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CONSTRUCTIONS AND EXTENSIONS OF FREE AND CONTROLLED ADDITIVE RELATIONS

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Abstract. By using several auxiliary results on relations and their intersection convolutions, we give some conditions in order that certain additive partial selection relations Φ of a relation F of one group X to another Y could be extended to certain total additive selection relations Ψ of the relation $F + \Phi(0)$.

The results obtained extend some Hahn-Banach type extension theorems of B. Rodríguez-Salinas and L. Bou; Z. Gajda, A. Smajdor and W. Smajdor; and the second author. Moreover, they can be used to prove certain forms of the Hyers–Ulam type selection theorems of Z. Gajda and R. Ger; R. Badora; and the second author.

Contents

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Introduction

The origin of the following generalization of the classical Hahn–Banach extension theorem goes back to R. Kaufman [37]. It is a particular case of [16, Corollary 1.3] by B. Fuchssteiner. (For some more readable treatments, see also Fuchssteiner and Lusky [18] and Száz [71].)

Theorem 1. *If p is a subadditive function of a commutative semigroup X to $\mathbb{R}$ and φ is an additive function of a subsemigroup V of X to $\mathbb{R}$ such that:*

(1) $\varphi(x) \leq p(x)$ for all $x \in V$;

(2) $\varphi(x+y) \leq p(x) + \varphi(y)$ for all $x \in X$ and $y \in V$ with $x+y \in V$;

then φ can be extended to an additive function ψ of X to $\mathbb{R}$ such that $\psi(x) \leq p(x)$ for all $x \in X$.

Remark 2. To see the necessity of condition (2), note that if ψ is as above, then

$$\varphi(x+y) = \psi(x+y) = \psi(x) + \psi(y) \leq p(x) + \varphi(y)$$

for all $x \in X$ and $y \in V$ with $x+y \in V$.

In [26], to have a close analogue of Theorem 1, we have proved the following generalization of the classical Hyers–Ulam stability theorem [33]. (For its former direct generalizations, see Rätz [53].)

Theorem 2. *If f is an ε–approximately additive function of a commutative semigroup X to a Banach space Y, for some $\varepsilon \geq 0$, in the sense that*

$$\| f(x+y) - f(x) - f(y) \| \leq \varepsilon$$

for all $x, y \in X$, and φ is a 2–homogeneous function of a subsemigroup V of X to Y which is δ–near to f, for some $\delta \geq 0$, in the sense that

$$\| f(x) - \varphi(x) \| \leq \delta$$

for all $x \in V$, then φ can be extended to an additive function ψ of X to Y that is ε–near to f.

Remark 2. To see that this theorem is somewhat more general than that of Hyers and Ulam, note that if in particular X has a zero element 0, then $\| f(0) \| \leq \varepsilon$. Thus, $\varphi = \{(0, 0)\}$ is an additive function of the subgroup $\{0\}$ of X to Y such that φ is ε–near to f. Therefore, by the above theorem, there exists an additive function ψ of X to Y which is ε–near to f.

Moreover, we can note that if p and φ are as in Theorem 1, then by defining a relation F of X to $\mathbb{R}$ such that

$$F(x) = \,]-\infty, \; p(x)]$$

for all $x \in X$, we have $\varphi(x) \in F(x)$ for all $x \in V$.

While, if f and φ are as in Theorem 2, then by defining a relation F of X to Y such that

$$F(x) = f(x) + B_\delta(0)\,, \qquad \text{with} \qquad B_\delta(0) = \{ y \in Y: \ \ \| y \| \le \delta \}\,,$$

for all $x \in X$, we again have $\varphi(x) \in F(x)$ for all $x \in V$.

Therefore, the essence of Theorems 1 and 2 is nothing else but the observation that an additive partial selection function φ of a certain relation F of X to $\mathbb{R}$ and Y, respectively, can be extended to a total additive selection function of ψ of F.

The corresponding fact in connection with the classical Hahn–Banach extension theorem was already recognized by B. Rodríguez-Salinas and L. Bou [54]. (For some further developments, see Ioffe [36], Z. Gajda, A. Smajdor and W. Smajdor [22], W. Smajdor and J. Szczawińska [58], and Száz [60].)

Moreover, W. Smajdor [57] and Z. Gajda and R. Ger [20] observed that the essence of the classical Hyers–Ulam stability theorem is the existence of an additive selection function of a certain relation. (For some further developments, see Gajda [19], Badora [2], Badora, Ger and Páles [4], Popa [51], and Száz [65].)

In [60], by using a particular case the intersection convolution

$$(F * G)(x) = \bigcap \left\{ F(u) + G(v) : \quad x = u + v\,, \quad F(u) \ne \emptyset\,, \quad G(v) \ne \emptyset \right\}$$

of relations F and G, the second author has proved the following generalization of [54, Theorem 1] of B. Rodríguez-Salinas and L. Bou.

Theorem 3. *If F is a sublinear relation of one vector space X to another Y, and there exists a translation-invariant Nachbin system $\mathcal{A}$ in Y such that $F(x) \in \mathcal{A}$ for all $x \in X$, then each semi-linear partial selection relation Φ of F can be extended to a total linear selection relation Ψ of $F + \Phi(0)$.*

Remark 3. Here, a family $\mathcal{A}$ of sets is called a Nachbin system if each subfamily $\mathcal{B}$ of $\mathcal{A}$, such that any two members of $\mathcal{B}$ intersect, has a nonvoid intersection.

Now, by improving the arguments of [60], we shall prove the following generalization of [22, Theorem 1] of Z. Gajda, A. Smajdor and W. Smajdor.

Theorem 4. *If F is an odd $\mathbb{N}$-subhomogeneous subadditive relation of a commutative group X to a vector space Y over $\mathbb{Q}$, and there exists an admissible Nachbin system $\mathcal{A}$ in Y such that $F(x) \in \mathcal{A}$ for all $x \in X$, then each odd $\mathbb{N}$–semi-subhomogeneous superadditive partial selection relation Φ of F can be extended to a total $\mathbb{Z} \setminus \{0\}$–homogeneous additive selection relation Ψ of $F + \Phi(0)$.*

Remark 4. Here, a Nachbin system $\mathcal{A}$ in Y is called admissible if in addition to its translation-invariance, we also have $n^{-1} A \in \mathcal{A}$ for all $n \in \mathbb{N}$ and $A \in \mathcal{A}$.

Unfortunately, by using the convolutional method of second author, we have not been able to extend Theorem 3 to commutative semigroups. However, the several auxiliary results leading to Theorem 4 are much more general than those used in the proof Theorem 3. They are mostly formulated in terms of semigroups.

1. A few basic facts on relations

A subset F of a product set $X \times Y$ is called a relation on X to Y. If in particular $F \subset X^2$, then we may simply say that F is a relation on X. The same terminology can also be used when Y need not be specified.

If F is a relation on X to Y, then for any $x \in X$ and $A \subset X$ the sets $F(x) = \{ y \in Y : (x, y) \in F \}$ and $F[A] = \bigcup_{a \in A} F(a)$ are called the images of x and A under F, respectively.

Moreover, the sets $D_F = \{ x \in X : F(x) \neq \emptyset \}$ and $R_F = F[D_F]$ are called the domain and range of F, respectively. If in particular $D_F = X$, then we say that F is a relation of X to Y, or that F is a total relation on X to Y.

If F is a relation on X to Y, then the values $F(x)$, where $x \in X$, uniquely determine F since we have $F = \bigcup_{x \in X} \{x\} \times F(x)$. Therefore, the inverse relation F^{-1} can be defined such that $F^{-1}(y) = \{ x \in X : y \in F(x) \}$ for all $y \in Y$.

Moreover, if in addition G is a relation on Y to Z, then the composition relation $G \circ F$ can be defined such that $(G \circ F)(x) = G[F(x)]$ for all $x \in X$. Thus, we also have $(G \circ F)[A] = G[F[A]]$ for all $A \subset X$.

While, if in addition G is a relation on Z to W, then the box product relation $F \boxtimes G$ can be defined such that $(F \boxtimes G)(x, z) = F(x) \times G(z)$ for all $x \in X$ and $z \in Z$. Thus, we have $(F \boxtimes G)[A] = G \circ A \circ F^{-1}$ for all $A \subset X \times Z$.

In particular, a relation f on X to Y is called a function if for each $x \in D_f$ there exists $y \in Y$ such that $f(x) = \{y\}$. In this case, by identifying singletons with their elements, we may simply write $f(x) = y$ in place of $f(x) = \{y\}$.

If F is a relation on X to Y and $A_i \subset X$ for all $i \in I$, then in general we only have $F\left[\bigcup_{i \in I} A_i\right] = \bigcup_{i \in I} F[A_i]$. However, if in particular f is a function, then all set-theoretic operations are preserved under the relation f^{-1}.

If F is a relation on X to Y, then a subset Φ of F is called a partial selection relation of F. Thus, we also have $D_\Phi \subset D_F$. Therefore, a partial selection relation Φ of F may be called total if $D_\Phi = D_F$.

The total selection relations of a relation F will usually be simply called the selection relations of F. Thus, the Axiom of Choice can be briefly expressed by saying that every relation F has a selection function.

If F is a relation on X to Y and $U \subset D_F$, then the relation $F \,|\, U = F \cap (U \times Y)$ is called the restriction of F to U. Moreover, F and G are relations on X to Y such that $D_F \subset D_G$ and $F = G \,|\, D_F$, then G is called an extension of F.

2. A few basic facts on groupoids

Definition 2.1. If X is a set, then a function $+$ of X^2 to X is called an operation in X. And the ordered pair $X(+) = (X, +)$ is called a groupoid.

Remark 2.2. In this case, we may simply write $x + y$ in place of $+(x, y)$ for all $x, y \in X$. Moreover, we may also simply write X in place of $X(+)$.

Instead of groupoids, it is usually sufficient to consider only semigroups (associative grupoids) or even monoids (semigroups with zero). However, several definitions on semigroups can be naturally extended to groupoids.

Definition 2.3. If X is a groupoid and $x \in X$, then we define $1\,x = x$. Moreover, if $n \in \mathbb{N}$ such that $n\,x$ is already defined, then we define $(\,n+1\,)\,x = n\,x + x$.

Now, by induction, we can easily prove the following two theorems.

Theorem 2.4. *If X is a semigroup, then for any $x \in X$ and $m, n \in \mathbb{N}$ we have*

(1) $(\,m+n\,)\,x = m\,x + n\,x\,,$ *(2)* $(n\,m)\,x = n\,(m\,x)\,.$

Proof. To prove (2), note that $(1\,m)x = m\,x = m(1\,x)$. Moreover, if $(n\,m)x = n(m\,x)$(2) holds for some $n \in \mathbb{N}$, then by (1) we also have

$$((\,n+1\,)\,m)x = (\,n\,m+m\,)x = (n\,m)x + m\,x = n(m\,x) + m\,x = (n+1)(m\,x)\,.$$

Theorem 2.5. *If X is a semigroup, then for any $n \in \mathbb{N}$ and $x, y \in X$, with $x+y = y+x$, we have*

$$n\,(x+y) = n\,x + n\,y\,.$$

Proof. For this, we must first note that $x+1\,y = x+y = y+x = 1\,y+x$. Moreover, if $x+n\,y = n\,y+x$ for some $n \in \mathbb{N}$, then we also have

$$x + (\,n+1)y = x + n\,y + y = n\,y + x + y = n\,y + y + x = (\,n+1\,)y + x\,.$$

Therefore, $x + n\,y = n\,y + x$ holds for all $n \in \mathbb{N}$.

Now, we can also note that $1(\,x+y\,) = x+y = 1\,x+1\,y$. Moreover, if $n(x+y) = n\,x + n\,y$ for some $n \in \mathbb{N}$, then by the above observation we also have

$$\begin{aligned}(\,n+1\,)(x+y\,) &= n(\,x+y\,) + x + y = n\,x + n\,y + x + y = \\ &= n\,x + x + n\,y + y = (\,n+1\,)x + (\,n+1\,)y\,.\end{aligned}$$

Therefore, the required assertion is also true.

Definition 2.6. If in particular X is a groupoid with zero, then we also define $0\,x = 0$ for all $x \in X$.

Moreover, if more specially X is a group, then we also define $(-n\,)\,x = -(\,n\,x\,)$ for all $x \in X$ and $n \in \mathbb{N}$.

Now, by induction, we can also easily prove the following

Theorem 2.7. *If X is a group, then for any $x \in X$ and $n \in \mathbb{N}$ we have*

$$(-n\,)\,x = n\,(-x)\,.$$

Moreover, by using this observation and the above results, we can prove the following two theorems.

Theorem 2.8. *If X is a group, then for any $x \in X$ and $k\,, l \in \mathbb{Z}$ we have*

(1) $(\,k\,l\,)\,x = k\,(\,l\,x\,)\,,$ *(2)* $(\,k+l\,)\,x = k\,x + l\,x\,.$

Theorem 2.9. *If X is a group, then for any $k \in \mathbb{Z}$ and $x, y \in X$, with $x+y = y+x$, we have*

$$k(x+y) = kx + ky.$$

Proof. To check this, note that by Theorems 2.7 and 2.5 we have

$$(-n)(x+y) = n\big(-(x+y)\big) = n\big(-(y+x)\big) = \\ = n(-x-y) = n(-x) + n(-y) = (-n)x + (-n)y$$

for all $n \in \mathbb{N}$.

Remark 2.10. Thus, a commutative group X is already a module over the ring $\mathbb{Z}$ of integers.

Definition 2.11. For some $n \in \mathbb{N}$, a subset U of gruopoid X is called

(1) n–cancellable if $nx = ny$ implies $x = y$ for all $x, y \in U$;

(2) n–divisible if for each $x \in U$ there exists $y \in U$ such that $x = ny$.

