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INCLUSIONS ON BOX AND TOTALIZATION RELATIONS

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Abstract. In this paper, we shall only study inclusions on box and totalization relations

$$\Gamma_{(A,B)} = A \times B \qquad \text{and} \qquad \tilde{F} = F \cup \Gamma_{(D_F^c,\, Y)},$$

where $A \subset X$, $B \subset Y$ and F is a relation on X to Y with domain D_F. The set and relation theoretic operations, and several algebraic and topological properties of these relations, will be studied elsewhere.

This line of investigations is mainly motivated by the fact that the relations

$$\tilde{\Gamma}_{(A,B)} = \widetilde{\Gamma_{(A,B)}} \qquad \text{and} \qquad \tilde{\Gamma}_A = \tilde{\Gamma}_{(A,A)}$$

play an important role in the uniformization of various topological structures such as proximities, closures and topologies, for instance. Moreover, the relations $\tilde{F}$ can be used to prove a useful reduction theorem for the intersection convolution of relations. The latter operation allows of a natural treatment of the Hahn-Banach type extension theorems.

1. A few basic facts on relations

A subset F of a product set $X \times Y$ is called a relation on X to Y. If in particular F is a relation on X to itself, then we may simply say that F is a relation on X. Thus, a relation F on X to Y is also a relation on $X \cup Y$.

If F is a relation on X to Y, then for any $x \in X$ and $A \subset X$ the sets $F(x) = \{ y \in Y : \ (x, y) \in F \}$ and $F[A] = \bigcup_{a \in A} F(a)$ are called the images of x and A under F, respectively.

Moreover, the sets $D_F = \{ x \in X : \ F(x) \neq \emptyset \}$ and $R_F = F[X]$ are called the domain and range of F, respectively. If in particular $X = D_F$, then we say that F is a relation of X to Y, or that F is a total relation on X to Y. While, if $Y = R_F$, then we say that F is a relation on X onto Y.

If F is a relation on X to Y and $U \subset D_F$, then the relation $F \,|\, U = F \cap (U \times Y)$ is called the restriction of F to U. Moreover, if F and G are relations on X to Y such that $D_F \subset D_G$ and $F = G \,|\, D_F$, then G is called an extension of F.

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In particular, a relation f on X to Y is called a function if for each $x \in D_f$ there exists $y \in Y$ such that $f(x) = \{y\}$. In this case, by identifying singletons with their elements, we may simply write $f(x) = y$ instead of $f(x) = \{y\}$.

Concerning relations, we shall only quote here the following basic theorems.

Theorem 1.1. *If F is a relation on X to Y, then*

$$F = \bigcup_{x \in X} \{x\} \times F(x) = \bigcup_{x \in D_F} \{x\} \times F(x).$$

Remark 1.2. By this theorem, a relation F on X to Y can be naturally defined by specifying $F(x)$ for all $x \in X$, or by specifying D_F and $F(x)$ for all $x \in D_F$.

Corollary 1.3. *If F and G are relations on X to Y, then the following assertions are equivalent:*

(1) $F \subset G$;

(2) $F(x) \subset G(x)$ *for all* $x \in X$; *(3)* $F(x) \subset G(x)$ *for all* $x \in D_F$.

Corollary 1.4. *If F and G are relations on X to Y, then the following assertions are equivalent:*

(1) $F = G$;

(2) $F(x) = G(x)$ *for all* $x \in X$;

(3) $D_F = D_G$ *and* $F(x) = G(x)$ *for all* $x \in D_F$.

Theorem 1.5. *If F is a relation on X to Y, then for any $A, B \subset X$ we have*

(1) $F[A \cap B] \subset F[A] \cap F[B]$; *(2)* $F[A \cup B] = F[A] \cup F[B]$.

Hint. To check this, note first that $F[A] \subset F[B]$ whenever $A \subset B$.

Theorem 1.6. *If F is a relation on X to Y, then for any $A, B \subset X$ we have*

$$F[A] \setminus F[B] \subset F[A \setminus B].$$

Corollary 1.7. *If F is a relation on X onto Y, then for any $A \subset X$ we have*

$$F[A]^c \subset F[A^c].$$

Remark 1.8. If in particular the inverse $F^{-1} = \big\{ (y, x) : \ (x, y) \in F \big\}$ of F is a function, then the equality also holds in Theorems 1.5 and 1.6 and Corollary 1.7.

Theorem 1.9. *If F and G are relations on X to Y, then for any $x \in X$ we have*

(1) $\big(F \cap G\big)(x) = F(x) \cap G(x)$; *(2)* $\big(F \cup G\big)(x) = F(x) \cup G(x)$.

Theorem 1.10. *If F and G are relations on X to Y, then for any $A \subset X$ we have*

(1) $\big(F \cap G\big)[A] \subset F[A] \cap G[A]$; (2) $\big(F \cup G\big)[A] = F[A] \cup G[A]$.

Hint. To check this, note first that $F[A] \subset G[A]$ whenever $F \subset G$.

Remark 1.11. Theorems 1.5, 1.9 and 1.10 can be naturally extended to arbitrary families of sets and relations.

Theorem 1.12. *If F and G are relations on X to Y, then for any $x \in X$ and $A \subset X$ we have*

(1) $(F \setminus G)(x) = F(x) \setminus G(x)$; (2) $F[A] \setminus G[A] \subset (F \setminus G)[A]$.

Corollary 1.13. *If F is a relation on X to Y, then for any $x \in X$ and $A \subset X$, with $A \neq \emptyset$, we have*

(1) $F^c(x) = F(x)^c$; (2) $F[A]^c \subset F^c[A]$.

Theorem 1.14. *If F is a relation on X to Y, then for any $A \subset X$ we have*

$$F^c[A]^c = \bigcap_{x \in A} F(x).$$

Proof. By DeMorgan's law and Corollary 1.13, we have

$$F^c[A]^c = \Big(\bigcup_{x \in A} F^c(x)\Big)^c = \bigcap_{x \in A} F^c(x)^c = \bigcap_{x \in A} F(x).$$

2. Box and totalization relations

Definition 2.1. For any $A \subset X$ and $B \subset Y$, we define

$$\Gamma_{(A,\,B)} = A \times B.$$

Remark 2.2. In particular, we shall also write

$$\Gamma_A = \Gamma_{(A,\,A)} \qquad \text{and} \qquad \Gamma_{(a,\,B)} = \Gamma_{(\{a\},\,B)}$$

for any $a \in X$.

Theorem 2.3. *If $A \subset X$ and $B \subset Y$, then for any $x \in X$ we have*

$$\Gamma_{(A,\,B)}(x) = \begin{cases} B & \text{if} \quad x \in A, \\ \emptyset & \text{if} \quad x \notin A. \end{cases}$$

Proof. By the corresponding definitions, for any $y \in Y$, we have

$$y \in \Gamma_{(A,\,B)}(x) \iff (x,\,y) \in \Gamma_{(A,\,B)} \iff (x,\,y) \in A \times B \iff x \in A, \quad y \in B.$$

Therefore, the required equality is also true.

Remark 2.4. Thus, in particular if $A \subset X$, then for any $x \in X$ we have

$$\Gamma_A(x) = \begin{cases} A & \text{if} \quad x \in A\,, \\ \emptyset & \text{if} \quad x \notin A\,. \end{cases}$$

Theorem 2.5. *If $A \subset X$ and $B \subset Y$, then for any $U \subset X$ we have*

$$\Gamma_{(A,\,B)}\,[\,U\,] = \begin{cases} \emptyset & \textit{if} \quad U \subset A^c, \\ B & \textit{if} \quad U \not\subset A^c. \end{cases}$$

Proof. By Theorems 1.5 and 2.3, we have

$$\begin{aligned} \Gamma_{(A,\,B)}\,[\,U\,] &= \Gamma_{(A,\,B)}\,[\,(\,U \cap A\,) \cup (\,U \setminus A\,)\,] = \Gamma_{(A,\,B)}\,[\,U \cap A\,] \cup \Gamma_{(A,\,B)}\,[\,U \setminus A\,] \\ &= \Big(\bigcup_{x \in U \cap A} \Gamma_{(A,\,B)}(\,x\,) \Big) \cup \Big(\bigcup_{x \in U \setminus A} \Gamma_{(A,\,B)}(\,x\,) \Big) \\ &= \bigcup_{x \in U \cap A} B = \begin{cases} \emptyset & \text{if} \quad U \cap A = \emptyset\,, \\ B & \text{if} \quad U \cap A \neq \emptyset\,. \end{cases} \end{aligned}$$

Hence, since $U \cap A = \emptyset \iff U \subset A^c$, it is clear that the required equality is also true.

Remark 2.6. Thus, in particular if $A \subset X$, then for any $U \subset X$ we have

$$\Gamma_A\,[\,U\,] = \begin{cases} \emptyset & \text{if} \quad U \subset A^c, \\ A & \text{if} \quad U \not\subset A^c. \end{cases}$$

Definition 2.7. For any relation F on one set X to another Y, we define

$$\tilde{F} = F \,\cup\, \Gamma_{(\,D_F^c\,,\,Y\,)}\,.$$

Remark 2.8. If $Y \neq \emptyset$, then the relation $\tilde{F}$ may be called the natural totalization of F. Its usefulness will be cleared up by the forthcoming results.

In particular, for any $A\,,\,B \subset X$, the totalizations

$$\tilde{\Gamma}_A = \widetilde{\Gamma_A} \qquad \text{and} \qquad \tilde{\Gamma}_{(A,\,B)} = \widetilde{\Gamma_{(A,\,B)}}$$

may be called the Davis–Pervin and the Hunsaker–Lindgren relations on X, respectively.

The latter relations play an important role in the generalized uniformization of various topological structures such as proximities, closures, topologies, and filters, for instance. (See [2], [14], [21] and [1, pp. 42, 193], [6], [16].)

While, the relations $\tilde{F}$ can be used to prove a useful reduction theorem for the intersection convolution of relations [22]. The latter operation allows of a natural treatment of the Hahn-Banach type extension theorems. (See [17] and [5].)

