with $a \in \mathrm{lb}\,(B)$, such that $V \subset \Gamma_{(a,\,B)}^c[\,\{u\}\,]^c$, i.e., $\Gamma_{(a,\,B)}^c(u) \subset V^c$. Moreover, by Corollary 7.5, the latter inclusion is equivalent to the requirements that $u = a$ and $V \subset B$. Therefore, (1) and (2) are also equivalent.

Remark 7.8. Now, by establishing the basic properties of the relations $\mathrm{Lb}_{\mathcal{R}}$ and $\mathrm{lb}_{\mathcal{R}}$ for a relator $\mathcal{R}$ on X to Y, we can give some necessary and sufficient conditions on the relations Lb and lb on $\mathcal{P}(Y)$ to $\mathcal{P}(X)$ and X, respectively, in order that the equalities $\mathrm{Lb} = \mathrm{Lb}_{\mathcal{R}_{\mathrm{Lb}}}$ and $\mathrm{lb} = \mathrm{lb}_{\mathcal{R}_{\mathrm{lb}}}$ could be true.

Moreover, for a relator $\mathcal{R}$ on X to Y, we can investigate the validity the equalities $\mathrm{Lb}_{\mathcal{R}} = \mathrm{Lb}_{\mathcal{R}_{\mathrm{Lb}_{\mathcal{R}}}}$ and $\mathrm{lb}_{\mathcal{R}} = \mathrm{int}_{\mathcal{R}_{\mathrm{lb}_{\mathcal{R}}}}$. And, for a relator $\mathcal{R}$ on X to

Y, we may look for the largest relator $\mathcal{S}$ on X to Y such that the equality $\mathrm{Lb}_{\mathcal{R}} = \mathrm{Lb}_{\mathcal{S}}$, resp. $\mathrm{lb}_{\mathcal{R}} = \mathrm{lb}_{\mathcal{S}}$ could be true.

In addition to Theorems 7.6 and 7.7, we can also easily prove the following three closely related theorems.

Theorem 7.9. *If $l \subset \mathcal{P}(X)$, then for any nonvoid subset U of X the following assertions are equivalent:*

(1) $U \in l_{\mathcal{R}_l}$; *(2)* $U \subset A$ *for some* $A \in l$.

Proof. By the corresponding definition, we have (1) if and only if $U \in \mathrm{Lb}_{\mathcal{R}_l}(U)$. This means that there exists $A \in l$ such that $U \subset \Gamma_A^c[U]^c$, i.e., $\Gamma_A^c[U] \subset U^c$. Moreover, by Remark 7.2, the latter inclusion is equivalent to the requirement that $U \subset A$. Therefore, (1) and (2) are also equivalent.

Theorem 7.10. *If $\mathcal{L} \subset \mathcal{P}(X)$, then for any nonvoid subset U of X the following assertions are equivalent:*

(1) $U \in \mathcal{L}_{\mathcal{R}_{\mathcal{L}}}$; *(2)* $U \subset A$ *for some* $A \in \mathcal{L}$.

Proof. By the corresponding definition, we have (1) if and only if $U \subset \mathrm{lb}_{\mathcal{R}_{\mathcal{L}}}(U)$. That is, for each $u \in U$, we have $u \in \mathrm{lb}_{\mathcal{R}_{\mathcal{L}}}(U)$, i.e., $\{u\} \in \mathrm{Lb}_{\mathcal{R}_{\mathcal{L}}}(U)$. This means that there exists $A_u \in \mathcal{L}$ such that $U \subset \Gamma_{A_u}^c(\{u\})^c$, i.e., $\Gamma_A^c(u) \subset U^c$. Moreover, by Remark 7.4, the latter inclusion is equivalent to the requirement that $u \in A_u$ and $U \subset A_u$. Hence, it is clear that (1) and (2) are also equivalent.

Theorem 7.11. *If $\mathfrak{L} \subset \mathcal{P}(X)$, then for any nonvoid subset U of X the following assertions are equivalent:*

(1) $U \in \mathfrak{L}_{\mathcal{R}_{\mathfrak{L}}}$; *(2)* $U \subset A$ *for some* $A \in \mathfrak{L}$.

Proof. By the corresponding definitions, we have (1) if and only if $\mathrm{lb}_{\mathcal{R}_{\mathfrak{L}}}(U) \neq \emptyset$. That is, there exist $x \in X$ such that $x \in \mathrm{lb}_{\mathcal{R}_{\mathfrak{L}}}(U)$. By the proof of Theorem 7.10, the latter inclusion is equivalent to the requirement that there exists $A \in \mathfrak{L}$ such that $x \in A$ and $U \subset A$. Hence, it is clear that (1) and (2) are also equivalent.

Remark 7.12. Now, by establishing the basic properties of the families $l_{\mathcal{R}}$, $\mathcal{L}_{\mathcal{R}}$, and $\mathfrak{L}_{\mathcal{R}}$ for a relator $\mathcal{R}$ on X, we can give some necessary and sufficient conditions on a family $\mathcal{A}$ of subsets of X in order that the equality $\mathcal{A} = l_{\mathcal{R}_{\mathcal{A}}}$, $\mathcal{A} = \mathcal{L}_{\mathcal{R}_{\mathcal{A}}}$, resp. $\mathcal{A} = \mathfrak{L}_{\mathcal{R}_{\mathcal{A}}}$ could be true.

Moreover, for a relator $\mathcal{R}$ on X, we can investigate the validity the equalities $l_{\mathcal{R}} = l_{\mathcal{R}_{l_{\mathcal{R}}}}$, $\mathcal{L}_{\mathcal{R}} = \mathcal{L}_{\mathcal{R}_{\mathcal{L}_{\mathcal{R}}}}$ and $\mathfrak{L}_{\mathcal{R}} = \mathfrak{L}_{\mathcal{R}_{\mathfrak{L}_{\mathcal{R}}}}$. And, for a relator $\mathcal{R}$ on X, we may look for the largest relator $\mathcal{S}$ on X such that the equality $l_{\mathcal{R}} = l_{\mathcal{S}}$, $\mathcal{L}_{\mathcal{R}} = \mathcal{L}_{\mathcal{S}}$, resp. $\mathfrak{L}_{\mathcal{R}} = \mathfrak{L}_{\mathcal{S}}$ could be true.

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