Now, U may, for instance, be naturally called A–divisible, for some $A \subset \mathbb{N}$, if it is n–divisible for all $n \in A$.

Remark 2.12. Note that if both (1) and (2) hold, then U is already uniquely n–divisible in the sense that for each $x \in U$ there exists a unique $y \in U$ such that $x = ny$. Therefore, $n^{-1}x$ can be defined by this y.

Remark 2.13. Moreover, it is also worth noticing that if U is an n–cancellable subset of groupoid X, with zero, such that $0 \in U$, then $nx = 0$ implies $x = 0$ for all $x \in U$. Namely, if $x \in U$ such that $nx = 0$, then we also have $nx = n0$, and hence $x = 0$.

In this respect, we can also easily prove the following two theorems.

Theorem 2.14. *If X is a commutative group, then for each $n \in \mathbb{N}$ the following assertions are equivalent:*

(1) X is n–cancellable; *(2) $nx = 0$ implies $x = 0$ for all $x \in X$.*

Proof. To prove that (2) also implies (1), note that if $x, y \in X$ such that $nx = ny$, then by Theorems 2.5 and 2.7 we also have

$$n(x-y) = nx + n(-y) = nx - ny = 0.$$

Hence, if (2) holds then we can infer that $x - y = 0$, and thus $x = y$. Therefore, (1) also holds.

Theorem 2.15. *If X is an $\mathbb{N}$–cancellable group, then $kx = lx$ implies $k = l$ for all $x \in X \setminus \{0\}$ and $k, l \in \mathbb{Z}$.*

Proof. Assume on the contrary that there exist $x \in X \setminus \{0\}$ and $k, l \in \mathbb{Z}$ such that $kx = lx$, but $k \neq l$. Then, by using Theorem 2.8, we can see that

$$(k-l)x = kx - lx = 0 \qquad \text{and} \qquad (l-k)x = lx - kx = 0.$$

Moreover, we have either $k < l$ or $l < k$, and thus either $l - k \in \mathbb{N}$ or $k - l \in \mathbb{N}$. Hence, by Remark 2.13, it follows that $x = 0$. This contradiction proves the theorem.

Remark 2.16. By using an analogue of Definition 2.11, we can also easily see that if X is an $\mathbb{N}$–divisible (resp. $\mathbb{N}$–cancellable) group, then it is also $\mathbb{Z} \setminus \{0\}$–divisible (resp. $\mathbb{Z} \setminus \{0\}$–cancellable).

Concerning groupoids, in the sequel, we shall also need the following

Definition 2.17. If X is a groupoid, then for any $A, B \subset X$ and $n \in \mathbb{N}$ we define

$$A + B = \big\{ a + b : \quad a \in A , \quad b \in B \big\} \qquad \text{and} \qquad n\, A = \big\{ n\, a : \quad a \in A \big\} .$$

Remark 2.18. If in particular X is a group, then for any $A \subset X$ and $k \in \mathbb{Z}$ we also define $k\, A = \big\{ k\, a : \quad a \in A \big\}$.

And, for any A, $B \subset X$, we also write $-A = (-1)\, A$ and $A - B = A + (-B\,)$ despite that the family $\mathcal{P}(X)$ of all subsets of X is only a monoid.

Remark 2.19. If more specially X is a vector space over $\mathbb{K}$, with $\mathbb{K} = \mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$, then for any $A \subset X$ and $\lambda \in \mathbb{K}$ we also define $\lambda\, A = \big\{ \lambda\, a : \quad a \in A \big\}$.

Thus, only two axioms of a vector space may fail to hold for $\mathcal{P}\,(X)$. Namely, only the one point subsets of X can have additive inverses. Moreover, in general we only have $(\, \lambda + \mu\,)\, A \subset \lambda\, A + \mu\, A$.

Definition 2.20. For some $\lambda \in \mathbb{K}$, a subset A of a vector space X over $\mathbb{K}$ is called λ–affine if

$$\lambda\, A + (\, 1 - \lambda\,) A \subset A .$$

Remark 2.21. Now, the set A may be naturally called Λ–affine, for some $\Lambda \subset \mathbb{K}$, if it is λ–affine for all $\lambda \in \Lambda$.

In particular, the set A may be naturally called affine if it is $\mathbb{K}$–affine. Moreover, A may be naturally called convex if it is $[\, 0\, ,\, 1\,] \cap \mathbb{K}$–affine.

The importance of affine subsets is apparent from the following

Theorem 2.22. *For a subset A of a vector space X over $\mathbb{K}$, the following assertions are equivalent:*

(1) A is an affine subset of X;

(2) $A - a$ is a linear subset of X for all $a \in A$;

(3) $A = x + V$ for some point x and linear subset V of X;

(4) $(\, \lambda + \mu\,)\, A = \lambda\, A + \mu\, A$ for all λ, $\mu \in \mathbb{K}$ with $\lambda + \mu \neq 0$.

Proof. Only the implication (1) $\Longrightarrow$ (2) requires a nontrivial proof. For this, assume that (1) holds and $a \in A$. Define $B = A - a$, and suppose that $\lambda \in \mathbb{K}$ and x, $y \in B$. Then, there exist u, $v \in A$ such that $x = u - a$ and $y = v - a$. Hence, by using (1), we can already infer that

$$\lambda\, x = \lambda\, u - \lambda\, a = \lambda\, u + (1 - \lambda)\, a - a \in A - a = B$$

and

$$x + y = u + v - 2a = 2 \Big(2^{-1}\, u + \big(1 - 2^{-1}\big)\, v \Big) + (1 - 2)\, a \; - \; a \in A - a = B .$$

Therefore, B is a subspace of X, and thus (2) also holds.

Remark 2.23. If A is a convex subset of X, then by using a similar argument we can only show that $(\lambda+\mu)A = \lambda A + \mu A$ for all $\lambda, \mu \in \mathbb{K}$ with $\lambda, \mu \geq 0$ and $\lambda+\mu \neq 0$.

3. The most important additivity properties of relations

Definition 3.1. Let F be a relation on one groupoid X to another Y and let Ω be a relation on X. Then, F is called

(1) Ω–subadditive if $F(x+y) \subset F(x)+F(y)$ for all $(x, y) \in \Omega$;

(2) Ω–superadditive if $F(x)+F(y) \subset F(x+y)$ for all $(x, y) \in \Omega$.

Remark 3.2. Now, the relation F may, for instance, be naturally called superadditive if it is X^2–superadditive. Note that thus F is superadditive if and only if $F+F \subset F$.

Moreover, if in particular F is a reflexive superadditive relation of X to itself, then F is already a translation relation in the sense that $x+F(y) \subset F(x+y)$ for all $x, y \in X$.

Now, we can also briefly formulate the following

Definition 3.3. A relation F on one groupoid X to another Y is called

(1) semi-subadditive if it is D_F^2–subadditive;

(2) left-quasi-subadditive if it is $D_F \times X$–subadditive;

(2) right-quasi-subadditive if it is $X \times D_F$–subadditive.

Remark 3.4. Now, the relation F may be naturally called quasi-subadditive if it both left-quasi-subadditive and right-quasi-subadditive. In the sequel, we shall see that quasi-subadditivity is also a quite natural additivity property.

Definition 3.5. A relation F on a groupoid X with zero to an arbitrary groupoid Y is called

(1) left-zero-subadditive if it is $\{0\}\times X$–subadditive;

(2) left-zero-superadditive if it is $\{0\}\times X$–superadditive.

Remark 3.6. The right-zero-subadditive and right-zero-superadditive relations are defined analogously by using the relation $X \times \{0\}$.

Now, the relation F may, for instance, be naturally called zero-subadditive if it is both left-zero-subadditive and right-zero-subadditive.

By the corresponding definitions, we evidently have the following

Theorem 3.7. *A relation F on one groupoid X with zero to another Y, then*

(1) F is zero-subadditive if $0 \in F(0)$;

(2) F is zero-superadditive if $F(0) \subset \{0\}$.

Proof. Namely, if for instance $0 \in F(0)$, then

$$F(x) = F(x)+\{0\} \subset F(x)+F(0) \qquad \text{and} \qquad F(x) = \{0\}+F(x) \subset F(0)+F(x)$$

for all $x \in X$. Therefore, (1) is true.

Definition 3.8. A relation F on a group X to a groupoid Y is called inversion-subadditive (resp. inversion-superadditive) it is Ω–subadditive (resp. Ω–superadditive) with $\Omega = \{ (x, -x) : \ x \in X \}$.

Remark 3.9. Moreover, the relation F may also be naturally called inversion-semi-subadditive if it is $\Omega \,|\, D_F$–subadditive with the above Ω .

Note that in this case, we have not only $F(0) \subset F(x) + F(-x)$, but also $F(0) \subset F(-x) + F(x)$ for all $x \in D_F$. Namely, if $F(0) \neq \emptyset$, then by the first inclusion we also have $F(-x) \neq \emptyset$, and thus $-x \in D_F$ for all $x \in D_F$.

Definition 3.10. For some $n \in \mathbb{N}$, a relation F on one groupoid X to another Y is called

(1) n–subhomogeneous if $F(n\,x) \subset n\,F(x)$ for all $x \in X$;

(2) n–superhomogeneous if $n\,F(x) \subset F(n\,x)$ for all $x \in X$.

Remark 3.11. Now, the relation F may also be naturally called n–semi-subhomogeneous if $F(n\,x) \subset n\,F(x)$ for all $x \in D_F$.

Moreover, the relation F may, for instance, be naturally called n–homogeneous if it is both n–subhomogeneous and n–superhomogeneous.

And the relation F may, for instance, be naturally called A–subhomogeneous, for some $A \subset \mathbb{N}$, if it is n–subhomogeneous for all $n \in A$.

Theorem 3.12. *If F is a superadditive relation on one groupoid X to another Y , then F is $\mathbb{N}$–superhomogeneous.*

Proof. Namely, for any $x \in X$, we have $1\,F(x) = F(x) = F(1\,x)$. Moreover, if $n \in \mathbb{N}$ such that $n\,F(x) \subset F(n\,x)$, then we also have

$$(n+1)F(x) \subset n\,F(x) + F(x) \subset F(n\,x) + F(x) \subset F(n\,x + x) = F\big((n+1)\,x\big) .$$

Hence, it is clear that in particular we also have

Corollary 3.13. *If f is an additive function of one groupoid X to another Y, then f is $\mathbb{N}$–homogeneous.*

Remark 3.14. Note that if F is a relation on one groupoid X with zero to another Y such that $0 \in F(0)$, then we have $0\,F(x) \subset \{0\} \subset F(0) = F(0\,x)$ for all $x \in X$.

In addition to Definition 3.10, we shall also need the following

Definition 3.15. A relation F of one group X to another Y is called odd if $F(-x) = -F(x)$ for all $x \in X$.

Remark 3.16. Quite similarly, a relation F on a group X to a set Y may be naturally called even if $F(-x) = F(x)$ for all $x \in X$.

Now, in contrast to subodd and superodd functionals [8] , the subodd and superodd relations need not be introduced since we have the following

Theorem 3.17. *If F is a relation on one group X to another Y, then the following assertions are equivalent:*

(1) F *is odd;*

(2) $F(-x) \subset -F(x)$ *for all* $x \in X$*;*

(3) $-F(x) \subset F(-x)$ *for all* $x \in X$.

Proof. Note that if (2) holds, then we also have

$$-F(x) = -F\big(-(-x)\big) \subset -\big(-F(-x)\big) = F(-x)$$

for all $x \in X$. Therefore, (3), and thus (1) also holds.

Remark 3.18. Now, we can also state that the relation F is odd if and only if $-F \subset F$, and thus $-F = F$.

In this respect, it is also worth mentioning that an even relation on one group to another is odd if and only if its inverse is even.

By using an analogue of Definition 3.10, we can also prove the following

Theorem 3.19. *If F is an odd $\mathbb{N}$–subhomogeneous ($\mathbb{N}$–superhomogeneous) relation on one group X to another Y, then F is $\mathbb{Z} \setminus \{0\}$–subhomogeneous ($\mathbb{Z} \setminus \{0\}$–superhomogeneous).*

Proof. In the superhomogeneous case, for any $x \in X$ and $n \in \mathbb{N}$, we also have

$$(-n)\, F(x) = -\big(\, n\, F(x)\big) \subset -F(n\, x) = F(-n\, x) = F((-n)\, x)\, .$$

Now, as an immediate consequence of Theorems 3.12 and 3.19, we can also state

Theorem 3.20. *If F is an odd superadditive relation on one group X to another Y, then F is $\mathbb{Z} \setminus \{0\}$–superhomogeneous.*

Remark 3.21. Thus, if in addition $0 \in F(0)$ also holds, then by Remark 3.14 we can also state that F is $\mathbb{Z}$–superhomogeneous.

Hence, it is clear that in particular we also have

Corollary 3.22. *If f is an additive function of one group X to another Y, then f is $\mathbb{Z}$–homogeneous.*

Proof. Namely, $f(0) = f(0) + f(0)$, and thus $f(0) = 0$. Moreover, $f(x) + f(-x) = f(0) = 0$, and thus $f(-x) = -f(x)$ for all $x \in X$. Therefore, Theorem 3.20 and Remark 3.21 can be applied.