Theorem 2.9. *If F is a relation on X to Y, then for any $U \subset X$ we have*

$$\tilde{F}[U] = \begin{cases} F[U] & \text{if} \quad U \subset D_F, \\ Y & \text{if} \quad U \not\subset D_F. \end{cases}$$

Proof. By Definition 2.7 and Theorems 1.10 and 2.5, we have

$$\tilde{F}[U] = \big(F \cup \Gamma_{(D_F^c, Y)}\big)[U] = F[U] \cup \Gamma_{(D_F^c, Y)}[U] = F[U] \cup \begin{cases} \emptyset & \text{if} \quad U \subset D_F, \\ Y & \text{if} \quad U \not\subset D_F. \end{cases}$$

Hence, it is clear that the required equality is also true.

Corollary 2.10. *If F is a relation on X to Y, then for some $U \subset X$ we have $F[U] = \tilde{F}[U]$ if and only if either $U \subset D_F$ or $F[U] = Y$.*

Corollary 2.11. *If F is a relation on X to Y, then for some $x \in X$ we have $F(x) = \tilde{F}(x)$ if and only if either $x \in D_F$ or $F(x) = Y$.*

From Theorem 2.9, it is clear that in particular we also have

Theorem 2.12. *If F is a relation on X to Y, then for any $x \in X$ we have*

$$\tilde{F}(x) = \begin{cases} F(x) & \text{if} \quad x \in D_F, \\ Y & \text{if} \quad x \notin D_F. \end{cases}$$

Corollary 2.13. *If F is a relation on X to Y, then $\tilde{F}$ is an extension of F such that $F = \tilde{F}$ if and only if $F(x) = Y$ for all $x \in D_F^c$.*

Corollary 2.14. *If F is a relation on X to Y and $Y \neq \emptyset$, then $F = \tilde{F}$ if and only if F is total.*

Theorem 2.15. *If $A \subset X$ and $B \subset Y$, then*

$$\tilde{\Gamma}_{(A, B)} = \begin{cases} \Gamma_{(X, Y)} & \text{if} \quad B = \emptyset, \\ \Gamma_{(A, B)} \cup \Gamma_{(A^c, Y)} & \text{if} \quad B \neq \emptyset. \end{cases}$$

Proof. If $B \neq \emptyset$, then by Theorem 2.3 it is clear that $A = D_{\Gamma_{(A, B)}}$. Hence, by Definition 2.7, we can see that $\tilde{\Gamma}_{(A, B)} = \Gamma_{(A, B)} \cup \Gamma_{(A^c, Y)}$.

While, if $B = \emptyset$, then we can note that $\Gamma_{(A, B)} = \Gamma_{(A, \emptyset)} = A \times \emptyset = \emptyset$. Hence, since $\emptyset = D_\emptyset$, we can already see that $\tilde{\Gamma}_{(A, B)} = \tilde{\emptyset} = \emptyset \cup \Gamma_{(\emptyset^c, Y)} = \Gamma_{(X, Y)}$.

Remark 2.16. Thus, in particular if A is a nonvoid subset of X, then

$$\tilde{\Gamma}_A = \Gamma_A \cup \Gamma_{(A^c, X)}.$$

Moreover, we can easily see that the latter equality is also true for $A = \emptyset$.

Theorem 2.17. *If* $A \subset X$ *and* $B \subset Y$ *such that* $B \neq \emptyset$ *, then for any* $U \subset X$ *, with* $U \neq \emptyset$ *, we have*

$$\tilde{\Gamma}_{(A,\,B)}\,[\,U\,] \,=\, \begin{cases} B & \textit{if} \quad U \subset A\,, \\ Y & \textit{if} \quad U \not\subset A\,. \end{cases}$$

Proof. Because of $B \neq \emptyset$ and Theorem 2.3, we have $A = D_{\Gamma_{(A,\,B)}}$. Now, by Theorem 2.9, we can see that

$$\tilde{\Gamma}_{(A,\,B)}\,[\,U\,] \,=\, \begin{cases} \Gamma_{(A,\,B)}\,[\,U\,] & \text{if} \quad U \subset A\,, \\ Y & \text{if} \quad U \not\subset A\,. \end{cases}$$

Moreover, if $U \subset A$, then because of $U \neq \emptyset$ we can note that $U \not\subset A^c$. Therefore, by Theorem 2.5, we have $\Gamma_{(A,\,B)}\,[\,U\,] \,=\, B$. Hence, it is clear that the required equality also is true.

Remark 2.18. Thus, in particular if A and U are nonvoid subsets of X, then

$$\tilde{\Gamma}_{A}\,[\,U\,] \,=\, \begin{cases} A & \text{if} \quad U \subset A\,, \\ X & \text{if} \quad U \not\subset A\,. \end{cases}$$

Moreover, we can easily see that the latter equality is also true for $A = \emptyset$.

By Theorem 2.17, it is clear that in particular we also have

Corollary 2.19. *If* $A \subset X$ *and* $B \subset Y$ *such that* $B \neq \emptyset$ *, then for any* $x \in X$ *we have*

$$\tilde{\Gamma}_{(A,\,B)}(x) = \begin{cases} B & \textit{if} \quad x \in A\,, \\ Y & \textit{if} \quad x \notin A\,. \end{cases}$$

Remark 2.20. Thus, in particular if A is a nonvoid subset of X and $x \in X$, then

$$\tilde{\Gamma}_{A}(x) = \begin{cases} A & \text{if} \quad x \in A\,, \\ X & \text{if} \quad x \notin A\,. \end{cases}$$

Moreover, we can easily see that the latter equality is also true for $A = \emptyset$.

Remark 2.21. Note that if $A \subset X$ and $\emptyset$ is considered as a subset of Y, then by Theorems 2.15 and 2.3 we have $\tilde{\Gamma}_{(A,\,\emptyset)}(x) = \Gamma_{(X,\,Y)}(x) = Y$ for all $x \in X$. Therefore, the assumption $B \neq \emptyset$ is indispensable in Corollary 2.19 and Theorem 2.17.

3. Inclusions on box relations

Theorem 3.1. *If* $A \subset X$ *and* $B \subset Y$*, then for any* $U \subset X$ *and* $V \subset Y$ *the following assertions are equivalent:*

(1) $\Gamma_{(A,\,B)}\,[\,U\,] \subset V$ *;* *(2)* $U \subset A^c$ *or* $B \subset V$ *.*

Proof. If $U \not\subset A^c$, then by Theorem 2.5 we have $\Gamma_{(A,\,B)}\,[\,U\,] \,=\, B$. Hence, it is clear that (1) implies (2). By Theorem 2.5, the converse implication is even more obvious.

Remark 3.2. Thus, in particular if $A \subset X$, then for any $U, V \subset X$ we have $\Gamma_A[U] \subset V$ if and only if either $U \subset A^c$ or $A \subset V$.

Corollary 3.3. *If $A \subset X$ and $B \subset Y$, then for any $x \in X$ and $V \subset Y$ the following assertions are equivalent:*

(1) $\Gamma_{(A,B)}(x) \subset V$ *;* *(2)* $x \notin A$ *or* $B \subset V$ *.*

Remark 3.4. Thus, in particular if $A \subset X$, then for any $x \in X$ and $V \subset X$ we have $\Gamma_A(x) \subset V$ if and only if either $x \notin A$ or $A \subset V$.

Theorem 3.5. *If $A \subset X$ and $B \subset Y$, then for any relation F on X to Y the following assertions are equivalent:*

(1) $\Gamma_{(A,B)} \subset F$ *;* *(2)* $B \subset F^c[A]^c$*;*

(3) for any $x \in A$ we have $B \subset F(x)$;

(4) for any $x \in X$ we have either $x \notin A$ or $B \subset F(x)$.

Proof. By Corollaries 1.3 and 3.3 and Theorem 1.14, we can see that

$$\begin{aligned}
\Gamma_{(A,B)} \subset F \iff \forall\, x \in X: \quad \Gamma_{(A,B)}(x) \subset F(x) \\
\iff \forall\, x \in X: \quad x \notin A \quad \text{or} \quad B \subset F(x) \iff \forall\, x \in A: \quad B \subset F(x) \\
\iff B \subset \bigcap_{x \in A} F(x) \iff B \subset F^c[A]^c.
\end{aligned}$$

Remark 3.6. Thus, in particular if $A \subset X$, then for any relation F on X we have $\Gamma_A \subset F$ if and only $A \subset F^c[A]^c$, or equivalently $A \subset F(x)$ for all $x \in A$.

Analogously to Theorem 3.1, we can also easily establish the following

Theorem 3.7. *If $A \subset X$ and $B \subset Y$, then for any $U \subset X$ and $V \subset Y$, with $V \neq \emptyset$, the following assertions are equivalent:*

(1) $V \subset \Gamma_{(A,B)}[U]$ *;* *(2)* $U \not\subset A^c$ *and* $V \subset B$ *.*

Proof. If (1) holds, then because of $V \neq \emptyset$ and Theorem 2.5 we necessarily have $U \not\subset A^c$ and $V \subset \Gamma_{(A,B)}[U] = B$. Therefore, (2) also holds. By Theorem 2.5, the converse implications is even more obvious.

Remark 3.8. Thus, in particular if $A \subset X$, then for any $U \subset X$ and $V \subset Y$, with $V \neq \emptyset$, we have $V \subset \Gamma_A[U]$ if and only if $U \not\subset A^c$ and $V \subset A$.

Corollary 3.9. *If $A \subset X$ and $B \subset Y$, then for any $x \in X$ and $V \subset Y$, with $V \neq \emptyset$, the following assertions are equivalent:*

(1) $V \subset \Gamma_{(A,B)}(x)$ *;* *(2)* $x \in A$ *and* $V \subset B$ *.*

Remark 3.10. Thus, in particular if $A \subset X$, then for any $x \in X$ and $V \subset Y$, with $V \neq \emptyset$, we have $V \subset \Gamma_A(x)$ if and only if $x \in A$ and $V \subset A$.

Theorem 3.11. *If* $A \subset X$ *and* $B \subset Y$, *then for any relation* F *on* X *to* Y *the following assertions are equivalent:*

(1) $F \subset \Gamma_{(A,\,B)}$ *;* *(2)* $D_F \subset A$ *and* $R_F \subset B$ *;*

(3) for any $x \in D_F$ *we have* $x \in A$ *and* $F(x) \subset B$.

Proof. By Corollaries 1.3 and 3.9, we can see that

$$F \subset \Gamma_{(A,\,B)} \iff \forall\ x \in D_F : \quad F(x) \subset \Gamma_{(A,\,B)}(x)$$
$$\iff \forall\ x \in D_F : \quad x \in A \quad \text{and} \quad F(x) \subset B \iff D_F \subset A \quad \text{and} \quad R_F \subset B\,.$$

Remark 3.12. Thus, in particular if $A \subset X$, then for any relation F on X we have $F \subset \Gamma_A$ if and only if $D_F \cup R_F \subset A$, or equivalently for any $x \in D_F$ we have $x \in A$ and $F(x) \subset A$.

Now, as an immediate consequence of Theorems 3.1 and 3.7, we can also state

Theorem 3.13. *If* $A \subset X$ *and* $B \subset Y$, *then for any* $U \subset X$ *and* $V \subset Y$, *with* $V \neq \emptyset$, *the following assertions are equivalent:*

(1) $V = \Gamma_{(A,\,B)}[\,U\,]$ *;* *(2)* $U \not\subset A^c$ *and* $B = V$.

Proof. If (1) holds, then by Theorem 3.7 we have $U \not\subset A^c$ and $V \subset B$. Hence, by Theorem 3.1, it is clear that $B \subset V$ also holds. Thus, (1) implies (2). By the above mentioned theorems, the converse implication is even more obvious.

Remark 3.14. Thus, in particular if $A \subset X$, then for any $U \subset X$ and $V \subset Y$, with $V \neq \emptyset$, we have $V = \Gamma_A[\,U\,]$ if and only if $U \not\subset A^c$ and $A = V$.

Corollary 3.15. *If* $A \subset X$ *and* $B \subset Y$, *then for any* $x \in X$ *and* $V \subset Y$, *with* $V \neq \emptyset$, *the following assertions are equivalent:*

(1) $V = \Gamma_{(A,\,B)}(x)$ *;* *(2)* $x \in A$ *and* $B = V$.

Remark 3.16. Thus, in particular if $A \subset X$, then for any $x \subset X$ and $V \subset Y$, with $V \neq \emptyset$, we have $V = \Gamma_A(x)$ if and only if $x \in A$ and $A = V$.

In principle, the following theorem can also be proved with the help of Corollaries 1.4 and 3.15. However, it can now be, more easily, proved with the help of Theorems 3.5 and 3.11.

Theorem 3.17. *If* $A \subset X$ *and* $B \subset Y$, *then for any relation* F *on* X *to* Y *the following assertions are equivalent:*

(1) $F = \Gamma_{(A,\,B)}$ *;* *(2)* $D_F \subset A$ *and* $R_F \subset B \subset F^c[\,A\,]^c$*;*

(3) for any $x \in D_F$ *we have* $x \in A$, *and for any* $x \in A$ *we have* $F(x) = B$.

Proof. By Theorems 3.11 and 3.5, it is clear that (1) and (2) are equivalent. On the other hand, if (1) holds, then by Theorem 3.11, for any $x \in D_F$, we have $x \in A$, and for any $x \in X$ we have $F(x) \subset B$. Moreover, by Theorem 3.5, for any $x \in A$ we also have $B \subset F(x)$. Therefore, (3) also holds. By the above mentioned theorems, the converse implication is even more obvious.

Remark 3.18. Thus, in particular if $A \subset X$, then for any relation F on X we have $F = \Gamma_A$ if and only if $D_F \cup R_F \subset A \subset F^c\,[A]^c$, or equivalently for any $x \in D_F$ we have $x \in A$, and for any $x \in A$ we have $F(x) = A$.

By using Theorems 3.1 and 3.7, we can also easily prove the following

Theorem 3.19. *If $A, C \subset X$ and $B, D \subset Y$ such that $B \neq \emptyset$, then for any $U, V \subset X$ the following assertions are equivalent:*

(1) $\Gamma_{(A,B)}[U] \subset \Gamma_{(C,D)}[V]$; *(2)* $U \subset A^c$ *or* $(V \not\subset C^c$ *and* $B \subset D)$.

Proof. By Theorem 3.1, we have (1) if and only if $U \subset A^c$ or $B \subset \Gamma_{(C,D)}[V]$. Moreover, by Theorem 3.7, the latter inclusion holds if and only if $V \not\subset C^c$ and $B \subset D$. Therefore, the equivalence of (1) and (2) is also true.

Remark 3.20. Thus, in particular if $A, B \subset X$ such that $A \neq \emptyset$, then for any $U, V \subset X$ we have $\Gamma_A[U] \subset \Gamma_B[V]$ if and only if either $U \subset A^c$ or $(V \not\subset B^c$ and $A \subset B)$.

Corollary 3.21. *If $A, C \subset X$ and $B, D \subset Y$ such that $B \neq \emptyset$, then for any $x, y \in X$ the following assertions are equivalent:*

(1) $\Gamma_{(A,B)}(x) \subset \Gamma_{(C,D)}(y)$; *(2)* $x \notin A$ *or* $\big(y \in C$ *and* $B \subset D\big)$.

Remark 3.22. Thus, in particular if $A, B \subset X$ such that $A \neq \emptyset$, then for any $x, y \in X$ we have $\Gamma_A(x) \subset \Gamma_B(y)$ if and only if either $x \notin A$ or $(y \in B$ and $A \subset B)$.

Theorem 3.23. *If $A, C \subset X$ and $B, D \subset Y$ such that $A \neq \emptyset$ and $B \neq \emptyset$, then the following assertions are equivalent:*

(1) $\Gamma_{(A,B)} \subset \Gamma_{(C,D)}$; *(2)* $A \subset C$ *and* $B \subset D$.

Proof. By Theorem 3.11, we have (1) if and only if $D_{\Gamma_{(A,B)}} \subset C$ and $R_{\Gamma_{(A,B)}} \subset D$. Moreover, by Theorem 2.3, we now also have $A = D_{\Gamma_{(A,B)}}$ and $B = R_{\Gamma_{(A,B)}}$. Therefore, (1) and (2) are also equivalent.

Remark 3.24. Thus, in particular for any $A, B \subset X$, with $A \neq \emptyset$, we have $\Gamma_A \subset \Gamma_B$ if and only if $A \subset B$.

Theorem 3.25. *If $A, C \subset X$ and B and D are nonvoid subsets of Y, then for any $U, V \subset X$ the following assertions are equivalent:*

(1) $\Gamma_{(A,B)}[U] = \Gamma_{(C,D)}[V]$;

(2) $(U \subset A^c$ *and* $V \subset C^c)$ *or* $(U \not\subset A^c,\ V \not\subset C^c$ *and* $B = D)$.

Proof. By Theorem 3.19, we have

$$\Gamma_{(A,B)}[U] \subset \Gamma_{(C,D)}[V] \iff U \subset A^c \quad \text{or} \quad (V \not\subset C^c \ \text{ and } \ B \subset D)$$

and

$$\Gamma_{(C,D)}[V] \subset \Gamma_{(A,B)}[U] \iff V \subset C^c \quad \text{or} \quad (U \not\subset A^c \ \text{ and } \ D \subset B).$$

Hence, it is clear that the equivalence (1) and (2) is also true.

Remark 3.26. Thus, in particular if A and B are nonvoid subsets of X, then for any $U, V \subset X$ we have $\Gamma_A[U] = \Gamma_B[V]$ if and only if either $(U \subset A^c$ and $V \subset B^c)$ or $(U \not\subset A^c,\ V \not\subset B^c$ and $A = B)$.

Corollary 3.27. *If $A, C \subset X$ and B and D are nonvoid subsets of Y, then for any $x, y \in X$ the following assertions are equivalent:*

(1) $\Gamma_{(A,B)}(x) = \Gamma_{(C,D)}(y)$*;*

(2) $(x \notin A$ *and* $y \notin C)$ *or* $(x \in A,\ y \in C$ *and* $B = D)$.

Remark 3.28. Thus, in particular if A and B are nonvoid subsets of X, then for any $x, y \in X$ we have $\Gamma_A(x) = \Gamma_B(y)$ if and only if either $(x \notin A$ and $y \notin B)$ or $(x \in A,\ y \in B$ and $A = B)$.

Finally, as an immediate consequence of Theorem 3.23, we can also state

Theorem 3.29. *If A and C are nonvoid subsets of X and B and D are novoid subsets of Y, then the following assertions are equivalent:*

(1) $\Gamma_{(A,B)} = \Gamma_{(C,D)}$*;* *(2)* $A = C$ *and* $B = D$.

Remark 3.30. Thus, in particular for any nonvoid subsets A and B of X we have $\Gamma_A = \Gamma_B$ if and only if $A = B$.

4. Inclusions on totalization relations

Theorem 4.1. *If F is a relation on X to Y, then for any $U \subset X$ and $V \subset Y$, with $V \neq Y$, the following assertions are equivalent:*

(1) $\tilde{F}[U] \subset V$*;* *(2)* $U \subset D_F$ *and* $F[U] \subset V$.

Proof. If (1) holds, then by Theorem 2.9 and the assumption $V \neq Y$, it is clear that $U \subset D_F$ and $F[U] = \tilde{F}[U] \subset V$. Therefore, (2) also holds. By Theorem 2.9, the converse implication is even more obvious.

Corollary 4.2. *If F is a relation on X to Y, then for any $x \in X$ and $V \subset Y$, with $V \neq Y$, the following assertions are equivalent:*

(1) $\tilde{F}(x) \subset V$*;* *(2)* $x \in D_F$ *and* $F(x) \subset V$.