In addition to Theorem 3.12, it is also worth mentioning the following

Theorem 3.23. *If F is a subadditive relation on a groupoid X to a vector space Y over $\mathbb{K}$ such that $F(x)$ is n^{-1}–affine for all $n \in \mathbb{N}$ and $x \in X$, then F is $\mathbb{N}$–subhomogeneous.*

Proof. Namely, for any $x \in X$, we have $1\, F(x) = F(x) = F(1\, x)$. Moreover, if $n \in \mathbb{N}$ such that $F(n\, x) \subset n\, F(x)$, then we also have

$$F\big((n+1)x\big) = F(n\, x + x) \subset F(n\, x) + F(x) \subset n\, F(x) + F(x) = F(x) + n\, F(x) =$$
$$= (\, n+1\,)\Big(\, (\, n+1\,)^{-1} F(x) + \big(\, 1 - (\, n+1\,)^{-1}\big)\, F(x)\Big) \subset (\, n+1\,)\, F(x)\, .$$

Now, as an immediate consequence of Theorems 3.23 and 3.19, we can also state

Theorem 3.24. *If F is an odd subadditive relation on a group X to a vector space Y over $\mathbb{K}$ such that $F(x)$ is n^{-1}–affine for all $n \in \mathbb{N}$ and $x \in X$, then F is $\mathbb{Z} \setminus \{0\}$–subhomogeneous.*

Remark 3.25. Note that if F is an inversion-subadditive relation on a group X to a groupoid Y with zero such that $F(0) \subset \{0\}$, then we also have $F(0\,x) = F(0) \subset 0\,F(x)$ for all $\in X$. Namely, if $F(x) = \emptyset$ for some $x \in X$, then because of $F(0) \subset F(x) + F(-x)$, we also have $F(0) = \emptyset$.

4. Further important homogeneity properties of relations

In addition to Definition 3.15, it is also worth introducing the following

Definition 4.1. A relation F on a group X to a groupoid Y with zero is called quasi-odd if $0 \in F(x) + F(-x)$ for all $x \in D_F$.

Remark 4.2. Thus, an odd relation is, in particular, quasi-odd. Moreover, each reflexive relation F of a symmetric subset D of group X to itself is quasi-odd.

Furthermore, we can also state that if F is an inversion-semi-subadditive relation on a group X to a groupoid Y with zero such that $0 \in F(0)$, then F is quasi-odd.

Now, as an improvement of [67, Theorem 3.6], we can also prove

Theorem 4.3. *If F is a nonvoid, quasi-odd and superadditive relation on a group X to a monoid Y, then $0 \in F(0)$ and F is quasi-additive.*

Proof. If $x \in D_F$, then $0 \in F(x) + F(-x) \subset F(0)$. Moreover,

$$F(x+y) \subset F(x) + F(-x) + F(x+y) \subset F(x) + F(y)$$

for all $y \in X$. The case $x \in X$ and $y \in D_F$ can be treated quite similarly.

Moreover, as a simple reformulation of Definition 4.1, we can also state

Theorem 4.4. *A relation F on one group X to another Y, then the following assertions are equivalent:*

(1) F is quasi-odd; *(2) $-F(x) \cap F(-x) \neq \emptyset$ for all $x \in D_F$.*

Definition 4.5. A partial selection relation Φ of a relation F on one group X to another Y is called odd-like if $-\Phi(x) \subset F(-x)$ for all $x \in X$.

Remark 4.6. Note that if Φ is an odd partial selection relation of F, then $-\Phi(x) = \Phi(-x) \subset F(-x)$ for all $x \in X$. Therefore, Φ is odd-like.

Moreover, if Φ is a partial selection relation of F and F is odd, then $-\Phi(x) \subset -F(x) = F(-x)$ for all $x \in X$. Therefore, Φ is again odd-like.

Now, in addition to Theorem 4.4, we can also easily establish the following

Theorem 4.7. *If F is a relation on one group X to another Y, then the following assertions are equivalent:*

(1) F is quasi-odd; *(2) F has an odd-like selection function.*

Proof. For instance, if (1) holds, then by Theorem 4.4 we have $-F(x) \cap F(-x) \neq \emptyset$, and hence $F(x) \cap (-F(-x)) \neq \emptyset$ for all $x \in D_F$. Thus, by the Axiom of Choice, there exists a function φ of D_F to Y such that $\varphi(x) \in F(x) \cap (-F(-x))$, and hence $\varphi(x) \in F(x)$ and $-\varphi(x) \in F(-x)$ for all $x \in D_F$. Therefore, φ is an odd-like selection function of F, and thus (2) also holds.

Remark 4.8. Necessary and sufficient conditions in order that a relation on one group to another could have an odd selection function have been given in [25].

Definition 4.9. A selection relation Φ of a relation F on a groupoid X with zero to an arbitrary one Y is called

(1) left-representing $F(x) = \Phi(x) + F(0)$ for all $x \in X$;

(2) right-representing if $F(x) = F(0) + \Phi(x)$ for all $x \in X$.

Remark 4.10. Now, a selection relation Φ of F may be naturally called representing if it both left-representing and right-representing. However, this terminology differs from the earlier one [73].

The importance of quasi-odd relations is also quite obvious from the following

Theorem 4.11. *If F is a right-zero-superadditive and inversion-superadditive relation on one group X to another Y and Φ is an odd-like selection relation of F, then Φ is a left-representing selection relation of F.*

Proof. For any $x \in X$, we have $\Phi(x) + F(0) \subset F(x) + F(0) \subset F(x)$ and

$$F(x) \subset \Phi(x) - \Phi(x) + F(x) \subset \Phi(x) + F(-x) + F(x) \subset \Phi(x) + F(0).$$

Therefore, $F(x) = \Phi(x) + F(0)$, and thus the required assertion is also true.

Remark 4.12. Note that if φ is a selection function of a left-zero-superadditive relation F on a groupoid X with zero to an arbitrary one Y such that $F(x) \subset \varphi(x) + F(0)$ for all $x \in X$, then we have

$$F(0) + \varphi(x) \subset F(0) + F(x) \subset F(x) \subset \varphi(x) + F(0)$$

for all $x \in X$.

Therefore, if in particular Y is a group, then for any $x \in X$ and $u \in \varphi^{-1}(-\varphi(x))$ we also have

$$\begin{aligned} \varphi(x) + F(0) = \varphi(x) + F(0) - \varphi(x) + \varphi(x) = \varphi(x) + F(0) + \varphi(u) + \varphi(x) \subset \\ \subset \varphi(x) + \varphi(u) + F(0) + \varphi(x) = \varphi(x) - \varphi(x) + F(0) + \varphi(x) = F(0) + \varphi(x). \end{aligned}$$

Therefore, if in addition $-\varphi[X] \subset \varphi[X]$ also holds, then we have $\varphi(x) + F(0) = F(0) + \varphi(x)$ for all $x \in X$. Thus, φ is a representing selection function of F whenever F is, in particular, zero-superadditive.

However, it is now more important to note that, as an immediate consequence of Theorems 4.7 and 4.11, we can also state

Corollary 4.13. *If F is a quasi-odd and inversion-superadditive relation on one group X to another Y such that $F(0) \subset \{0\}$, then F is a function.*

Proof. Now, by Theorem 4.7, F has an odd-like selection function φ. Moreover, by Theorem 3.7, F is zero-superadditive. Thus, by Theorem 4.11, we have $F(x) = \varphi(x) + F(0) \subset \varphi(x) + \{0\} = \varphi(x)$ for all $x \in X$. Therefore, $F \subset \varphi$, and thus $F = \varphi$ also holds.

Remark 4.14. Some deeper sufficient conditions in order that a relation should be a function have been given by Nikodem and Popa [44].

Analogously to Definition 3.10, we may also naturally introduce the following

Definition 4.15. For some $\lambda \in \mathbb{K}$, a relation F on one vector space X over $\mathbb{K}$ to another Y is called

(1) λ–subhomogeneous if $F(\lambda x) \subset \lambda F(x)$ for all $x \in X$;

(2) λ–superhomogeneous if $\lambda F(x) \subset F(\lambda x)$ for all $x \in X$.

Remark 4.16. Now, F may be naturally called λ–homogeneous if it is both λ–subhomogeneous and λ–superhomogeneous.

Moreover, F may, for instance, be naturally called A–subhomogeneous, for some $A \subset \mathbb{K}$, if it is λ–subhomogeneous for all $\lambda \in A$.

In particular, F will be called subhomogeneous if it is $\mathbb{K} \setminus \{0\}$-subhomogeneous. Namely, the 0–subhomogeneity is a too restrictive property.

Now, as a counterpart of Theorem 3.17, we can prove the following

Theorem 4.17. *If F is a subhomogeneous (superhomogeneous) relation on one vector space X over $\mathbb{K}$ to another Y, then F is homogeneous.*

Proof. If F is subhomogeneous, then we also have

$$\lambda F(x) = \lambda F(\lambda^{-1} \lambda x) \subset \lambda \lambda^{-1} F(\lambda x) = F(\lambda x)$$

for all $x \in X$ and $\lambda \in \mathbb{K} \setminus \{0\}$. Therefore, F is also superhomogeneous.

Remark 4.18. Now, we can also state that a relation F on one vector space X over $\mathbb{K}$ to another Y is homogeneous if and only if $\lambda F \subset F$ for all $\lambda \in \mathbb{K} \setminus \{0\}$, and thus $\lambda F = F$ for all $\lambda \in \mathbb{K} \setminus \{0\}$.

Definition 4.19. A relation F on one vector space X over $\mathbb{K}$ to another Y is called

(1) sublinear if it is both homogeneous and subadditive;

(2) superlinear if it is both homogeneous and superadditive.

Remark 4.20. Quite similarly, the relation F may be naturally called linear if it is both homogeneous and additive.

Moreover, F may, for instance, be naturally called quasi-linear if it is both homogeneous and quasi-additive. Namely, by Theorem 4.3, we have the following

Theorem 4.21. *If F is a nonvoid superlinear relation on one vector space X over $\mathbb{K}$ to another Y, then $0 \in F(0)$ and F is quasi-linear.*

Remark 4.22. Now, we can also state a nonvoid relation F on one vector space X over $\mathbb{K}$ to another Y is quasi-linear if and only if F is a linear subspace of the product space $X \times Y$. Thus, our present terminology differs from the earlier one [64].

5. Direct sum decompositions of groupoids

Definition 5.1. If U and V are subsets of a groupoid X such that for every $x \in X$ there exists a unique pair $(u_x, v_x) \in U \times V$ such that

$$x = u_x + v_x ,$$

then we say that X is the direct sum of U and V and we write $X = U \oplus V$.

Remark 5.2. Here, we could naturally assume that U and V are subgroupoids of X in the sense that they are closed under addition.

Definition 5.3. Two subsets U and V of a groupoid X will be called here commuting if $u + v = v + u$ for all $u \in U$ and $v \in V$.

Remark 5.4. In this case, we should rather say that U and V are elementwise commuting. Since the sets U and V are usually called commuting if $U+V = V+U$.

Note that if U and V are commuting in the sense of Definition 5.3, then we have not only $U + V = V + U$, but also $u + V = V + u$ and $U + v = v + U$ for all $u \in U$ and $v \in V$.

Theorem 5.5. *If U and V are commuting subgroupoids of a semigroup X such that $X = U \oplus V$, then the mappings*

$$x \ \mapsto \ u_x \qquad \textit{and} \qquad x \ \mapsto \ v_x ,$$

where $x \in X$, are additive. Thus, in particular, they are $\mathbb{N}$–homogeneous.

Proof. If $x, y \in X$, then by the assumed associativity and commutativity properties of the addition in X we have

$$x + y = (u_x + v_x) + (u_y + v_y) = (u_x + u_y) + (v_x + v_y) .$$

Therefore, since $u_x + u_y \in U$ and $v_x + v_y \in V$, the equalities

$$u_{x+y} = u_x + u_y \qquad \text{and} \qquad v_{x+y} = v_x + v_y$$

are also true. Now, by Corollary 3.13, it is clear that the second statement of the theorem is also true.

Remark 5.6. If in particular X is a group, then by Corollary 3.22 we can also state that the above mappings are $\mathbb{Z}$–homogeneous.

Thus, if $0 \in U \cap V$, then U and V are subgroups of X. Namely, if for instance $x \in U$, then because of $x = x + 0$ and $0 \in V$, we necessarily have $u_x = x$. Therefore, $-x = -u_x = u_{-x} \in U$ also holds.

Remark 5.7. If more specially X is a vector space over $\mathbb{K}$ and U and V are subspaces of X, then we can immediately see that the corresponding mappings are $\mathbb{K}$–homogeneous.

Namely, for any $x \in X$ and $\lambda \in \mathbb{K}$, we have $\lambda\, x = \lambda\,(u_x + v_x) = \lambda\, u_x + \lambda\, v_x$. Hence, by using that $\lambda\, u_x \in U$ and $\lambda\, v_x \in V$, we can already infer that $u_{\lambda\, x} = \lambda\, u_x$ and $v_{\lambda\, x} = \lambda\, v_x$.

In addition to Theorem 5.5, it is also worth proving the following two theorems.

Theorem 5.8. *If U and V are subsets of a semigroup X such that $X = U + V$, then the following assertions are equivalent:*

(1) X is commutative;

(2) U and V are commutative and commuting.

Proof. Note that if $x\,, y \in X$, then because of $X = U + V$ there exist $u\,, \omega \in U$ and $v\,, w \in V$ such that $x = u + v$ and $y = \omega + w$. Hence, if (2) holds, we can already see that

$$x + y = u + v + \omega + w = u + \omega + v + w = \omega + u + w + v = \omega + w + u + v = y + x\,.$$

Therefore, (1) also holds.

Theorem 5.9. *If U and V are subsets of a groupoid X such that $X = U \oplus V$, then the following assertions are equivalent:*

(1) U and V are commuting;

(2) $u + V = V + u$ and $v + U = U + v$ for all $u \in U$ and $v \in V$;

(3) $u + V \subset V + u$ and $v + U \subset U + v$ for all $u \in U$ and $v \in V$;

(4) $V + u \subset u + V$ and $U + v \subset v + U$ for all $u \in U$ and $v \in V$.

Proof. Note that if for instance (3) holds, then for any $u \in U$ and $v \in V$ we have $u + v \in u + V \subset V + u$. Therefore, there exists $w \in V$ such that $u + v = w + u$. Moreover, again by (3), we can see that $w + u \in w + U \subset U + w$. Therefore, there exists $\omega \in U$ such that $w + u = \omega + w$. Thus, we also have $u + v = \omega + w$. Hence, by using that $X = U \oplus V$, we can infer that $u = \omega$ and $v = w$. Therefore, $u + v = v + u$, and thus (1) is also true.

In the sequel, we shall also use the following

Definition 5.10. A subgroupoid U of a monoid X is called a submonoid of X if $0 \in U$.

Moreover, a submonoid U of a monoid X is called a subgroup of X if each member of U has an additive inverse in U.

Remark 5.11. Thus, a subset U of a monoid X is a subgroup of X if and only if $0 \in U$ and U is a group with the restriction of the addition in X to U^2.

Now, we can also easily prove the following two theorems.

Theorem 5.12. *If U is a subgroup of a monoid X, then for any $V \subset X$ the following assertions are equivalent:*

(1) $u + V = V + u$ *for all* $u \in U$;

(2) $u + V \subset V + u$ *for all* $u \in U$;

(3) $V + u \subset u + V$ *for all* $u \in U$.

Proof. Note that if for instance (2) holds, then

$$V + u = 0 + V + u = u - u + V + u \subset u + V - u + u = u + V + 0 = u + V$$

also holds for all $u \in U$.

Theorem 5.13. *If U and V are subgroups of a monoid X, then the following assertions are equivalent:*

(1) $X = U \oplus V$; *(2)* $X = U + V$ *and* $U \cap V = \{0\}$.

Proof. Note that if (1) holds and $x \in U \cap V$, then because of $x = 0 + x$ and $x = x + 0$ we have $u_x = 0$ and $v_x = 0$. Therefore, $x = u_x + v_x = 0$ also holds.

While, if (2) holds and $x = u_1 + v_1$ and $x = u_2 + v_2$ for some $u_1, u_2 \in U$ and $v_1, v_2 \in V$, then $u_1 + v_1 = u_2 + v_2$, and thus $-u_2 + u_1 = v_2 - v_1$. Hence, since $-u_2 + u_1 \in U$ and $v_2 - v_1 \in V$, we can already infer that $-u_2 + u_1 = 0$ and $v_2 - v_1 = 0$. Therefore, $u_1 = u_2$, and $v_1 = v_2$ also hold.

Remark 5.14. In this theorem, we could naturally assume that X is also a group. Namely, if U and V are subgroups of monoid X such that $X = U + V$, then for any $x \in X$ there exist $u \in U$ and $v \in V$ such that $x = u + v$. Hence, by taking $y = -v - u$, we can see that $x + y = 0$ and $y + x = 0$. Therefore, $-x = y$, and thus X is also a group.

Now, a as useful consequence of Theorem 5.13, we can also state

Corollary 5.15. *If V is an $\mathbb{N}$–divisible subgroup of an $\mathbb{N}$–cancellable group X and $a \in X \setminus V$ such that, under the notation*

$$U = \mathbb{Z}\, a = \big\{\, k\, a : \quad k \in \mathbb{Z} \,\big\},$$

we have $X = U + V$, then we actually have $X = U \oplus V$.

Proof. By Theorem 2.8, it is clear that U is also a subgroup of X. Thus, by Theorem 5.13, it is enough to show only that $U \cap V \subset \{0\}$. That is, $x \in U \cap V$ implies $x = 0$.

For this, note that if $x \in U$, then there exists $k \in \mathbb{Z}$ such that $x = k\, a$. Moreover, if $x \neq 0$, then $k \neq 0$. Therefore, if $x \in V$ also holds, then by Remark 2.16 there exists $v \in V$ such that $x = k\, v$. Thus, we have $k\, a = k\, v$. Hence, by Remark 2.16, it follows that $a = v$, and thus $a \in V$. This contradiction proves the required assertion.

From the above proof, it is clear that more specially we also have

Corollary 5.16. *If V is a subspace of a vector space X over $\mathbb{K}$ and $a \in X \setminus V$ such that, under the notation*

$$U = \mathbb{K}\, a = \left\{ \lambda\, a : \quad \lambda \in \mathbb{K} \right\},$$

we have $X = U + V$, then we actually have $X = U \oplus V$.

In connection with direct sums, we shall also need the following basic theorem which is usually proved with the help of Hamel bases [74 , p. 43] .

Theorem 5.17. *If X is an vector space over $\mathbb{K}$, then for each subspace V of X there exists a subspace U of X such that $X = U \oplus V$.*

Proof. Denote by $\mathcal{A}$ the family of all subspaces A of X such that $A \cap V = \{0\}$. Then, $\{0\} \in \mathcal{A}$, and thus $\mathcal{A}$ is a nonvoid partially ordered set with set inclusion. Thus, by the Hausdorff maximal principle, there exists a nonvoid maximal linearly ordered subset $\mathcal{U}$ of $\mathcal{A}$. Define, $U = \bigcup \mathcal{U}$. Then, it is clear that $0 \in U$. Moreover, we can easily see that $U + U \subset U$. Namely, if $x_1, x_2 \in U$, then by the definition of U there exist $U_1, U_2 \in \mathcal{U}$ such that $x_1 \in U_1$ and $x_2 \in U_2$. Moreover, since $\mathcal{U}$ is linearly ordered we have either $U_2 \subset U_1$ or $U_1 \subset U_2$. Therefore, either

$$x_1 + x_2 \in U_1 + U_2 \subset U_1 + U_1 \subset U_1 \subset U$$

or

$$x_1 + x_2 \in U_1 + U_2 \subset U_2 + U_2 \subset U_2 \subset U$$

holds. Hence, since $0\, U = \{0\} \subset U$ and

$$\lambda\, U = \lambda \bigcup_{A \in \mathcal{U}} A = \bigcup_{A \in \mathcal{U}} \lambda\, A \subset \bigcup_{A \in \mathcal{U}} A = U$$

also holds for all $\lambda \in \mathbb{K}$ with $\lambda \neq 0$, it is clear that U is a subspace of X.

Moreover, we can also easily see that

$$U \cap V = \Big(\bigcup_{A \in \mathcal{U}} A \Big) \cap V = \bigcup_{A \in \mathcal{U}} A \cap V = \bigcup_{A \in \mathcal{U}} \{0\} = \{0\}.$$

Therefore, by Theorem 5.13, we need only show that $X = U + V$. For this, assume on the contrary that there exists $a \in X$ such that $a \notin U + V$. Moreover, define

$$U^* = \mathbb{K}\, a + U, \qquad \text{where} \qquad \mathbb{K}\, a = \left\{ \lambda\, a : \ \lambda \in \mathbb{K} \right\}.$$

Then, it is clear that U^* is a subspace of X which properly contains U. Moreover, we can easily see that $x \in U^* \cap V$ implies $x = 0$. Namely, if $x \in U^*$, then there exist $\lambda \in \mathbb{K}$ and $u \in U$ such that $x = \lambda\, a + u$. Moreover, if in addition $x \in V$ and $x \neq 0$, then we necessarily have $\lambda \neq 0$, since $U \cap V = \{0\}$. Now, we can already see that

$$\lambda\, a = x - u \in V - U \subset U + V, \ \text{ and thus } \ a \in \lambda^{-1} (U + V) = \lambda^{-1} V + \lambda^{-1} U \subset U + V,$$

which is a contradiction. Therefore, we have $U^* \cap V = \{0\}$, and thus $U^* \in \mathcal{A}$. Hence, it is clear that such that $\mathcal{U}^* = \mathcal{U} \cup \{ U^* \}$ is also a linearly ordered subset of $\mathcal{A}$. However, this contradicts the maximality of $\mathcal{U}$ since $U^* \notin \mathcal{U}$.

Remark 5.18. In a former unpublished manuscript, by using a more delicate argument, the second author proved that if X is an $\mathbb{N}$–cancellable commutative group such that each subgroup of X is $\mathbb{N}$–divisible, then for each subgroup V of X there exists a subgroup U of X such that $X = U \oplus V$.

However, the first author has later observed that this statement is quite nonsense since if X is a group such that each subgroup of X is $\mathbb{N}$–divisible, then we necessarily have $X = \{0\}$. Namely, if there exists $a \in X$ such that $a \neq 0$, then it can be shown that $U = \mathbb{Z}\, a$ is a subgroup of X such that U is not $\mathbb{N}$–divisible.

6. Constructions of additive relations on line sets

Theorem 6.1. *Let X and Y be monoids. Suppose that $a \in X \setminus \{0\}$, $b \in Y$ and $\emptyset \neq C \subset Y$ such that*

(1) $b + C = C + b$ *and* $C = C + C$,

(2) $n\, a = m\, a$ *implies* $n\, b + C = m\, b + C$ *for all* $n, m \in \{0\} \cup \mathbb{N}$.

Then, there exists a unique additive relation F of the monoid $U = \{n\, a\}_{n=0}^{\infty}$ to Y such that $F(0) = C$ and $F(a) = b + C$.

Proof. If F is as above, then by induction we can see that

$$F\,(n\, a) = n\, b + C$$

for all $n \in \{0\} \cup \mathbb{N}$. Namely, $F(\,0\, a\,) = F(0) = C = 0\, b + C$. Moreover, if $n \in \{0\} \cup \mathbb{N}$ such that the required equality holds, then we also have

$$\begin{aligned} F((n+1)\, a) &= F(n\, a + a) = F(n\, a) + F(a) = \\ &= n\, b + C + b + C = n\, b + b + C + C = (n+1)\, b + C\,. \end{aligned}$$

Therefore, the unicity part of the theorem is true.

To prove the existence part of the theorem, note that by (2) we may unambiguously define a relation F of U to Y such that

$$F(na) = n\, b + C$$

for all $n \in \{0\} \cup \mathbb{N}$. Thus, we evidently have $F(0) = C$ and $F(a) = b + C$. Moreover, by induction, we can see that

$$n\, b + C = C + n\, b$$

for all $n \in \{0\} \cup \mathbb{N}$. Hence, it is clear that

$$\begin{aligned} F(\,n\, a + m\, a\,) &= F\big((n+m)a\big) = (n+m)\, b + C = \\ &= n\, b + m\, b + C + C = n\, b + C + m\, b + C = F(n\, a) + F(m\, a) \end{aligned}$$

for all $n, m \in \{0\} \cup \mathbb{N}$. Therefore, F is additive.

Remark 6.2. If in particular C is m–divisible for some $m \in \mathbb{N}$, then in addition to $m C \subset C$ we also have $C \subset m C$. Therefore, if Y is commutative, then we can see that

$$F\big(m\,(n\,a)\big) = F\big((m\,n)\,a\big) = (m\,n)\,b + C =$$
$$= m\,(n\,b) + m\,C = m\,(\,n\,b + C\,) = m\,F\,(\,n\,a\,)$$

for all $n \in \mathbb{N}$. Thus, F is m–homogeneneous.

Analogously to the above theorem, we can also prove the following

Theorem 6.3. *Let X and Y be groups. Suppose that $a \in X \setminus \{0\}$, $b \in Y$ and C is a subgroup of Y such that*

(1) $b + C = C + b$,

(2) $n\,a = 0$ *implies* $n\,b \in C$ *for all* $n \in \mathbb{N}$.

Then, there exists a unique odd additive relation F of the group $U = \mathbb{Z}\,a$ to Y such that $F(0) = C$ and $F(a) = b + C$.

Proof. Now, if F is as above, then in addition to our former observation on F we can see that

$$F((-n)a\,) = F(\,-n\,a\,) = -F(\,n\,a\,) =$$
$$= -(\,n\,b + C\,) = -(\,C + n\,b\,) = -n\,b - C = (-n)b + C$$

for all $n \in \mathbb{N}$. Therefore, the unicity part of the theorem is true.

Quite similarly, we can also note that if $n \in \mathbb{N}$ such that $(-n\,)\,a = 0$, then we also have $-n\,a = 0$, and thus $n\,a = 0$. Hence, by (2), it follows that $n\,b \in C$. Thus, $(-n\,)\,b = -n\,b \in -C \subset C$ also holds. Therefore, (2) is now equivalent to the requirement that $k\,a = 0$ implies $k\,b \in C$ for all $k \in \mathbb{Z} \setminus \{0\}$.

Now, to prove the existence part of the theorem, we can note that if $k\,,\,l \in \mathbb{Z}$ such that $k\,a = l\,a$, then

$$(-l + k\,)\,a = (-l\,)a + k\,a = -l\,a + k\,a = 0\,.$$

Hence, by the above mentioned extension of (2), it follows that

$$-l\,b + k\,b = (\,-l\,)\,b + k\,b = (\,-l + k\,)\,b \in C\,.$$

Now, since C is a subgroup of X, we can already see that

$$-l\,b + k\,b + C = C \qquad \text{and thus} \qquad k\,b + C = l\,b + C\,.$$

Therefore, we may unambiguously define a relation F of U to Y such that

$$F(ka) = k\,b + C$$

for all $k \in \mathbb{Z}$. Thus, we evidently have $F(0) = C$ and $F(a) = b + C$. Moreover, in addition to our former observation on b and C, we can see that

$$(-n\,)\,b + C = -n\,b - C = -(\,C + n\,b\,) = -(\,n\,b + C\,) = -C - n\,b = C + (-n)\,b$$

for all $n \in \mathbb{N}$. Hence, it is clear that

$$F(-k\,a) = F\big((-k)a\big) = (-k)b + C = \\ = -k\,b - C = -(\,C + k\,b) = -(k\,b + C\,) = -F(\,k\,a\,)$$

and

$$F(\,k\,a + l\,a\,) = F\big((k+l)a\big) = (k+l)\,b + C = \\ = k\,b + l\,b + C + C = k\,b + C + l\,b + C = F(k\,a) + F(l\,a)$$

for all $k, l \in \mathbb{Z}$. Therefore, F is odd and additive.