Theorem 4.3. *For any relations F and G on X to Y, the following assertions are equivalent:*

(1) $\tilde{F} \subset G$*;*

(2) for any $x \in X$, with $G(x) \neq Y$, we have $x \in D_F$ and $F(x) \subset G(x)$;

(3) for any $x \in X$ we have either $G(x) = Y$ or $\big(x \in D_F$ and $F(x) \subset G(x)\big)$;

(4) for any $x \in D_F$ we have $F(x) \subset G(x)$, and for any $x \in D_F^c$ we have $G(x) = Y$.

Proof. Define $A = \{ x \in X : \quad G(x) \neq Y \}$. Then, by Corollaries 1.3 and 4.2, we can see that

$$\tilde{F} \subset G \iff \forall\ x \in X : \quad \tilde{F}(x) \subset G(x)$$

$$\iff \forall\ x \in A : \quad \tilde{F}(x) \subset G(x) \iff \forall\ x \in A : \quad x \in D_F \text{ and } F(x) \subset G(x).$$

Therefore, (1) and (2) are equivalent. Moreover, we can easily check that (2) is equivalent to (3), and (3) is equivalent to (4).

Analogously to Theorem 4.1, we can also easily establish the following

Theorem 4.4. *If F is a relation on X to Y, then for any $U \subset X$ and $V \subset Y$ the following assertions are equivalent:*

(1) $V \subset \tilde{F}[U]$; *(2)* $U \not\subset D_F$ *or* $V \subset F[U]$.

Proof. If $U \subset D_F$, then by Theorem 2.9 we have $\tilde{F}[U] = F[U]$. Hence, it is clear that (1) implies (2). By Theorem 2.9, the converse implication is also quite obvious.

Corollary 4.5. *If F is a relation on X to Y, then for any $x \in X$ and $V \subset Y$ the following assertions are equivalent:*

(1) $V \subset \tilde{F}(x)$; *(2)* $x \notin D_F$ *or* $V \subset F(x)$.

Theorem 4.6. *For any relations F and G on X to Y, the following assertions are equivalent:*

(1) $G \subset \tilde{F}$;

(2) for any $x \in D_F$ *we have* $G(x) \subset F(x)$;

(3) for any $x \in D_F \cap D_G$ *we have* $G(x) \subset F(x)$;

(4) for any $x \in X$ *we have either* $x \notin D_F$ *or* $G(x) \subset F(x)$;

(5) for any $x \in D_G$ *we have either* $x \notin D_F$ *or* $G(x) \subset F(x)$.

Proof. By Corollaries 1.3 and 4.5, it is clear that

$$G \subset \tilde{F} \iff \forall\ x \in D_G : \quad G(x) \subset \tilde{F}(x)$$

$$\iff \forall\ x \in D_G : \quad x \notin D_F \text{ or } G(x) \subset F(x).$$

Therefore, (1) and (5) are equivalent. Moreover, we can easily see that (2) is equivalent to both (3) and (4), and (4) is equivalent to (5).

Now, as an immediate consequence of Theorems 4.1 and 4.4, we can also state

Theorem 4.7. *If F is a relation on X to Y, then for any $U \subset X$ and $V \subset Y$, with $V \neq Y$, the following assertions are equivalent:*

(1) $V = \tilde{F}[U]$; *(2)* $U \subset D_F$ *and* $V = F[U]$.

Proof. Namely, by Theorem 4.1 and 4.4, we have

$$\tilde{F}[U] \subset V \iff U \subset D_F \text{ and } F[U] \subset V$$

and

$$V \subset \tilde{F}[U] \iff U \not\subset D_F \text{ or } V \subset F[U].$$

Hence, it is clear that (1) and (2) are also equivalent.

Corollary 4.8. *If F is a relation on X to Y, then for any $x \in X$ and $V \subset Y$, with $V \neq Y$, the following assertions are equivalent:*

(1) $V = \tilde{F}(x)$; *(2)* $x \in D_F$ *and* $V = F(x)$.

Now, as an immediate consequence of Theorems 4.3 and 4.6, we can also state

Theorem 4.9. *For any relations F and G on X to Y, the following assertions are equivalent:*

(1) $\tilde{F} = G$;

(2) for any $x \in D_F$ we have $F(x) = G(x)$, and for any $x \in D_F^{c}$, we have $G(x) = Y$;

(3) for any $x \in X$ we have $\big(x \notin D_F$ and $G(x) = Y \big)$ or $\big(F(x) = Y$ and $G(x) = Y \big)$ or $\big(x \in D_F$ and $F(x) = G(x) \big)$.

However, it is now more interesting to note that, by using Theorems 4.1 and 4.4, we can also easily prove the following

Theorem 4.10. *If F and G are relations on X to Y, then for any $U \subset X$ and $V \subset Y$, with $G[V] \neq Y$, the following assertions are equivalent:*

(1) $\tilde{F}[U] \subset \tilde{G}[V]$; *(2)* $V \not\subset D_G$ *or* $\big(U \subset D_F$ *and* $F[U] \subset G[V] \big)$.

Proof. By Theorem 4.4, we have

$$\tilde{F}[U] \subset \tilde{G}[V] \quad \Longleftrightarrow \quad V \not\subset D_G \quad \text{or} \quad \tilde{F}[U] \subset G[V] .$$

Moreover, by Theorem 4.4, we have

$$\tilde{F}[U] \subset G[V] \quad \Longleftrightarrow \quad U \subset D_F \quad \text{and} \quad F[U] \subset G[V] .$$

Therefore, the equivalence of (1) and (2) is also true.

Corollary 4.11. *If F and G are relations on X to Y, then for any $x, y \in X$, with $G(y) \neq Y$, the following assertions are equivalent:*

(1) $\tilde{F}(x) \subset \tilde{G}(y)$; *(2)* $y \notin D_G$ *or* $(x \in D_F$ *and* $F(x) \subset G(y))$.

Now, analogously to Theorem 4.3, we can also easily establish the following

Theorem 4.12. *For any relations F and G on X to Y, the following assertions are equivalent:*

(1) $\tilde{F} \subset \tilde{G}$;

(2) for any $x \in X$, with $G(x) \neq Y$, we have either $x \notin D_G$ or $\big(x \in D_F$ and $F(x) \subset G(x) \big)$;

(3) for any $x \in X$ we have $G(x) = Y$ or $x \notin D_G$ or $\big(x \in D_F$ and $F(x) \subset G(x) \big)$;

(4) for any $x \in D_F \cap D_G$ we have $F(x) \subset G(x)$, and for any $x \in D_G \setminus D_F$ we have $G(x) = Y$.

Moreover, as an immediate consequence of Theorem 4.10, we can also state

Theorem 4.13. *If F and G are relations on X to Y, then for any $U \subset X$ and $V \subset Y$, with $F[U] \neq Y$ and $G[V] \neq Y$, the following assertions are equivalent:*

(1) $\tilde{F}[U] = \tilde{G}[V]$;

(2) $\big(U \not\subset D_F$ *and* $V \not\subset D_G \big)$ *or* $\big(U \subset D_F,\ V \subset D_G$ *and* $F[U] = G[V] \big)$.

Proof. By Theorem 4.10, we have

$$\tilde{F}[U] \subset \tilde{G}[V] \iff V \not\subset D_G \quad \text{or} \quad \big(U \subset D_F \quad \text{and} \quad F[U] \subset G[V] \big).$$

and

$$\tilde{G}[V] \subset \tilde{F}[U] \iff U \not\subset D_F \quad \text{or} \quad \big(V \subset D_G \quad \text{and} \quad G[V] \subset F[U] \big).$$

Hence, it is clear that the equivalence of (1) and (2) is also true.

Corollary 4.14. *If F and G are relations on X to Y, then for any $x, y \in X$, with $F(x) \neq Y$ and $G(y) \neq Y$, the following assertions are equivalent:*

(1) $\tilde{F}(x) = \tilde{G}(y)$;

(2) $\big(x \notin D_F$ *and* $y \notin D_G \big)$ *or* $\big(x \in D_F,\ y \in D_G$ *and* $F(x) = G(y) \big)$.

Theorem 4.15. *For any relations F and G on X to Y, the following assertions are equivalent:*

(1) $\tilde{F} = \tilde{G}$;

(2) *for any* $x \in D_F \cap D_G$ *we have* $F(x) = G(x)$, *for any* $x \in D_F \setminus D_G$ *we have* $F(x) = Y$, *and for any* $x \in D_G \setminus D_F$ *we have* $G(x) = Y$;

(3) *for any* $x \in X$ *we have* $x \notin D_F \cup D_G$ *or* $(x \notin D_F$ *and* $G(x) = Y)$ *or* $(x \notin D_G$ *and* $F(x) = Y)$ *or* $(x \in D_F \cap D_G$ *and* $F(x) = G(x))$.

Proof. By Theorem 4.12, it is clear that (1) and (2) are equivalent. Moreover, we can easily see that (2) and (3) are also equivalent.

5. Inclusions on totalizations of box relations

In principle, the following theorem can be naturally derived from Theorems 4.1 and 3.1. However, it can be more easily proved with the help of Theorem 2.17.

Theorem 5.1. *If $A \subset X$ and $B \subset Y$, then for any $U \subset X$ and $V \subset Y$, with $U \neq \emptyset$ and $V \neq Y$, the following assertions are equivalent:*

(1) $\tilde{\Gamma}_{(A,\,B)}[U] \subset V$; (2) $U \subset A$ *and* $\emptyset \neq B \subset V$.

Proof. By Theorems 2.15 and 2.3 and the condition $U \neq \emptyset$, it is clear that $\tilde{\Gamma}_{(A,\,\emptyset)}[U] = \Gamma_{(X,\,Y)}[U] = Y$. Therefore, if (1) holds, then because of $V \neq Y$ we necessarily have $B \neq \emptyset$. Hence, by Theorem 2.17 and the assumption $V \neq Y$, we can see that $U \subset A$ and $B = \tilde{\Gamma}_{(A,\,B)}[U] \subset V$. Therefore, (2) also holds. By Theorem 2.17, the converse implication is even more obvious.