Remark 6.4. If in particular C is $\mathbb{N}$–divisible and Y is commutative, then by Remark 6.2 F is $\mathbb{N}$–homogeneneous. Thus, by Theorem 3.19, F is also $\mathbb{Z}\setminus\{0\}$–homogeneous.

Remark 6.5. If more specially, X and Y are vector spaces over $\mathbb{K}$, $a \in X\setminus\{0\}$, $b \in Y$ and C is a subspace of Y, then we can quite similarly see that there exists a unique linear relation F of the space $U = \mathbb{K}\,a$ to Y such that $F(0) = C$ and $F(a) = b + C$.

7. Constructions of additive relations on sum sets

Analogously to Definition 5.3, we may also introduce the following

Definition 7.1. Two relations F and G of some subsets U and V of a set X to a groupoid Y, respectively, will be called here commuting if $F(u) + G(v) = G(v) + F(u)$ for all $u \in U$ and $v \in V$.

Remark 7.2. In this case, more precisely, we should say that F and G are pointwise commuting with respect to the addition in $\mathcal{P}(Y)$. Namely, two relations F and G are usually called commuting if $F \circ G = G \circ F$.

Now, in addition to Theorems 6.1, we can also prove the following

Theorem 7.3. *Suppose that U and V are commuting submonoids of a monoid of X such that*

$$X = U \oplus V\,.$$

Moreover, assume that F and G are commuting additive relations of U and V to a semigroup Y, respectively, such that $F(0) = G(0)$. Then there exists a unique additive relation H of X to Y that extends both F and G.

Proof. If H is as above, then it is clear that

$$H(x) = H\,(u_x + v_x) = H(u_x) + H(v_x) = F(u_x) + G(v_x)$$

for all $x \in X$. Therefore, the unicity part of the theorem is true.

To prove the existence part of the theorem, define a relation H of X to Y such that

$$H(x) = F(u_x) + G(v_x)$$

for all $x \in X$. Then, by Theorem 5.5, for any $s \in U$ and $t \in V$ we have

$$H(s+t) = F(u_{s+t}) + G(v_{s+t}) =$$
$$= F(u_s + u_t) + G(v_s + v_t) = F(s+0) + F(0+t) = F(s) + G(t).$$

Hence, in particular it is clear that

$$H(s) = H(s+0) = F(s) + G(0) = F(s) + F(0) = F(s)$$

and

$$H(t) = H(0+t) = F(0) + G(t) = G(0) + G(t) = G(t).$$

Therefore, H extends both F and G.

Moreover, by taking $\omega \in U$ and $w \in V$, we can also easily see that

$$H\big((s+t) + (\omega + w)\big) = H\big((s+\omega) + (t+w)\big) =$$
$$= F(s+\omega) + G(t+w) = \big(F(s) + F(\omega)\big) + \big(G(t) + G(w)\big) =$$
$$= \big(F(s) + G(t)\big) + \big(F(\omega) + G(w)\big) = H(s+t) + H(\omega + w).$$

Therefore, H is also additive.

Remark 7.4. If in particular F and G are n–homogeneous, for some $n \in \mathbb{N}$, and Y is commutative, then for any $s \in U$ and $t \in V$ we have

$$H\big(n(s+t)\big) = H(ns + nt) = F(ns) + G(nt) =$$
$$= nF(s) + nG(t) = n\big(F(s) + G(t)\big) = nH(s+t).$$

Therefore, H is also n–homogeneous.

However, it is now more interesting that we can also prove the following

Theorem 7.5. *Suppose that U and V are commuting subgroups of a group X such that*

$$X = U + V.$$

Moreover, assume that F and G are commuting additive relations of U and V to a group Y, respectively, such that $F(x) = G(x)$ for all $x \in U \cap V$. Then there exists a unique additive relation H of X to Y that extends both F and G.

Proof. If H is as above, then we again have

$$H(u+v) = H(u) + H(v) = F(u) + G(v)$$

for all $u \in U$ and $v \in V$. Hence, the unicity part of the theorem is quite obvious.

To prove the existence part of the theorem, note that if $u_1, u_2 \in U$ and $v_1, v_2 \in V$ such that

$$u_1 + v_1 = u_2 + v_2,$$

then $-u_2 + u_1 = v_2 - v_1$. Hence, it is clear that, in addition to $-u_2 + u_1 \in U$ and $v_2 - v_1 \in V$, we also have

$$-u_2 + u_1 \in V \qquad \text{and} \qquad v_2 - v_1 \in U.$$

Thus, in particular $F(v_2-v_1)=G(-u_2+u_1)$ also holds. Now, we can already observe that

$$F(u_1)=F(u_2+v_2-v_1)=F(u_2)+F(v_2-v_1)=F(u_2)+G(-u_2+u_1)$$

and

$$G(v_1)=G(-u_1+u_2+v_2)=G\big(-(-u_2+u_1)+v_2\big)=G\big(-(-u_2+u_1)\big)+G(v_2),$$

and thus

$$\begin{aligned}&F(u_1)+G(v_1)=\\ &=F(u_2)+G(-u_2+u_1)+G\big(-(-u_2+u_1)\big)+G(v_2)=F(u_2)+G(0)+G(v_2)=\\ &\qquad\qquad=F(u_2)+G(v_2).\end{aligned}$$

Therefore, we may unambiguously define a relation H of $X=U+V$ to Y such that

$$H(u+v)=F(u)+G(v)$$

for all $u\in U$ and $v\in V$. Now, quite similarly as in the proof of Theorem 7.3, we can see that H has the required properties.

Remark 7.6. If if in particular F and G are odd, then we also have

$$\begin{aligned}&H\big(-(u+v)\big)=H\big(-(v+u)\big)=H(-u+(-v))=F(-u)+G(-v)=\\ &\quad=G(-v)+F(-u)=-G(v)+\big(-F(u)\big)=-\big(F(u)+G(v)\big)=-H(u+v)\end{aligned}$$

for all $u\in U$ and $v\in V$. Therefore, H is also odd.

Remark 7.7. Moreover, if in particular F and G are n–homogeneous, for some $n\in\mathbb{N}$, and Y is commutative, then quite similarly as in Remark 7.4 we can see that H is also n–homogeneous.

Therefore, if in particular F and G are odd and $\mathbb{N}$–homogeneous and Y is commutative, then by the above observations and Theorem 3.19 we can also state that H is $\mathbb{Z}\setminus\{0\}$–homogeneous.

Remark 7.8. Moreover, if more specially X and Y are vector spaces over $\mathbb{K}$, U and V are subspaces of X, and F and G are homogeneous, then we can also easily see that H is also homogeneous.

8. One-step extensions of additive relations

Now, as an immediate consequence of Theorems 6.1 and 7.3, we can also state

Theorem 8.1. *Let X and Y be commutative monoids. Suppose that G is an additive relation of a submonoid V of X to Y. Moreover, assume that $a\in X\setminus V$ and $b\in Y$ such that*

(1) $X=U\oplus V$ *holds with* $U=\big\{n\,a\big\}_{n=0}^{\infty}$,

(2) $n\,a=m\,a$ *implies* $n\,b+G(0)=m\,b+G(0)$ *for all* $n,m\in\{0\}\cup\mathbb{N}$.

Then, there exists a unique additive relation H of X to Y extending G such that $H(a) = b + G(0)$.

Proof. In particular, we have $G(0) \neq \emptyset$ and $G(0) = G(0) + G(0)$. Thus, by Theorem 6.1, there exists a unique additive relation F of U to Y such that $F(0) = G(0)$ and $F(a) = b + G(0)$. Moreover, by Theorem 7.3, there exists a unique additive relation H of X to Y which extends both F and G. Therefore, the required assertion is also true.

Remark 8.2. If in particular G is n–homogeneous for some $n \in \mathbb{N}$, then by Remarks 6.2 and 7.4, H is also n–homogeneous.

Moreover, as an immediate consequence of Theorems 6.3 and 7.3, we can also state

Theorem 8.3. *Let X and Y be commutative groups. Suppose that G is an odd superadditive relation of a subgroup V of X to Y. Moreover, assume that $a \in X \setminus V$ and $b \in Y$ such that*

(1) $X = U \oplus V$ holds with $U = \mathbb{Z}\, a$,

(2) $n\, a = 0$ implies $n\, b \in G(0)$ for all $n \in \mathbb{N}$.

Then, there exists a unique odd additive relation H of X to Y extending G such that $H(a) = b + G(0)$.

Proof. In particular, we have $G(0) \neq \emptyset$ and $G(0) - G(0) \subset G(0) + G(0) \subset G(0)$. Thus, $G(0)$ is a subgroup of Y. Therefore, by Theorem 6.3, there exists a unique odd additive relation F of U to Y such that $F(0) = G(0)$ and $F(a) = b + G(0)$.

Moreover, from Theorem 4.3, we can see that G is quasi-additive. Therefore, by Theorem 7.3, there exists a unique additive relation H of X to Y which extends both F and G. Moreover, by Remark 7.6, it is clear that H is also odd. Therefore, the required assertion is true.

Remark 8.4. If in particular X is $\mathbb{N}$–cancellable, then by Remark 2.13 we have $n\, a \neq 0$ for all $n \in \mathbb{N}$. Therefore, (2) automatically holds.

Moreover, if in addition V is $\mathbb{N}$–divisible, then by Corollary 5.15 the equality $X = U + V$ already implies that $X = U \oplus V$. Therefore, instead of (1) it is enough to assume only that $X = U + V$.

Remark 8.5. While, if in particular G is $\mathbb{N}$–subhomogeneous, then in particular $G(0) = G(\,n\,0\,) \subset n\, G(0)$ for all $n \in \mathbb{N}$. Thus, $G(0)$ is $\mathbb{N}$–divisible. Therefore, by Remark 6.2, F is $\mathbb{N}$–homogeneous. Moreover, from Theorem 3.12 we can see that G is also $\mathbb{N}$–homogeneous. Hence, by Remark 7.7, we can state that H is $\mathbb{Z} \setminus \{0\}$–homogeneous.

Remark 8.6. Moreover, if more specially X and Y are vector spaces over $\mathbb{K}$, G is a linear relation of a subspace V of X, and $a \in X \setminus V$ such that $X = U + V$ holds with $U = \mathbb{K}\, a$, then we can quite similarly see that for any $b \in Y$ there exists a unique linear relation H of X to Y extending G such that $H(a) = b + G(0)$.

However, it is now more interesting that, by using Theorems 6.3 and 7.5, we can prove the following

Theorem 8.7. *Let X and Y be commutative groups. Suppose that G is an odd $\mathbb{N}$–subhomogeneous superadditive relation of a subgroup V of X to Y. Moreover, assume that $a \in X \setminus V$ and $b \in Y$ such that*

(1) $X = U + V$ holds with $U = \mathbb{Z}\, a$,

(2) $n\, b \in G(n\, a)$ and Y is n–cancellable for some $n \in \mathbb{N}$.

Then, there exists a unique $\mathbb{Z} \setminus \{0\}$–homogeneous additive relation H of X to Y extending G such that $H(a) = b + \Phi(0)$.

Proof. Define

$$L = \big\{ k \in \mathbb{Z} : \quad k\, a \in V \big\} .$$

Then, it can be easily seen that L is an ideal of $\mathbb{Z}$. Moreover, if n is as in (2), then we have $n \in L$. Now, by Theorems 3.12 and 3.19, it is clear that

$$n\,(k\,b) = k\,(n\,b) \in k\,G\,(n\,a) = G\,\big(k\,(n\,a)\big) = G\,\big(n\,(k\,a)\big) = n\,G\,(k\,a)$$

for all $k \in L \setminus \{0\}$. Hence, by using the n–cancellabilty of Y and the inclusion $0 \in G(0)$, we can infer that

$$k\,b \in G(\,k\,a\,)$$

for all $k \in L$. Now, by Remark 4.6 and Theorem 4.11, it it is clear that

$$G\,(\,k\,a\,) = k\,b + G(0)$$

also holds for all $k \in L$.

Moreover, we can note that if $m \in \mathbb{Z}$ such that $m\,a = 0$, then $m \in L$. Therefore, $m\,b \in G\,(\,m\,a\,) = G(0)$. Thus, by Theorem 6.3 and Remark 6.4, there exists a unique $\mathbb{Z}\setminus\{0\}$–homogeneous additive relation F of U to Y such that $F(0) = G(0)$ and $F(a) = b + G(0)$. Moreover, by the proof of Theorem 6.3, we have

$$F\,(\,k\,a\,) = k\,b + G(0)$$

for all $k \in \mathbb{Z}$. Thus, in particular,

$$F(\,k\,a\,) = k\,b + G(0) = G(\,k\,a\,)$$

for all $k \in L$. Hence, by the definition of L , it follows that $F(x) = G(x)$ for all $x \in U \cap V$. Moreover, from Theorem 4.3 we can see that G is quasi-additive. Thus, by Theorem 7.5 and Remark 7.7, there exists a unique $\mathbb{Z}\setminus\{0\}$–homogeneous additive relation H of X to Y that extends both F and G .

Remark 8.8. If X and Y are commutative monoids, G is an additive relation of a submonoid V of X to Y, and $a \in X \setminus V$ such that $X = U \oplus V$ holds with $U = \big\{ n\,a \big\}_{n=0}^{\infty}$, then by Theorem 8.1 there exists a unique additive relation H of X to Y extending G such that $H\,(a) = 0 + G(0) = G(0)$. Moreover, by the proof of Theorem 6.1, we have $H\,(\,n\,a\,) = n\,0 + G(0) = G(0)$ for all $n \in \{0\} \cup \mathbb{N}$.