Remark 5.2. Thus, in particular if $A \subset X$, then for any U, $V \subset X$, with $U \neq \emptyset$ and $V \neq X$, we have $\tilde{\Gamma}_A[U] \subset V$ if and only if $U \subset A \subset V$.

Corollary 5.3. *If $A \subset X$ and $B \subset Y$, then for any $x \in X$ and $V \subset Y$, with $V \neq Y$, the following assertions are equivalent:*

(1) $\tilde{\Gamma}_{(A,B)}(x) \subset V$; *(2)* $x \in A$ *and* $\emptyset \neq B \subset V$.

Remark 5.4. Thus, in particular if $A \subset X$, then for any $x \in X$ and $V \subset X$, with $V \neq X$, we have $\tilde{\Gamma}_A(x) \subset V$ if and only if $x \in A \subset V$.

Theorem 5.5. *If $A \subset X$ and $B \subset Y$ such that $B \neq \emptyset$, then for any relation F on X on Y the following assertions are equivalent:*

(1) $\tilde{\Gamma}_{(A,B)} \subset F$; *(2)* $B \subset F^c[A]^c$ *and* $Y = F^c[A^c]^c$;

(3) for any $x \in X$, with $F(x) \neq Y$, we have $x \in A$ and $B \subset F(x)$;

(4) for any $x \in X$ we have either $F(x) = Y$ or $\big(x \in A$ and $B \subset F(x)\big)$;

(5) for any $x \in A$ we have $B \subset F(x)$, and for any $x \in A^c$, we have $F(x) = Y$.

Proof. By Corollaries 1.3 and 5.3, and the proof of Theorem 4.3, it is clear that (1) and (3) are equivalent. Moreover, we can easily see that (3) is equivalent to (4), and (4) is equivalent to (5). Finally, by Theorem 1.14, it is clear that (2) is only a concise reformulation of (5).

Remark 5.6. Thus, in particular if A is a nonvoid subset of X, then for any relation F on X we have $\tilde{\Gamma}_A \subset F$ if and only if $A \subset F^c[A]^c$ and $X = F^c[A^c]^c$, or equivalently for any $x \in X$, with $F(x) \neq X$, we have $x \in A$ and $A \subset F(x)$.

Analogously to Theorem 5.1, we can also easily prove the following

Theorem 5.7. *If $A \subset X$ and $B \subset Y$ such that $B \neq \emptyset$, then for any $U \subset X$ and $V \subset Y$, with $U \neq \emptyset$, the following assertions are equivalent:*

(1) $V \subset \tilde{\Gamma}_{(A,B)}[U]$; *(2)* $U \not\subset A$ *or* $V \subset B$.

Proof. If $U \subset A$, then by Theorem 2.17 we have $\tilde{\Gamma}_{(A,B)}[U] = B$. Hence, it is clear that (1) implies (2). By Theorem 2.17, the converse implication is also quite obvious.

Remark 5.8. Thus, in particular if A is a nonvoid subset of X, then for any U, $V \subset X$, with $U \neq \emptyset$, we have $V \subset \tilde{\Gamma}_A[U]$ if and only if either $U \not\subset A$ or $V \subset A$.

Corollary 5.9. *If $A \subset X$ and $B \subset Y$ such that $B \neq \emptyset$, then for any $x \in X$ and $V \subset Y$ the following assertions are equivalent:*

(1) $V \subset \tilde{\Gamma}_{(A,B)}(x)$; *(2)* $x \notin A$ *or* $V \subset B$.

Remark 5.10. Thus, in particular if A is a nonvoid subset of X, then for any $x \in X$ and $V \subset X$ we have $V \subset \tilde{\Gamma}_A(x)$ if and only if either $x \notin A$ or $V \subset A$.

Now, as an immediate consequence of Corollaries 1.3 and 5.9, we can also state

Theorem 5.11. *If $A \subset X$ and $B \subset Y$ such that $B \neq \emptyset$, then for any relation F on X on Y the following assertions are equivalent:*

(1) $F \subset \tilde{\Gamma}_{(A,\,B)}$; *(2)* $F\,[\,A\,] \subset B$;

(3) for any $x \in A$ *we have* $F(x) \subset B$;

(4) for any $x \in X$ *we have either* $x \notin A$ *or* $F(x) \subset B$.

Remark 5.12. Thus, in particular if A is a nonvoid subset of X, then for any relation F on X we have $F \subset \tilde{\Gamma}_A$ if and only if $F\,[\,A\,] \subset A$, or equivalently $F(x) \subset A$ for all $x \in A$.

Now, as an immediate consequence of Theorems 5.1 and 5.7, we can also easily establish

Theorem 5.13. *If $A \subset X$ and $B \subset Y$ such that $B \neq \emptyset$, then for any $U \subset X$ and $V \subset Y$, with $U \neq \emptyset$ and $V \neq Y$, the following assertions are equivalent:*

(1) $V = \tilde{\Gamma}_{(A,\,B)}\,[\,U\,]$; *(2)* $U \subset A$ *and* $B = V$.

Proof. If (1) holds, then by Theorem 5.1, we have $U \subset A$ and $B \subset V$. Hence, by Theorem 5.7, it is clear that $V \subset B$ also holds. Thus, (1) implies (2). By the the above mentioned theorems, the converse implication is even more obvious.

Remark 5.14. Thus, in particular if A is a nonvoid subset of X, then for any $U\,,\,V \subset X$, with $U \neq \emptyset$ and $V \neq Y$, we have $V = \tilde{\Gamma}_A\,[\,U\,]$ if and only if $U \subset A$ and $A = V$.

Corollary 5.15. *If $A \subset X$ and $B \subset Y$ such that $B \neq \emptyset$, then for any $x \in X$ and $V \subset Y$, with $V \neq Y$, the following assertions are equivalent:*

(1) $V = \tilde{\Gamma}_{(A,\,B)}(x)$; *(2)* $x \in A$ *and* $B = V$.

Remark 5.16. Thus, in particular if A is a nonvoid subset of X, then for any $x \in X$ and $V \subset X$, with $V \neq X$, we have $V = \tilde{\Gamma}_A(x)$ if and only if $x \in A$ and $A = V$.

Now, as an immediate consequence of Theorems 5.5 and 5.11, we can also state

Theorem 5.17. *If $A \subset X$ and $B \subset Y$ such that $B \neq \emptyset$, then for any relation F on X on Y the following assertions are equivalent:*

(1) $F = \tilde{\Gamma}_{(A,\,B)}$;

(2) $F\,[\,A\,] \subset B \subset F^c[\,A\,]^c$ *and* $Y = F^c[\,A^c]^c$;

(3) for any $x \in A$ *we have* $F(x) = B$, *and for any* $x \in A^c$ *we have* $F(x) = Y$

(4) for $x \in X$ *we have* $\big(x \notin A$ *and* $F(x) = Y \big)$ *or* $\big(F(x) = Y$ *and* $B = Y \big)$ *or* $\big(x \in A$ *and* $F(x) = B \big)$.

Remark 5.18. Thus, in particular if A is a nonvoid subset of X, then for any relation F on X we have $F = \tilde{\Gamma}_A$ if and only if $F\,[\,A\,] \subset A \subset F^c[\,A\,]^c$ and $X = F^c[\,A^c]^c$, or equivalently for any $x \in A$ we have $F(x) = A$, and for any $x \in A^c$ we have $F(x) = X$.

By using Theorems 5.1 and 5.7, we can also easily prove the following

Theorem 5.19. *If $A, C \subset X$ and B and D are nonvoid subsets of Y such that $D \neq Y$, then for any nonvoid subsets U and V of X the following assertions are equivalent:*

(1) $\tilde{\Gamma}_{(A,B)}[U] \subset \tilde{\Gamma}_{(C,D)}[V]$*;* *(2)* $V \not\subset C$ *or* $\big(U \subset A$ *and* $B \subset D\big)$.

Proof. By Theorem 5.7, we have

$$V \subset \tilde{\Gamma}_{(A,B)}[U] \iff U \not\subset A \quad \text{or} \quad V \subset B.$$

Moreover, by Theorem 5.1, we have

$$\tilde{\Gamma}_{(A,B)}[U] \subset V \iff U \subset A \quad \text{and} \quad B \subset V.$$

Therefore, (1) and (2) are also equivalent.

Remark 5.20. Thus, if in particular A and B are nonvoid subsets of X such that $B \neq X$, then for any nonvoid subsets U and V of X we have $\tilde{\Gamma}_A[U] \subset \tilde{\Gamma}_B[V]$ if and only if either $V \not\subset B$ or $\big(U \subset A$ and $A \subset B\big)$.

Corollary 5.21. *If $A, C \subset X$ and B and D are nonvoid subsets of Y such that $D \neq Y$, then for any $x, y \in X$ the following assertions are equivalent:*

(1) $\tilde{\Gamma}_{(A,B)}(x) \subset \tilde{\Gamma}_{(C,D)}(y)$*;* *(2)* $y \notin C$ *or* $\big(x \in A$ *and* $B \subset D\big)$.

Remark 5.22. Thus, if in particular A and B are nonvoid subsets of X such that $B \neq X$, then for any $x, y \in X$ we have $\tilde{\Gamma}_A(x) \subset \tilde{\Gamma}_B(y)$ if and only if either $y \notin B$ or $\big(x \in A$ and $A \subset B\big)$.

Theorem 5.23. *If $A, C \subset X$ such that $C \neq \emptyset$ and B and D are nonvoid subset of Y such that $D \neq Y$, then the following assertions are equivalent:*

(1) $\tilde{\Gamma}_{(A,B)} \subset \tilde{\Gamma}_{(C,D)}$*;* *(2)* $C \subset A$ *and* $B \subset D$*;*

Proof. By Corollaries 1.3 and 5.21, we can see that

$$\begin{aligned}\tilde{\Gamma}_{(A,B)} \subset \tilde{\Gamma}_{(C,D)} &\iff \forall\ x \in X: \ \tilde{\Gamma}_{(A,B)}(x) \subset \tilde{\Gamma}_{(C,D)}(x)\\ &\iff \forall\ x \in X: \ x \notin C \quad \text{or} \quad \big(x \in A \quad \text{and} \quad B \subset D\big)\end{aligned}$$

even if $C = \emptyset$. Hence, it is clear that now (1) and (2) are also equivalent.