In this respect, it is also worth mentioning that, by using Theorem 7.3, we can easily prove the following

Theorem 8.9. *Suppose that U and V are commuting submonoids of a monoid of X such that*

$$X = U \oplus V \,.$$

Then, every additive relation G of V to a semigroup Y can be extended to an additive relation H of X to Y such that $H(u) = G(0)$ for all $u \in U$.

Proof. Define a relation F on U to Y such that $F(u) = G(0)$ for all $u \in U$. Then

$$F(\, u_1 + u_2\,) = G(0) = G(0) + G(0) = F(\, u_1\,) + F(\, u_2\,)$$

for all $u_1 ,\, u_2 \in U$, and

$$F(u) + G(v) = G(0) + G(v) = G(v) = G(v) + G(0) = G(v) + F(u)$$

for all $u \in U$ and $v \in V$. Thus, F is an additive relation of U to Y such that F and G are commuting. Hence, by Theorem 7.3, we can see that there exists a unique additive relation H of X to Y which extends both F and G. Thus, the required assertion is also true.

Remark 8.10. Note that if in particular X and Y are groups, U and V are subgroups of X, and G is odd, then F is also odd. Thus, by Remark 7.6, H is also odd.

By Theorems 5.17 and 8.9, it is clear that in particular we also have

Theorem 8.11. *If V is a subspace of a vector space X over $\mathbb{K}$, then any additive relation G of V to a semigroup Y can be extended to an additive relation H of X to Y such that $H(u) = 0$ for any u being in an algebraic complement U of V.*

Remark 8.12. Note that if in particular Y is also a vector space over $\mathbb{K}$ and G is linear, then the relation $F = U \times G(0)$ is also linear. Thus, by Remark 7.8, H is also linear.

9. The intersection convolution of relations

Definition 9.1. If X is a groupoid, then we define a relation Γ on X to $X^{\,2}$ such that

$$\Gamma\,(x) = \big\{ (u,\, v) \in X^{\,2} : \quad x = u + v \,\big\}$$

for all $x \in X$.

Moreover, for any $x \in X$ and $U ,\, V \subset X$, we define

$$\Delta\,(\, x,\, U,\, V\,) = \Gamma\,(x) \cap (U \times V\,)\,.$$

Remark 9.2. Note that thus Γ is just the inverse relation of the operation $+$ in X.

Moreover, for any $u\,,\, v \in X$, we have $(\,u\,,\, v) \in \Delta\,(\,x,\, U\,,\, V\,)$ if and only if $u \in U$ and $v \in V$ such that $x = u + v$.

Definition 9.3. If F and G are relations on one groupoid X to another Y, then we define a relation $F*G$ on X to Y such that

$$\big(F*G\big)(x)=\bigcap\,\big\{\,F(u)+G(v):\quad (u,\,v)\in\,\Delta\,(\,x,\,D_F\,,\,D_G)\,\big\}$$

for all $x\in X$. The relation $F*G$ will be called the intersection convolution of the relations F and G.

Remark 9.4. This definition has been introduced in [67] to improve the results of [60]. For some closely related notions, see also [40], [59], [6], [15] and [70].

The intersection convolution of relations is closely related not only the infimal convolution of functionals, but also to the global sum of relations [24] and the composition and box products of relations [68].

In particular, in [67,], the second author has proved the following

Theorem 9.5. *If F and G are relations on a group X to a groupoid Y, then for any $x\in X$ we have*

$$(\,F*G\,)(x)=\bigcap\,\big\{\,F\,(x-v)+G\,(v):\quad v\in(-D_F+x\,)\cap\,D_G\big\}=$$
$$=\bigcap\,\big\{\,F\,(u)+G\,(-u+x):\quad u\in\,D_F\cap\,(\,x-D_G\,)\big\}\,.$$

Hence, by using that $-X+x=X$ and $x-X=X$ for all $x\in X$, we can immediately derive

Corollary 9.6. *If F and G are relations on a group X to a groupoid Y, then for any $x\in X$ we have*

(1) $\big(F*G\big)(x)=\bigcap\limits_{v\in D_G}\big(F(x-v)+G(v)\big)$ *whenever F is total;*

(2) $\big(F*G\big)(x)=\bigcap\limits_{u\in D_F}\big(F\,(u)+G(-u+x)\big)$ *whenever G is total.*

Remark 9.7. The multiplicative form of the $D_G=X$ particular case of the first statement of the above corollary closely resembles to the definition of the ordinary convolution of integrable functions.

In the sequel, we shall also need the following consequences of the corresponding results of [13]. The direct proofs are included here for the reader's convenience.

Theorem 9.8. *If F is a relation on a monoid X to a groupoid Y and Φ is a semi-subadditive partial selection relation of F such that D_Φ is a subgroup of X, then $\Phi\subset F*\Phi$.*

Proof. If $x\in X$ and $u\in D_F$ and $v\in D_\Phi$ such that $x=u+v$, then since D_Φ is a subgroup of X we also have $u=x-v$. Therefore, $u\in D_\Phi$ also holds if $x\in D_\Phi$. Hence, since $\Phi(x)=\emptyset$ if $x\notin D_\Phi$, it is clear that

$$\Phi(x)=\Phi(u+v)\subset\Phi(u)+\Phi(v)\subset F\,(u)+\Phi(v)\,.$$

Therefore,

$$\Phi(x)\subset\bigcap\big\{\,F(u)+\Phi(v)\;:\quad(u,v)\in\Delta\,(x,\,D_F,\,D_\Phi)\big\}=(F*\Phi)(x)\,,$$

and thus the required inclusion is also true.

Remark 9.9. By [13, Example 6.1], a semi-additive partial selection relation Φ of a relation F of one group X to another Y can only be, in general, extended to an additive total selection relation of the relation $F+\Phi(0)$.

Therefore, it is also necessary to prove the following

Theorem 9.10. *If F is a relation on a groupoid X with zero to an arbitrary groupoid Y and Φ is a right-zero-subadditive partial selection relation of F, then Φ is also a partial selection relation of $F+\Phi(0)$.*

Proof. For any $x \in X$, we have

$$\Phi(x) \subset \Phi(x)+\Phi(0) \subset F(x)+\Phi(0) = \big(F+\Phi(0)\big)(x).$$

Therefore, the required assertion is also true.

Now, as an immediate consequence of Theorems 9.10 and 3.7, we can also state the next obvious

Corollary 9.11. *If F is a relation on one groupoid X with zero to another Y and Φ is a partial selection relation of F such that $0 \in \Phi(0)$, then Φ is also a partial selection relation of $F+\Phi(0)$.*

However, it is now more important to note that in addition to Theorem 9.8, we can also prove the following

Theorem 9.12. *If F is a relation on a groupoid X with zero to a semigroup Y, and moreover Φ is a left-zero-superadditive relation on X to Y and Ψ is a $D_F \times D_\Phi$–subadditive partial selection relation of $F+\Phi(0)$ such that $\Psi(v) \subset \Phi(v)$ for all $v \in D_\Phi$, then $\Psi \subset F * \Phi$.*

Proof. If $x \in X$ and $u \in D_F$ and $v \in D_\Phi$ such that $x=u+v$, then by the hypotheses of the theorem we have

$$\begin{aligned} \Psi(x) = \Psi(u+v) \subset \Psi(u)+\Psi(v) \subset \\ \subset \big(F+\Phi(0)\big)(u)+\Phi(v) = F(u)+\Phi(0)+\Phi(v) \subset F(u)+\Phi(v). \end{aligned}$$

Therefore,

$$\Psi(x) \subset \bigcap \big\{ F(u)+\Phi(v) : \quad (u,v) \in \Delta\,(x,\, D_F,\, D_\Phi)\big\} = (F*\Phi)(x),$$

and thus the required inclusion is also true.

From this theorem, we can immediately get the following

Corollary 9.13. *If F is a total and Φ is a left-zero-superadditive relation on a groupoid X with zero to a semigroup Y such that $\Phi(0) \neq \emptyset$ and there exists an $X \times D_\Phi$–subadditive total selection relation Ψ of $F+\Phi(0)$ such that $\Psi(v) \subset \Phi(v)$ for all $v \in D_\Phi$, then $X = D_{F*\Phi}$.*

Remark 9.14. This corollary gives an important necessary condition in order that a left-zero-additive partial selection relation Φ of an arbitrary relation F of a groupoid X with zero to a semigroup Y could be extended to an $X \times D_\Phi$–subadditive total selection relation Ψ of $F+\Phi(0)$.

10. Further inclusion properties of the intersection convolution

In addition to Theorem 9.10, we can also prove the following

Theorem 10.1. *If F and G are relations on one groupoid X with zero to another Y, then*

(1) $F \subset F + G(0)$ *if* $0 \in G(0)$ *;*

(2) $F + G(0) \subset F$ *if F is right-zero-superadditive and* $G(0) \subset F(0)$ *.*

Proof. If the conditions of (2) hold, then

$$\big(F + G(0)\big)(x) = F(x) + G(0) \subset F(x) + F(0) \subset F(x) .$$

for all $x \in X$. Therefore, the conclusion of (2) also even if Y does not have a zero element.

Now, as an immediate consequence of this theorem, we can also state

Corollary 10.2. *If F is a right-zero-superadditive and G is an arbitrary relation on one groupoid X with zero to another Y such that $0 \in G(0) \subset F(0)$, then $F = F + G(0)$.*

Moreover, in addition to Theorem 9.8, we can also prove the following

Theorem 10.3. *If F is a total and G is an arbitrary relation on a groupoid X with zero to an arbitrary groupoid Y such that $G(0) \neq \emptyset$, then $F * G \subset F + G(0)$.*

Proof. If $x \in X$, then $(x, 0) \in \Delta(x, D_F, D_G)$. Therefore,

$$\big(F * G\big)(x) = \bigcap \big\{ F(u) + G(v) : \quad (u, v) \in \Delta(x, D_F, D_G) \big\} \subset$$
$$\subset F(x) + G(0) = \big(F + G(0)\big)(x) .$$

Thus, the required inclusion is also true.

Now, as an immediate consequence of Theorems 9.8 and 10.3, we can also state

Corollary 10.4. *If F is a relation of a monoid X to a groupoid Y and Φ is a semi-subadditive partial selection relation of F such that D_Φ is a subgroup of X, then $\Phi \subset F * \Phi \subset F + \Phi(0)$.*

Moreover, in addition to Theorem 10.3, we can also prove the following

Theorem 10.5. *If F is a superadditive relation on a group X to a semigroup Y and Φ is an inversion-semi-subadditive partial selection relation of F, then $F + \Phi(0) \subset F * \Phi$.*

Proof. If $x \in X$, then by Remark 3.9 we have

$$\big(F + \Phi(0)\big)(x) = F(x) + \Phi(0) \subset$$
$$\subset F(x) + \Phi(-v) + \Phi(v) \subset F(x) + F(-v) + \Phi(v) \subset F(x - v) + \Phi(v)$$

for all $v \in D_\Phi$. Therefore, by Theorem 9.5, we also have

$$\big(F + \Phi(0)\big)(x) \subset \bigcap \big\{ F(x - v) + \Phi(v) : \quad v \in (-D_F + x) \cap D_\Phi \big\} = (F * \Phi)(x) .$$

Thus, the required inclusion also holds.

Now, as an immediate consequence of Theorems 10.3 and 10.5, we can also state

Corollary 10.6. *If F is a superadditive relation of a group X to a semigroup Y and Φ is an inversion-semi-subadditive partial selection relation of F such that $\Phi(0) \neq \emptyset$, then $F * \Phi = F + \Phi(0)$.*

Finally, we note that the following theorem is also true.

Theorem 10.7. *If F and G are relations on a groupoid X with zero to a semigroup Y, then*

(1) $F * G \subset \big(F + G(0)\big) * G$ *if* G *is left-zero-subadditive;*

(2) $\big(F + G(0)\big) * G \subset F * G$ *if* G *is left-zero-superadditive and* $G(0) \neq \emptyset$.

Proof. If the condition of (1) holds, then

$$F(u) + G(v) \subset F(u) + G(0) + G(v) = \big(F + G(0)\big)(u) + G(v)$$

for all $u, v \in X$. Therefore, for any $x \in X$, we have

$$\begin{aligned}\big(F * G\big)(x) &= \bigcap \big\{ F(u) + G(v) : \quad (u, v) \in \Delta(x, D_F, D_G) \big\} \subset \\ &\subset \bigcap \big\{ \big(F + G(0)\big)(u) + G(v) : \quad (u, v) \in \Delta(x, D_F, D_G) \big\} \subset \\ &\subset \bigcap \big\{ \big(F + G(0)\big)(u) + G(v) : \quad (u, v) \in \Delta(x, D_{F+G(0)}, D_G) \big\} = \\ &= \big(\big(F + G(0)\big) * G\big)(x).\end{aligned}$$

Therefore, the conclusion of (1) also holds.

While, if the conditions of (2) hold, then

$$\big(F + G(0)\big)(u) + G(v) = F(u) + G(0) + G(v) \subset F(u) + G(v)$$

for all $u, v \in X$. Therefore, for any $x \in X$, we have

$$\begin{aligned}\big(\big(F + G(0)\big) * G\big)(x) &= \\ &= \bigcap \big\{ \big(F + G(0)\big)(u) + G(v) : \quad (u, v) \in \Delta(x, D_{F+G(0)}, D_G) \big\} \subset \\ &\subset \bigcap \big\{ F(u) + G(v) : \quad (u, v) \in \Delta(x, D_{F+G(0)}, D_G) \big\} = \\ &= \bigcap \big\{ F(u) + G(v) : \quad (u, v) \in \Delta(x, D_F, D_G) \big\} = \big(F * G\big)(x).\end{aligned}$$

Therefore, the conclusion of (2) also holds.

Now, as an immediate consequence of the above theorem, we can also state

Corollary 10.8. *If F and G are relations on a groupoid X with zero to a semigroup Y such that G is left-zero-additive and $G(0) \neq \emptyset$, then $F * G = \big(F + G(0)\big) * G$.*

11. Additivity and homogeneity properties of the intersection convolution

Now, as an extension of [67 , Theorem 10.4 (2)] , we can prove the following

Theorem 11.1. *If F and G are relations on a monoid X to a semigroup Y such that D_G is a subgroup of X and G is superadditive, then for any $x, y \in X$ we have*

$$(F * G)(x) + G(y) \subset (F * G)(x + y) .$$

Proof. If $(u, v) \in \Delta (x + y, D_F, D_G)$, then $u \in D_F$ and $v \in D_G$ such that $x + y = u + v$. Hence, if in particular $y \in D_G$, we can see that $x = u + v - y$ and $v - y \in D_G$. Therefore, $(u, v - y) \in \Delta (x, D_F, D_G)$. Hence, it is clear that

$$\big(F * G\big)(x) = \bigcap \big\{ F(s) + G(t) : \quad (s, t) \in \Delta(x, D_F, D_G) \big\} \subset F(u) + G(v - y) .$$

Therefore,

$$(F * G)(x) + G(y) \subset F(u) + G(v - y) + G(y) \subset F(u) + G(v) .$$

Hence, it is clear that

$$\begin{aligned}(F * G)(x) + G(y) \subset \bigcap \big\{ F(u) + G(v) : \quad (u, v) \in \Delta(x + y, D_F, D_G) \big\} = \\ = (F * G)(x + y) .\end{aligned}$$

Thus, since $G(y) = \emptyset$ if $y \notin D_G$, the required inclusion is also true.

Simple applications of the above theorem give the following

Corollary 11.2. *If F and G are relations on one monoid X to another Y such that D_G is a subgroup of X and G is quasi-odd and superadditive, then for any $x \in X$ and $y \in D_G$ we have*

$$(F * G)(x + y) = (F * G)(x) + G(y) .$$

Proof. Because of $0 \in G(-y) + G(y)$ and Theorem 11.1, we have

$$(F * G)(x + y) \subset (F * G)(x + y) + G(-y) + G(y) \subset (F * G)(x) + G(y) .$$

Hence, by Theorem 11.1, it is clear that the required equality is also true.

Remark 11.3. Note that, if F and G are as above, then in particular we have

$$(F * G)(x) = (F * G)(x) + G(0) \qquad \text{and} \qquad (F * G)(y) = (F * G)(0) + G(y) ,$$

for all $x \in X$ and $y \in Y$.

Hence, if $0 \in (F * G)(0)$, we can infer that $G \subset F * G$. However, in general, $F * G$ need not be an extension of G. Namely, by Corollary 11.2, we also have $(F * G)(0) = (F * G)(-y) + G(y)$ for all $y \in Y$. Thus, in general $(F * G)(0) \neq \{0\}$.

Analogously to [67, Theorem 11.5], we can now prove the following two theorems.

Theorem 11.4. *If F and G are n–superhomogeneous relations on an n–cancellable commutative semigroup X to an arbitrary commutative semigroup Y, for some $n \in \mathbb{N}$, such that D_F and D_G are n–divisible, then $F * G$ is also n–superhomogeneous.*

Proof. If $x \in X$ and $(\omega, w) \in \Delta(n\,x, D_F, D_G)$, then $\omega \in D_F$ and $w \in D_G$ such that $n\,x = \omega + w$. Therefore, there exist $u \in D_F$ and $v \in D_G$ such that $\omega = n\,u$ and $w = n\,v$. Moreover, $n\,x = n\,u + n\,v = n\,(u+v)$, and thus $x = u+v$. Therefore, $(u, v) \in \Delta(x, D_F, D_G)$. Hence, it is clear that

$$\big(F * G\big)(x) = \bigcap \,\big\{\, F(s) + G(t) : \quad (s, t) \in \Delta(x, D_F, D_G)\,\big\} \subset F(u) + G(v)\,.$$

Now, we can also easily see that

$$n(F*G)(x) \subset n\big(F(u)+G(v)\big) = nF(u)+nG(v) \subset F(nu)+G(nv) = F(\omega)+G(w).$$

Hence, it is clear that

$$n\,(F*G)(x) \subset \bigcap \,\big\{\, F(\omega)+G(w) : \quad (\omega, w) \in \Delta(n\,x, D_F, D_G)\,\big\} = (F*G)(n\,x\,)\,.$$

Therefore, the required assertion is also true.

Theorem 11.5. *If F and G are n–subhomogeneous relations on a commutative semigroup X to an n–cancellable commutative semigroup Y, for some $n \in \mathbb{N}$, such that $n\,D_F \subset D_F$ and $n\,D_G \subset D_G$, then $F * G$ is also n–subhomogeneous.*

Proof. If $x \in X$ and $(u, v) \in \Delta(\,x, D_F, D_G)$, then $u \in D_F$ and $v \in D_G$ such that $x = u + v$. Hence, it follows that $n\,u \in D_F$, $n\,v \in D_G$ and $n\,x = n\,u + n\,v$. Therefore, $(\,n\,u\,, n\,v\,) \in \Delta(n\,x, D_F, D_G)$. Hence, it is clear that

$$\begin{aligned}\big(F * G\big)(\,n\,x\,) = \bigcap \,\big\{\, F(\omega) + G(w) : \quad (\omega, w) \in \Delta(\,n\,x, D_F, D_G)\,\big\} \subset \\ \subset F(\,n\,u) + G(\,n\,v\,) \subset n\,F(u) + n\,G(v) = n\,\big(\,F(u) + G(v)\big)\,.\end{aligned}$$

Hence, we can already infer that

$$\begin{aligned}\big(F * G\big)(\,n\,x\,) \subset \bigcap \,\big\{\, n\,\big(\,F(u) + G(v)\big) : \quad (u, v) \in \Delta(\,x, D_F, D_G)\,\big\} = \\ = n \bigcap \,\big\{\, F(u) + G(v) : \quad (u, v) \in \Delta(\,x, D_F, D_G)\,\big\} = n\,(F * G)(x)\,.\end{aligned}$$

Namely, by the n–cancellability of Y, the mapping $y \mapsto n\,y$, where $y \in Y$, is injective.

By [67, Theorem 11.3], we also have the following

Theorem 11.6. *If F and G are odd relations on one commutative group X to another Y, then $F * G$ is also odd.*

Therefore, from Theorems 11.4 and 11.5, by using Theorem 3.19, we can immediately get the following two theorems.

Theorem 11.7. *If F and G are odd $\mathbb{N}$–superhomogeneous relations on an $\mathbb{N}$–cancellable commutative group X to an arbitrary commutative group Y such that D_F and D_G are $\mathbb{N}$–divisible, then $F * G$ is $\mathbb{Z} \setminus \{0\}$–superhomogeneous.*

Theorem 11.8. *If F and G are odd $\mathbb{N}$–subhomogeneous relations on a commutative group X to an $\mathbb{N}$–cancellable commutative group Y, such that $n\,D_F \subset D_F$ and $n\,D_G \subset D_G$ for all $n \in \mathbb{N}$, then $F * G$ is $\mathbb{Z} \setminus \{0\}$–subhomogeneous.*

Remark 11.9. Note that if in addition to the conditions of Theorem 11.7 we also have $0 \in F(x) + G(-x)$ for all $x \in D_F \cap (-D_G)$, then by Theorem 9.5 we have

$$0 \in \bigcap \left\{ F(u) + G(-u) : \quad u \in D_F \cap (-D_G) \right\} = (F * G)(0)\,.$$

Therefore, by Remark 3.14, we can also state the relation $F * G$ is $\mathbb{Z}$–superhomogeneous.

Finally, we note that by [67 , Theorem 11.5], we also have the following

Theorem 11.10. *If F and G are homogeneous relations on one vector space X over $\mathbb{K}$ to another Y, then $F * G$ is also homogeneous.*

12. Constructions of additive selection relations on sum sets

In this section, by using the intersection convolution, we shall prove some partial generalizations of Theorems 7.3 and 7.5.

Theorem 12.1. *Let F be a relation of a monoid X to a semigroup Y. Suppose that U is a submonoid and V is a subgroup of X such that U and V are commuting and*

$$X = U \oplus V\,.$$

Moreover, assume that Θ and Φ are commuting additive relations of U and V to Y, respectively, such that

$$\Theta \subset F * \Phi \qquad \textit{and} \qquad \Theta(0) = \Phi(0)\,.$$

Then, there exists a unique additive selection relation Ψ of $F + \Phi(0)$ that extends both Θ and Φ.

Proof. Now, by Theorem 7.3, there exists a unique additive relation Ψ of X to Y that extends both Θ and Φ. Moreover, by the proof of Theorem 7.3, we have

$$\Psi\,(\,u + v\,) = \Theta(u) + \Phi(v)$$

for all $u \in U$ and $v \in V$.

Thus, we need only show that $\Psi \subset F + \Phi(0)$ also holds. For this, note that by the inclusion $\Theta \subset F * \Phi$ and Theorems 11.1 and 10.3 we have

$$\Psi\,(\,u + v\,) = \Theta(u) + \Phi(v) \subset (F * \Phi)(u) + \Phi(v) \subset$$
$$\subset (F * \Phi)(\,u + v\,) \subset \big(F + \Phi(0)\big)(\,u + v\,)$$

for all $u \in U$ and $v \in V$.

Remark 12.2. If in particular Θ and Φ are n–homogeneous, for some $n \in \mathbb{N}$, and Y is commutative, then by Remark 7.4 we can also state that Ψ is also n–homogeneous.

However, it is now more important to note that, as an immediate consequence of Theorems 12.1 and 9.12, we can also state

Corollary 12.3. *Let F be a relation of a monoid X to a semigroup Y. Suppose that V is a subgroup of X and Φ is an additive relation of V to Y. Moreover, assume that there exists a submonoid U of X such that U and V are commuting and $X = U \oplus V$. Then, the following assertions are equivalent:*

(1) Φ can be extended to an additive selection relation Ψ of $F + \Phi(0)$;

(2) there exists an additive relation Θ of U to Y such that Θ and Φ are commuting, and moreover $\Theta \subset F * \Phi$ and $\Theta(0) = \Phi(0)$.

Proof. If (2) holds, then by Theorem 12.1 there exists a unique additive selection relation Ψ of $F + \Phi(0)$ that extends both Θ and Φ. Thus, in particular, (1) also holds.

While, if (1) holds, then by Theorem 9.12 we have $\Psi \subset F * \Phi$. Therefore, the restriction $\Theta = \Psi \,|\, U$ of Ψ to U has the properties required in (2). Namely, now we also have

$$\Theta(u) + \Phi(v) = \Psi(u) + \Psi(v) = \Psi(\,u + v\,) = \Psi(\,v + u\,) = \Psi(v) + \Psi(u) = \Phi(v) + \Theta(u)$$

for all $u \in U$ and $v \in V$. Therefore, Θ and Φ are commuting.

By using Theorem 7.5 instead of Theorem 7.3, we can quite similarly prove the following

Theorem 12.4. *Let F be a relation of one group X to another Y. Suppose that U and V are commuting subgroups of X such that*

$$X = U + V\,.$$

Moreover, assume that Θ and Φ are commuting additive relations of U and V to Y, respectively, such that

$$\Theta \subset F * \Phi \qquad \text{and} \qquad \Theta(x) = \Phi(x) \quad \text{for all} \quad x \in U \cap V\,.$$

Then, there exists a unique additive selection relation Ψ of $F + \Phi(0)$ that extends both Θ and Φ.

Remark 12.5. If in particular Θ and Φ are odd, then by Remark 7.6 we can also state that Ψ is also odd.

Moreover, if in addition Θ and Φ are $\mathbb{N}$–homogeneous and Y is commutative, then by Remark 7.7 we can also state that Ψ is $\mathbb{Z} \setminus \{0\}$– homogeneous.

Remark 12.6. Moreover, if more specially X and Y are vector spaces over $\mathbb{K}$, U and V are subspaces of X, and Θ and Φ are homogeneous, then by Remark 7.8 we can also state that Ψ is also homogeneous.

However, it is again more important to note that, as an immediate consequence of Theorems 11.4 and 9.12, we can also state

Corollary 12.7. *Let F be a relation of one group X to another Y. Suppose that V is a subgroup of X and Φ is an additive relation of V to Y. Moreover, assume that there exists a subgroup U of X such that U and V are commuting and $X = U + V$. Then, the following assertions are equivalent:*

(1) Φ can be extended to an additive selection relation Ψ of $F + \Phi(0)$;

(2) there exists an additive relation Θ of U to Y such that Θ and Φ are commuting, and moreover $\Theta \subset F * \Phi$ and $\Theta(x) = \Phi(x)$ for all $x \in U \cap V$.

13. One-step extensions of additive partial selection relations

Now, as a partial extension of Theorems 8.1, we can also prove the following

Theorem 13.1. *Let F be a relation of one commutative monoid X to another Y. Suppose that Φ is an additive relation of a subgroup V of X to Y such that $\Phi \subset F$. Moreover, assume that $a \in X \setminus V$ and $b \in Y$ such that*

(1) $n\,b \in (F * \Phi)(n\,a)$ for all $n \in \mathbb{N}$;

(2) $X = U \oplus V$ holds with $U = \{n\,a\}_{n=0}^{\infty}$;

(3) $n\,a = m\,a$ implies $n\,b + \Phi(0) = m\,b + \Phi(0)$ for all $n, m \in \{0\} \cup \mathbb{N}$.