Remark 5.24. Thus, if in particular $A, B \subset X$ such that $\emptyset \neq B \neq X$, then $\tilde{\Gamma}_A \subset \tilde{\Gamma}_B$ if and only if $A = B$.

Namely, because of $\tilde{\Gamma}_\emptyset = X^2$ and $\tilde{\Gamma}_B \neq X^2$, the inclusion $\tilde{\Gamma}_A \subset \tilde{\Gamma}_B$ implies that $A \neq \emptyset$. (For some finer statements, see Levine [8, p. 99].)

Theorem 5.25. *If $A, C \subset X$ and B and D are proper, nonvoid subsets of Y, then for any nonvoid subsets U and V of X the following assertions are equivalent:*

(1) $\tilde{\Gamma}_{(A,B)}[U] = \tilde{\Gamma}_{(C,D)}[V]$*;*

(2) $\big(U \not\subset A$ *and* $V \not\subset C\big)$ *or* $\big(U \subset A,\ V \subset C$ *and* $B = D\big)$.

Proof. By Theorem 5.19, we have

$$\tilde{\Gamma}_{(A,B)}[U] \subset \tilde{\Gamma}_{(C,D)}[V] \iff V \not\subset C \quad \text{or} \quad \big(U \subset A \quad \text{and} \quad B \subset D\big)$$

and

$$\tilde{\Gamma}_{(C,D)}[V] \subset \tilde{\Gamma}_{(A,B)}[U] \iff U \not\subset A \quad \text{or} \quad \big(V \subset C \quad \text{and} \quad D \subset B\big).$$

Hence, it is clear that the required assertions are also equivalent.

Remark 5.26. Thus, in particular if A and B are proper, nonvoid subsets of X, then any nonvoid subsets U and V of X we have $\tilde{\Gamma}_A[U] = \tilde{\Gamma}_B[V]$ if and only if either $\big(U \not\subset A$ and $V \not\subset B\big)$ or $\big(U \subset A,\ V \subset B$ and $A = B\big)$.

Corollary 5.27. *If $A, C \subset X$ and B and D are proper, nonvoid subsets of Y, then for any $x, y \in X$ the following assertions are equivalent:*

(1) $\tilde{\Gamma}_{(A,B)}(x) = \tilde{\Gamma}_{(C,D)}(y)$ *;*

(2) $\big(x \notin A$ *and* $y \notin C\big)$ *or* $\big(x \in A,\ y \in C$ *and* $B = D\big)$.

Remark 5.28. Thus, in particular if A and B are proper, nonvoid subsets of X, then for any $x, y \in X$ we have $\tilde{\Gamma}_A(x) = \tilde{\Gamma}_B(y)$ if and only if either $\big(x \notin A$ and $y \notin B\big)$ or $\big(x \in A,\ y \in B$ and $A = B\big)$

Now, as an immediate consequence of Theorem 5.23, we can also state

Theorem 5.29. *If $A, C \subset X$ and B and D are proper, nonvoid subsets of Y, then the following assertions are equivalent:*

(1) $\tilde{\Gamma}_{(A,B)} = \tilde{\Gamma}_{(C,D)}$ *;* *(2)* $A = C$ *and* $B = D$.

Remark 5.30. Thus, in particular for any proper, nonvoid subsets A and B of X, we have $\tilde{\Gamma}_A = \tilde{\Gamma}_B$ if and only if $A = B$.

6. Further inclusions on totalization relations

Theorem 6.1. *If $A \subset X$ and $B \subset Y$ such that $B \neq \emptyset$, and F is a relation on X to Y, then for any $U, V \subset X$, with $U \neq \emptyset$ and $F[V] \neq Y$, the following assertions are equivalent:*

(1) $\tilde{\Gamma}_{(A,B)}[U] \subset \tilde{F}[V]$ *;* *(2)* $V \not\subset D_F$ *or* $\big(U \subset A$ *and* $B \subset F[V]\big)$.

Proof. By Theorem 4.4, we have

$$\tilde{\Gamma}_{(A,B)}[U] \subset \tilde{F}[V] \iff V \not\subset D_F \quad \text{or} \quad \tilde{\Gamma}_{(A,B)}[U] \subset F[V].$$

Moreover, by Theorem 5.1, we have

$$\tilde{\Gamma}_{(A,B)}[U] \subset F[V] \iff U \subset A \quad \text{and} \quad B \subset F[V].$$

Therefore, (1) and (2) are also equivalent.

Remark 6.2. Thus, in particular if A is a nonvoid subset of X and F is a relation on X, then for any $U, V \subset X$, with $U \neq \emptyset$ and $F[V] \neq X$, we have $\tilde{\Gamma}_A[U] \subset \tilde{F}[V]$ if and only if $V \not\subset D_F$ or $U \subset A \subset F[V]$.

Corollary 6.3. *If $A \subset X$ and $B \subset Y$ such that $B \neq \emptyset$, and F is a relation on X to Y, then for any $x, y \in X$, with $F(y) \neq Y$, the following assertions are equivalent:*

(1) $\tilde{\Gamma}_{(A,B)}(x) \subset \tilde{F}(y)$ *;* *(2)* $y \notin D_F$ *or* $\big(x \in A$ *and* $B \subset F(y)\big)$.

Remark 6.4. Thus, in particular if A is a nonvoid subset of X and F is a relation on X, then for any $x, y \in X$, with $F(y) \neq X$, we have $\tilde{\Gamma}_A(x) \subset \tilde{F}(y)$ if and only if $y \notin D_F$ or $x \in A \subset F(y)$.

Theorem 6.5. *If $A \subset X$ and $B \subset Y$ such that $B \neq \emptyset$ then for any relation F on X to Y the following assertions are equivalent:*

(1) $\tilde{\Gamma}_{(A,B)} \subset \tilde{F}$;

(2) for any $x \in D_F$ we have either $F(x) = Y$ or $\big(x \in A$ and $B \subset F(x)\big)$;

(3) for any $x \in X$ we have $x \notin D_F$ or $F(x) = Y$ or $\big(x \in A$ and $B \subset F(x)\big)$;

(4) for any $x \in X$, with $F(x) \neq Y$, we have either $x \notin D_F$ or $\big(x \in A$ and $B \subset F(x)\big)$.

Proof. Define $A = \big\{ x \in X : \ F(x) \neq Y \big\}$. Then, by Corollaries 1.3 and 6.3, we can see that

$$\tilde{\Gamma}_{(A,B)} \subset \tilde{F} \iff \forall\ x \in X : \ \tilde{\Gamma}_{(A,B)}(x) \subset \tilde{F}(x) \iff$$
$$\forall\ x \in A : \ \tilde{\Gamma}_{(A,B)}(x) \subset \tilde{F}(x) \iff x \notin D_F \quad \text{or} \quad \big(x \in A \quad \text{and} \quad B \subset F(x)\big).$$

Therefore, (1) and (4) are equivalent. Moreover, we can easily see that (3) is equivalent to both (2) and (4).

Remark 6.6. Thus, in particular if A is a nonvoid subset of X, then for any relation F on X we have $\tilde{\Gamma}_A \subset \tilde{F}$ if and only if for any $x \in X$, with $F(x) \neq X$, we have either $x \notin D_F$ or $x \in A \subset F(x)$.

Analogously to Theorem 6.1, we can also easily prove the following

Theorem 6.7. *If $A \subset X$ and $B \subset Y$ such that $B \neq \emptyset$ and $B \neq Y$, and F is a relation on X to Y, then for any $U, V \subset X$, with $U \neq \emptyset$, the following assertions are equivalent:*

(1) $\tilde{F}[V] \subset \tilde{\Gamma}_{(A,B)}[U]$; *(2)* $U \not\subset A$ *or* $\big(V \subset D_F$ *and* $F[V] \subset B\big)$.

Proof. By Theorem 5.7, we have

$$\tilde{F}[V] \subset \tilde{\Gamma}_{(A,B)}[U] \iff U \not\subset A \quad \text{or} \quad \tilde{F}[V] \subset B.$$

Moreover, by Theorem 4.1, we have

$$\tilde{F}[V] \subset B \iff V \subset D_F \quad \text{and} \quad F[V] \subset B.$$

Therefore, (1) and (2) are also equivalent.

Remark 6.8. Thus, in particular if A is a proper, nonvoid subset of X and F is a relation on X, then for any $U, V \subset X$, with $U \neq \emptyset$, we have $\tilde{F}[V] \subset \tilde{\Gamma}_A[U]$ if and only if $U \not\subset A$ or $\big(V \subset D_F$ and $F[V] \subset A\big)$.

Corollary 6.9. *If $A \subset X$ and $B \subset Y$ such that $B \neq \emptyset$ and $B \neq Y$, and F is a relation on X to Y, then for any $x, y \in X$ the following assertions are equivalent:*

(1) $\tilde{F}(y) \subset \tilde{\Gamma}_{(A,B)}(x)$; *(2)* $x \notin A$ *or* $\big(y \in D_F$ *and* $F(y) \subset B\big)$.

Remark 6.10. Thus, in particular if A is a proper, nonvoid subset of X and F is a relation on X, then for any $x\,,\,y \in X$, we have $\tilde{F}(y) \subset \tilde{\Gamma}_A(x)$ if and only if $x \notin A$ or $\big(y \in D_F$ and $F(y) \subset A\big)$.

Now, as an immediate consequence of Corollaries 1.3 and 6.9, we can also state

Theorem 6.11. *If $A \subset X$ and $B \subset Y$ such that $B \neq \emptyset$ and $B \neq Y$, then for any relation F on X to Y, the following assertions are equivalent:*

(1) $\tilde{F} \subset \tilde{\Gamma}_{(A,\,B)}$ *;* *(2)* $A \subset D_F$ *and* $F\,[\,A\,] \subset B$ *;*

(3) for any $x \in A$ *we have* $x \in D_F$ *and* $F(x) \subset B$ *;*

(4) for any $x \in X$ *we have either* $x \notin A$ *or* $\big(x \in D_F$ *and* $F(x) \subset B\big)$.

Remark 6.12. Thus, in particular if A is a proper, nonvoid subset of X, then for any relation F on X, we have $\tilde{F} \subset \tilde{\Gamma}_A$ if and only if $F\,[\,A\,] \subset A \subset D_F$, or equivalently for any $x \in A$ we have $x \in D_F$ and $F(x) \subset A$.

Now, as an immediate consequence of Theorems 6.1 and 6.7, we can also state

Theorem 6.13. *If $A \subset X$ and $B \subset Y$ such that $B \neq \emptyset$ and $B \neq Y$, and F is a relation on X to Y, then for any $U\,,\,V \subset X$, with $U \neq \emptyset$ and $F\,[\,V\,] \neq Y$, the following assertions are equivalent:*

(1) $\tilde{\Gamma}_{(A,\,B)}\,[\,U\,] = \tilde{F}\,[\,V\,]$ *;*

(2) $\big(U \not\subset A$ *and* $V \not\subset D_F\big)$ *or* $\big(U \subset A\,,\; V \subset D_F$ *and* $B = F\,[\,V\,]\big)$.

Proof. By Theorem 6.1, we have

$$\tilde{\Gamma}_{(A,\,B)}[\,U\,] \subset \tilde{F}\,[\,V\,] \iff V \not\subset D_F \quad \text{or} \quad \big(U \subset A \quad \text{and} \quad B \subset F\,[\,V\,]\big).$$

Moreover, by Theorem 6.7, we have

$$\tilde{F}\,[\,V\,] \subset \tilde{\Gamma}_{(A,\,B)}\,[\,U\,] \iff U \not\subset A \quad \text{or} \quad \big(V \subset D_F \quad \text{and} \quad F\,[\,V\,] \subset B\big).$$

Hence, it is clear (1) and (2) are also equivalent.

Remark 6.14. Thus, if in particular if A is a proper, nonvoid subset of X and F is a relation on X, then for any $U\,,\,V \subset X$, with $U \neq \emptyset$ and $F\,[\,V\,] \neq X$, we have $\tilde{\Gamma}_A[\,U\,] = \tilde{F}\,[\,V\,]$ if and only if $\big(U \not\subset A$ and $V \not\subset D_F\big)$ or $\big(U \subset A\,,\; V \subset D_F$ and $A = F\,[\,V\,]\big)$.

Corollary 6.15. *If $A \subset X$ and $B \subset Y$ such that $B \neq \emptyset$ and $B \neq Y$, and F is a relation on X to Y, then for any $x\,,\,y \in X$, with $F(x) \neq Y$, the following assertions are equivalent:*

(1) $\tilde{\Gamma}_{(A,\,B)}(x) = \tilde{F}(y)$ *;*

(2) $\big(x \notin A$ *and* $y \notin D_F\big)$ *or* $\big(x \in A\,,\; y \in D_F$ *and* $B = F(y)\big)$.

Remark 6.16. Thus, in particular if $A \subset X$ such that $A \neq \emptyset$ and $A \neq X$, and F is a relation on X, then for any $x\,,\,y \in X$, with $F(x) \neq X$, we have $\tilde{\Gamma}_A(x) = \tilde{F}(y)$ if and only if either $\big(x \notin A$ and $y \notin D_F\big)$ or $\big(x \in A\,,\; y \in D_F$ and $A = F(y)\big)$.

Now, as an immediate consequence of Corollaries 1.4 and 6.15, we can also state

Theorem 6.17. *If $A \subset X$ and $B \subset Y$ such that $B \neq \emptyset$ and $B \neq Y$, then for any relation F on X to Y, the following assertions are equivalent:*

(1) $\tilde{\Gamma}_{(A,\,B)} = \tilde{F}$ *;*

(2) for any $x \in A \cap D_F$ we have $F(x) = B$ and for any $x \in D_F \setminus A$ we have $F(x) = Y$;

(3) for any $x \in X$ we have $x \notin A \cup D_F$ or $\big(x \notin A$ and $F(x) = Y \big)$ or $\big(x \in A \cap D_F$ and $F(x) = B \big)$.

Remark 6.18. Thus, in particular if A is a proper, nonvoid subset of X, then for any relation F on X we have $\tilde{\Gamma}_A = \tilde{F}$ if and only if for any $x \in A \cap D_F$ we have $F(x) = A$ and for any $x \in D_F \setminus A$ we have $F(x) = X$.

7. Some applications to relator spaces

A family $\mathcal{R}$ on relations on one set X to another Y is called a relator on X to Y. Moreover, the ordered pair $(X, Y)(\mathcal{R}) = \big((X, Y), \mathcal{R}\big)$ is called a relator space. For the origins of this notion, see [15] and the references therein.

If in particular $\mathcal{R}$ is a relator on X to itself, then we may simply say that $\mathcal{R}$ is a relator on X. Moreover, by identifying singletons with their elements, we may naturally write $X(\mathcal{R})$ in place of $(X, X)(\mathcal{R})$.

Quite similarly, if R is a relation on X to Y, then we may simply write $(X, Y)(R)$ in place of $(X, Y)(\{R\})$. More generally, the same convention can also be applied when $\mathfrak{F}$ is a function of relators on X to Y.

Relator spaces of the simpler type $X(R)$ and $X(\mathcal{R})$ are substantial generalizations of ordered sets and uniform spaces [3]. However, they are insufficient to include the theory of context spaces [4], and to naturally express continuity properties of relations [18].

If $\mathcal{R}$ is a relator on X to Y, then for any $A \subset X$ and $B \subset Y$, we write:

(1) $A \in \operatorname{Int}_{\mathcal{R}}(B)$ if $R[A] \subset B$ for some $R \in \mathcal{R}$;

(2) $A \in \operatorname{Lb}_{\mathcal{R}}(B)$ if $B \subset R^c[A]^c$ for some $R \in \mathcal{R}$.

To see the appropriateness of the latter apparently very strange definition, recall that, by the corresponding definition and Theorem 1.14, we have

$$R[A] = \bigcup_{a \in A} R(a) \qquad \text{and} \qquad R^c[A]^c = \bigcap_{a \in A} R(a).$$

Thus, in particular $B \subset R^c[A]^c$ if and only if $B \subset R(a)$ for all $a \in A$. That is, $b \in R(a)$, i.e., $a\,R\,b$ for all $a \in A$ and $b \in B$. Therefore, A is a lower bound of B with respect to R.

In this respect, it is also worth noticing that $B \subset R^c[A]^c$ if and only if $R^c[A] \subset B^c$. Therefore,

$$\operatorname{Lb}_{\mathcal{R}}(B) = \operatorname{Int}_{\mathcal{R}^c}(B^c) \qquad \text{and} \qquad \operatorname{Int}_{\mathcal{R}}(B) = \operatorname{Lb}_{\mathcal{R}^c}(B^c),$$

where $\mathcal{R}^c = \{ R^c : \quad R \in \mathcal{R} \}$. Thus, in contrast to a common belief, the basic topological and order theoretic notions can be expressed in terms of each other. This fact, and the use of the notation Lb, was first put forward in [19].

Now, if $\mathcal{R}$ is a relator on X to Y, then for any $x \in X$ and $B \subset Y$, we may simply write:

(3) $x \in \operatorname{int}_{\mathcal{R}}(B)$ if $\{x\} \in \operatorname{Int}_R(B)$; (4) $x \in \operatorname{lb}_{\mathcal{R}}(B)$ if $\{x\} \in \operatorname{Lb}_R(B)$;

(5) $B \in \mathcal{E}_{\mathcal{R}}$ if $\operatorname{int}_{\mathcal{R}}(B) \neq \emptyset$; (6) $B \in \mathfrak{L}_{\mathcal{R}}$ if $\operatorname{lb}_{\mathcal{R}}(B) \neq \emptyset$.

Moreover, if in particular $\mathcal{R}$ is a relator on X, then for any $A \subset X$ we may also write:

(7) $A \in \tau_{\mathcal{R}}$ if $A \in \operatorname{Int}_{\mathcal{R}}(A)$; (8) $A \in \mathcal{T}_{\mathcal{R}}$ if $A \subset \operatorname{int}_{\mathcal{R}}(A)$;

(9) $A \in l_{\mathcal{R}}$ if $A \in \operatorname{Lb}_{\mathcal{R}}(A)$; (10) $A \in \mathcal{L}_{\mathcal{R}}$ if $A \subset \operatorname{lb}_{\mathcal{R}}(A)$.

The relations $\operatorname{Int}_{\mathcal{R}}$ and $\operatorname{int}_{\mathcal{R}}$ are called the proximal and topological interiors on Y to X induced by $\mathcal{R}$, respectively. While, the members of the families $\tau_{\mathcal{R}}$, $\mathcal{T}_{\mathcal{R}}$, and $\mathcal{E}_{\mathcal{R}}$ are called the proximally open, topologically open and fat subsets of $X(\mathcal{R})$, respectively.

The use of notation Int instead of $\Subset$ was first suggested in [15]. While, the fact that the fat sets are usually more important tools than the open ones was first stressed in [16], and at the Seventh Topological Symposium in Prague in 1991.

Now, by Remark 2.2 and Definition 2.1, we may naturally introduce the following generated relators.

Definition 7.1. For any family $\mathcal{A} \subset \mathcal{P}(X)$, we define

$$\mathcal{R}_{\mathcal{A}} = \{ \Gamma_A : \quad A \in \mathcal{A} \}.$$

Moreover, for any relations $\mathfrak{f}$ and $\mathfrak{F}$ on $\mathcal{P}(Y)$ to X and $\mathcal{P}(X)$, respectively, we define

$$\mathcal{R}_{\mathfrak{f}} = \{ \Gamma_{(a,B)} : \quad a \in \mathfrak{f}(B) \} \qquad \text{and} \qquad \mathcal{R}_{\mathfrak{F}} = \{ \Gamma_{(A,B)} : \quad A \in \mathfrak{F}(B) \}.$$

Remark 7.2. Note that if in particular $\mathfrak{F}_{\mathcal{A}}$ is the identity function of $\mathcal{A}$, then $\mathcal{R}_{\mathcal{A}} = \mathcal{R}_{\mathfrak{F}_{\mathcal{A}}}$.

While, if in particular $\mathfrak{F}_{\mathfrak{f}}(B) = \{ \{a\} : \quad a \in \mathfrak{f}(B) \}$ for all $B \subset Y$, then $\mathcal{R}_{\mathfrak{f}} = \mathcal{R}_{\mathfrak{F}_{\mathfrak{f}}}$.

Moreover, by Definition 2.7, we may also also naturally introduce the following totalization relator.

Definition 7.3. For any relator $\mathcal{R}$ on X to Y, we define

$$\tilde{\mathcal{R}} = \{ \tilde{R} : \quad R \in \mathcal{R} \}.$$

Remark 7.4. Now, for any family $\mathcal{A}$ of subsets of X and relation Int on X, the totalizations $\tilde{\mathcal{R}}_{\mathcal{A}}$ and $\tilde{\mathcal{R}}_{\operatorname{Int}}$ may be called the Davis–Pervin and the Hunsaker–Lindgren relators on X generated by $\mathcal{A}$ and Int, respectively.

By using Theorem 5.1, concerning an obvious generalization of the latter relator, we can easily prove the following

Theorem 7.5. *If* Int *is a relation on* $\mathcal{P}(Y)$ *to* $\mathcal{P}(X)$, *then for any* $U \subset X$ *and* $V \subset Y$, *with* $U \neq \emptyset$ *and* $V \neq Y$, *the following assertions are equivalent:*

(1) $U \in \operatorname{Int}_{\tilde{\mathcal{R}}_{\operatorname{Int}}}(V)$;

(2) $A \in \operatorname{Int}(B)$ *for some* $A \subset X$ *and* $B \subset Y$ *with* $U \subset A$ *and* $\emptyset \neq B \subset V$.

Proof. By the corresponding definitions, we have (1) if and only if there exist $A \subset X$ and $B \subset Y$, with $A \in \operatorname{Int}(B)$, such that $\tilde{\Gamma}_{(A,\,B)}[U] \subset V$. Moreover, from Theorem 5.1 we can see that the latter inclusion is equivalent to the requirements that $U \subset A$ and $\emptyset \neq B \subset V$. Hence, it is clear that (1) and (2) are also equivalent.

In principle, the following theorem can be derived from Theorem 7.5 by using Remark 7.2. However, it can be more easily proved with the help of Corollary 5.3.

Theorem 7.6. *If* int *is a relation on* $\mathcal{P}(Y)$ *to* X, *then for any* $x \in X$ *and* $V \subset Y$, *with* $V \neq Y$, *the following assertions are equivalent:*

(1) $x \in \operatorname{int}_{\tilde{\mathcal{R}}_{\operatorname{int}}}(V)$;

(2) $x \in \operatorname{int}(B)$ *for some* $B \subset Y$ *with* $\emptyset \neq B \subset V$.

Proof. By the corresponding definitions, we have (1) if and only if there exist $a \in X$ and $B \subset Y$, with $a \in \operatorname{int}(B)$, such that $\tilde{\Gamma}_{(\{a\},\,B)}(x) \subset V$. Moreover, from Corollary 5.2 we can see that the latter inclusion is equivalent to the requirements that $x \in \{a\}$ and $\emptyset \neq B \subset V$. This shows that $a = x$. Hence, it is clear that (1) and (2) are also equivalent.

Remark 7.7. Now, by establishing the basic properties of the relations $\operatorname{Int}_{\mathcal{R}}$ and $\operatorname{int}_{\mathcal{R}}$ for a relator $\mathcal{R}$ on X to Y, we can give some necessary and sufficient conditions on the relations Int and int on $\mathcal{P}(Y)$ to $\mathcal{P}(X)$ and X, respectively, in order that the equalities $\operatorname{Int} = \operatorname{Int}_{\tilde{\mathcal{R}}_{\operatorname{Int}}}$ and $\operatorname{int} = \operatorname{int}_{\tilde{\mathcal{R}}_{\operatorname{int}}}$ could be true.

Moreover, for a relator $\mathcal{R}$ on X to Y, we can investigate the validity the equalities $\operatorname{Int}_{\mathcal{R}} = \operatorname{Int}_{\tilde{\mathcal{R}}_{\operatorname{Int}_{\mathcal{R}}}}$ and $\operatorname{int}_{\mathcal{R}} = \operatorname{int}_{\tilde{\mathcal{R}}_{\operatorname{int}_{\mathcal{R}}}}$. And, for a relator $\mathcal{R}$ on X to Y, we may look for the largest relators $\mathcal{R}^{\#}$ and $\mathcal{R}^{\wedge}$ on X to Y such that the equalities $\operatorname{Int}_{\mathcal{R}} = \operatorname{Int}_{\mathcal{R}^{\#}}$ and $\operatorname{int}_{\mathcal{R}} = \operatorname{int}_{\mathcal{R}^{\wedge}}$ could be true.

However, it is now more convenient to note only that, by using Remarks 5.2 and 5.4, we can also easily prove the following theorems.

Theorem 7.8. *If* $\tau \subset \mathcal{P}(X)$, *then for any proper, nonvoid subset* U *of* X *the following assertions are equivalent:*

(1) $U \in \tau_{\tilde{\mathcal{R}}_{\tau}}$; *(2)* $U \in \tau$.

Proof. By the corresponding definitions, we have (1) if and only if there exists $A \in \tau$ such that $\tilde{\Gamma}_A[U] \subset U$. Moreover, from Remark 5.2 we can see that the latter inclusion is equivalent to the requirement that $U \subset A \subset U$, i.e., $A = U$. Hence, it is clear that (1) and (2) are also equivalent.

Theorem 7.9. *If $\mathcal{T} \subset \mathcal{P}(X)$, then for any proper, nonvoid subset U of X the following assertions are equivalent:*

(1) $U \in \mathcal{T}_{\tilde{\mathcal{R}}_{\mathcal{T}}}$; *(2)* $U = \bigcup \mathcal{B}$ *for some* $\mathcal{B} \subset \mathcal{T}$.

Proof. By the corresponding definitions, we have (1) if and only if for each $x \in U$ there exists $A_x \in \mathcal{T}$ such that $\tilde{\Gamma}_A(x) \subset U$. Moreover, from Remark 5.4 we can see that the latter inclusion is equivalent to the requirement that $x \in A_x \subset U$. This shows that $U = \bigcup_{x \in U} A_x$. Hence, it is clear that (1) and (2) are also equivalent.

Theorem 7.10. *If $\mathcal{E} \subset \mathcal{P}(X)$, then for any proper, nonvoid subset U of X the following assertions are equivalent:*

(1) $U \in \mathcal{E}_{\tilde{\mathcal{R}}_{\mathcal{E}}}$; *(2)* $A \subset U$ *for some* $A \in \mathcal{E}$ *with* $A \neq \emptyset$.

Proof. By the corresponding definitions, we have (1) if and only if there exist $x \in X$ and $A \in \mathcal{E}$ such that $\tilde{\Gamma}_A(x) \subset U$. Moreover, from Remark 5.4 we can see that the latter inclusion is equivalent to the requirements that $x \in A \subset U$. Hence, it is clear that (1) and (2) are also equivalent.

Remark 7.11. Now, by establishing the basic properties of the families $\tau_{\mathcal{R}}$, $\mathcal{T}_{\mathcal{R}}$, and $\mathcal{E}_{\mathcal{R}}$ for a relator $\mathcal{R}$ on X, we can give some necessary and sufficient conditions on the families τ , $\mathcal{T}$ and $\mathcal{E}$ of subsets of X in order that the equalities $\mathcal{A} = \tau_{\tilde{\mathcal{R}}_\tau}$, $\mathcal{A} = \mathcal{T}_{\tilde{\mathcal{R}}_{\mathcal{T}}}$ and $\mathcal{A} = \mathcal{E}_{\tilde{\mathcal{R}}_{\mathcal{E}}}$, respectively, could be true.

Moreover, for a relator $\mathcal{R}$ on X, we can investigate the validity the equalities $\tau_{\mathcal{R}} = \tau_{\tilde{\mathcal{R}}_{\tau_{\mathcal{R}}}}$, $\mathcal{T}_{\mathcal{R}} = \mathcal{T}_{\tilde{\mathcal{R}}_{\mathcal{T}_{\mathcal{R}}}}$ and $\mathcal{E}_{\mathcal{R}} = \mathcal{E}_{\tilde{\mathcal{R}}_{\mathcal{E}_{\mathcal{R}}}}$. And, for a relator $\mathcal{R}$ on X, we may look for the largest relators $\mathcal{R}^{\sharp}$, $\mathcal{R}^{\sqcap}$ and $\mathcal{R}^{\vartriangle}$ on X such that the equalities $\tau_{\mathcal{R}} = \tau_{\mathcal{R}^{\sharp}}$, $\mathcal{T}_{\mathcal{R}} = \mathcal{T}_{\mathcal{R}^{\sqcap}}$ and $\mathcal{E}_{\mathcal{R}} = \mathcal{E}_{\mathcal{R}^{\vartriangle}}$ could be true.

Unfortunately, by [9 , Example 5.3] , the relator $\mathcal{R}^{\sqcap}$ does not, in general, exist. Moreover, by [12 , Remark 6.20] , the operation $\sharp$ is not stable in the sense that in general $\{ X^2 \} \neq \{ X^2 \}^{\sharp}$. Therefore, we also need the modification relators $\mathcal{R}^{\# \infty}$ and $\mathcal{R}^{\wedge \infty}$ introduced in [9] and [10] .

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