Then, there exists a unique additive selection relation Ψ of $F + \Phi(0)$ extending Φ such that $\Psi(a) = b + \Phi(0)$.

Proof. In particular, we have $\Phi(0) \neq \emptyset$ and $\Phi(0) = \Phi(0) + \Phi(0)$. Thus, by Theorem 6.1, there exists a unique additive relation Θ of U to Y such that $\Theta(0) = \Phi(0)$ and $\Theta(a) = b + \Phi(0)$. Moreover, by the proof of Theorem 6.1, we have

$$\Theta(\,na\,) = n\,b + \Phi(0)$$

for all $n \in \{0\} \cup \mathbb{N}$.

Now, by using condition (1) and Theorem 11.1, we can see that

$$\Theta(na) = n\,b + \Phi(0) \subset (F * \Phi)(n\,a) + \Phi(0) \subset (F * \Phi)(n\,a)$$

for all $n \in \mathbb{N}$. Moreover, from Theorem 9.8, we can see that

$$\Theta\,(\,0\,a\,) = \Theta\,(0) = \Phi(0) \subset (F * \Phi)(0) = (F * \Phi)(\,0\,a\,)\,.$$

Therefore, we have $\Theta \subset F * \Phi$. Now, by Theorem 12.1, we can state that there exists a unique additive selection relation Ψ of $F + \Phi(0)$ that extends both Θ and Φ. Hence, it is clear that the required assertion is also true.

Remark 13.2. Note that now Φ is superadditive. Thus, by Theorem 3.12, Φ is $\mathbb{N}$–superhomogeneous. Therefore, if in particular X is $\mathbb{N}$–cancellable, V and X are $\mathbb{N}$–divisible, and F is $\mathbb{N}$–superhomogeneous, then by Theorem 11.4 the relation $F * \Phi$ is also $\mathbb{N}$–superhomogeneous. Thus,

$$n\,b \in n(\,F * \Phi\,)(a) \subset (\,F * \Phi\,)(\,n\,a\,)$$

for all $n \in \mathbb{N}$ and $b \in (\,F * \Phi\,)(a)$.

Analogously to the above theorem, we can also prove the following extension of Theorem 8.3.

Theorem 13.3. *Let F be an odd relation of one commutative group X to another Y. Suppose that Φ is an odd superadditive relation of a subgroup V of X to Y such that $\Phi \subset F$. Moreover, assume that $a \in X \setminus V$ and $b \in Y$ such that*

(1) $X = U \oplus V$ holds with $U = \mathbb{Z}\, a$;

(2) $n\, b \in (F * \Phi)(n\, a)$ for all $n \in \mathbb{N}$;

(3) $n\, a = 0$ implies $n\, b \in \Phi(0)$ for all $n \in \mathbb{N}$.

Then, there exists a unique odd additive selection relation Ψ of $F + \Phi(0)$ extending Φ such that $\Psi(a) = b + \Phi(0)$.

Proof. In particular, we have $\Phi(0) \neq \emptyset$ and $\Phi(0) - \Phi(0) \subset \Phi(0) + \Phi(0) \subset \Phi(0)$. Thus, $\Phi(0)$ is a subgroup of Y. Thus, by Theorem 6.3, there exists a unique odd additive relation Θ of U to Y such that $\Theta(0) = \Phi(0)$ and $\Theta(a) = b + \Phi(0)$. Moreover, by the proof of Theorem 6.3, we have

$$\Theta(ka) = k\, b + \Phi(0)$$

for all $k \in \mathbb{Z}$.

From Theorem 11.6, we know that the relation $F * \Phi$ is also odd. Thus, by condition (2), we also have

$$(-n)b = -(n\, b) \in -(F * \Phi)(n\, a) = (F * \Phi)\big(-(n\, a)\big) = (F * \Phi)\big((-n)a\big)$$

for all $n \in \mathbb{N}$. Therefore, (2) is now equivalent to the requirement that $k\, b \in (F * \Phi)(k\, a)$ for all $k \in \mathbb{Z} \setminus \{0\}$.

Now, by using this fact and Theorem 11.1, we can see that

$$\Theta(ka) = k\, b + \Phi(0) \subset (F * \Phi)(k\, a) + \Phi(0) \subset (F * \Phi)(k\, a)$$

for all $k \in \mathbb{Z} \setminus \{0\}$. Moreover, from Theorem 4.3 we can see that Φ is quasi-additive. Thus, by Theorem 9.8, we also have

$$\Theta(0\, a) = \Theta(0) = \Phi(0) \subset (F * \Phi)(0) = (F * \Phi)(0\, a).$$

Therefore, $\Theta \subset F * \Phi$. Now, by Theorem 12.1 and Remark 12.5, we can state that there exists a unique odd additive selection relation Ψ of $F + \Phi(0)$ that extends both Θ and Φ. Hence, it is clear that the required assertion is also true.

Remark 13.4. Note that if (2) holds, then by Theorem 10.3 we have

$$n\, b \in (F * \Phi)(n\, a) \subset \big(F + \Phi(0)\big)(n\, a) = F(n\, a) + \Phi(0)$$

for all $n \in \mathbb{N}$. Thus, if in particular $n \in \mathbb{N}$ such that $n\, a = 0$, and moreover $F(0) = \Phi(0)$, then we also have $n\, b \in F(0) + \Phi(0) = \Phi(0) + \Phi(0) \subset \Phi(0)$. Therefore, in this particular case (2) implies (3).

Remark 13.5. Now, in addition to the above theorem, it is also worth noticing that if in particular Φ is $\mathbb{N}$–subhomogeneous, then by Remark 8.5 we can also state that Ψ is $\mathbb{Z} \setminus \{0\}$–homogeneous.

Moreover, we can also easily prove the following

Theorem 13.6. *Let F be a homogeneous relation of one vector space X over $\mathbb{K}$ to another Y. Suppose that Φ is a linear relation of a subspace V of X to Y such that $\Phi \subset F$. Moreover, assume that $a \in X \setminus V$ such that*

$$X = U + V \qquad \textit{holds with} \qquad U = \mathbb{K}\, a\,.$$

Then, for any $b \in (F * \Phi)(a)$, there exists a unique linear selection relation Ψ of $F + \Phi(0)$ extending Φ such that $\Psi(a) = b + \Phi(0)$.

However, it is now more important to note that, by using Theorem 8.7, we can prove the following

Theorem 13.7. *Let F be an n–subhomogeneous relation of a commutative group X to an n-cancellable commutative group Y for some $n \in \mathbb{N}$. Suppose that Φ is an odd $\mathbb{N}$–subhomogeneous superadditive relation of a subgroup V of X to Y such that $\Phi \subset F$. Moreover, assume that $a \in X \setminus V$ and $b \in Y$ such that*

(1) $n\,b \in \Phi(n\,a)$;

(2) $X = U + V$ *holds with* $U = \mathbb{Z}\, a$.

Then, there exists a unique $\mathbb{Z} \setminus \{0\}$–homogeneous additive selection relation Ψ of F extending Φ such that $\Psi(a) = b + \Phi(0)$.

Proof. Now, by Theorem 8.7, there exists a unique $\mathbb{Z} \setminus \{0\}$–homogeneous additive relation Ψ of X to Y extending Φ such that $\Psi(a) = b + \Phi(0)$. Moreover, by the proof Theorem 8.7, we have

$$\Psi(k\,a + v) = k\,b + \Phi(v)$$

for all $k \in \mathbb{Z}$ and $v \in V$.

Now, we can already see that

$$\begin{aligned} n\,\Psi(k\,a + v) &= n\big(k\,b + \Phi(v)\big) = n(k\,b) + n\,\Phi(v) = k(n\,b) + \Phi(n\,v) \subset \\ &\subset k\,\Phi(n\,a) + \Phi(n\,v) = \Phi\big(k(n\,a)\big) + \Phi(n\,v) = \Phi\big(k(n\,a) + n\,v\big) \subset \\ &\subset F\big(k(n\,a) + n\,v\big) = F\big(n(k\,a) + n\,v\big) = F\big(n(k\,a + v)\big) \subset n\,F(ka + v)\,, \end{aligned}$$

for all $k \in \mathbb{Z} \setminus \{0\}$ and $v \in V$. Hence, by using the n–cancellability of Y, we can infer that

$$\Psi(k\,a + v) \subset F(k\,a + v)$$

for all $k \in \mathbb{Z} \setminus \{0\}$ and $v \in V$. Moreover, we can note that

$$\Psi(0\,a + v) = \Psi(v) = \Phi(v) \subset F(v) = F(0\,a + v)$$

for all $v \in V$. Therefore, $\Psi \subset F$, and thus Ψ is a selection relation of F.

Remark 13.8. Note that if in particular $X \neq U \oplus V$, then by Theorem 5.13 $U \cap V \neq \{0\}$. Thus, there exists $n \in \mathbb{N}$ such that $n\,a \in V$. Therefore, there exists $y \in Y$ such that $y \in \Phi(n\,a)$.

Now, if in addition Y is n–divisible, then we can state there exists $b \in Y$ such that $y = n\,b$. Hence, we can see that $n\,b = y \in \Phi(n\,a)$. Therefore, in this particular case condition (1) automatically holds.

However, note that if in particular V is $\mathbb{N}$–divisible and X is $\mathbb{N}$–cancellable, then by Corollary 5.16 $X = U + V$ implies that $X = U \oplus V$. Therefore, in this particular case, the above remark and Theorem 13.7 cannot be applied.

14. The main extension theorem of additive partial selection relations

Because of Theorems 8.7 and 8.3, it seems necessary to introduce the following

Definition 14.1. Let F be a relation of one group X to another Y. Suppose that Φ is a nonvoid odd $\mathbb{N}$–semi-subhomogeneous superadditive partial selection relation of F.

Then, Φ will be called admissible if every $\mathbb{Z}\setminus\{0\}$–semi-homogeneous quasi-additive partial selection relation Ψ of $F+\Phi(0)$, extending Φ, has the following properties:

(1) for each $a\in X\setminus D_\Psi$, with $\mathbb{N}\,a\cap D_\Psi\neq\emptyset$, there exist $b\in Y$ and $n\in\mathbb{N}$ such $n\,b\in\Psi(\,n\,a\,)$;

(2) for each $a\in X\setminus D_\Psi$, with $\mathbb{N}\,a\cap D_\Psi=\emptyset$, there exists $b\in Y$ such that for all $n\in\mathbb{N}$ we have $n\,b\in(\,F*\Psi\,)(\,n\,a\,)$.

Remark 14.2. Note that now D_Φ is a subgroup of X. Moreover, by Theorems 3.12 and 3.19, Φ is $\mathbb{Z}\setminus\{0\}$–semi-homogeneous. Furthermore, by Theorem 4.3, $0\in\Phi(0)$ and Φ is quasi-additive. Thus, in particular Φ is also a partial selection relation of $F+\Phi(0)$. Therefore, if Φ is admissible, then it also has the properties (1) and (2) with Φ in place of Ψ.

Remark 14.3. Moreover, if Ψ is as Definition 14.1, then by using Theorem 9.8 and Corollary 10.8 we can see that

$$\Psi\subset\big(\,F+\Phi(0)\big)*\Psi=\big(\,F+\Psi(0)\big)*\Psi=F*\Psi\,.$$

Hence, by taking $x\in D_\Psi$ and $y\in\Psi(x)$, we can already infer that

$$k\,y\in k\,\Psi(x)=\Psi\,(\,k\,x\,)\subset(\,F*\Psi\,)(\,k\,x\,)$$

for all $k\in\mathbb{Z}\setminus\{0\}$.

Remark 14.4. Therefore, if for each $a\in X\setminus D_\Psi$ there exists $b\in Y$ such that $n\,b\in(\,F*\Psi\,)(\,n\,a\,)$ for all $n\in\mathbb{N}$, then for each $x\in X$ there exists $y\in Y$ such that $n\,y\in(\,F*\Psi\,)(\,n\,x\,)$ for all $n\in\mathbb{N}$.

Moreover, we can also note that if in particular F is odd, then by Theorem 11.6 the relation $F*\Psi$ is also odd. Therefore, in the above case we also have $k\,y\in(\,F*\Psi\,)(\,k\,x\,)$ for all $k\in\mathbb{Z}\setminus\{0\}$.

Now, by using Theorems 13.7 and 13.3, we can prove the following

Theorem 14.5. *Suppose that F is an odd $\mathbb{N}$–subhomogeneous relation of a commutative group X to an $\mathbb{N}$–cancellable commutative group Y. Moreover, assume that Φ is an admissible nonvoid odd $\mathbb{N}$–semi-subhomogeneous superadditive partial selection relation of F. Then Φ can be extended to a total $\mathbb{Z}\setminus\{0\}$–homogeneous additive selection relation Ψ of $F+\Phi(0)$.*

Proof. Denote by $\mathcal{F}$ the family of all odd $\mathbb{N}$–semi-subhomogeneous superadditive partial selection relations of $F+\Phi(0)$ that extends Φ. Then, by Remark 14.2, we have $\Phi\in\mathcal{F}$. Thus, $\mathcal{F}$ is a nonvoid partially ordered set with the ordinary set inclusion. Therefore, by the Hausdorff maximality principle, there exists a nonvoid maximal linearly ordered subset $\mathcal{G}$ of $\mathcal{F}$. Define $\Psi = \bigcup \mathcal{G}$.

Then, since $G \subset F + \Phi(0)$ for all $G \in \mathcal{G}$, it is clear that $\Psi \subset F + \Phi(0)$. Thus, Ψ is also a partial selection relation of $F + \Phi(0)$. Moreover, we can also note that

$$\Psi(x) = \bigcup_{G \in \mathcal{G}} G(x)$$

for all $x \in X$. Hence, since each member of $\mathcal{G}$ is an extension of Φ and $\mathcal{G} \neq \emptyset$, it is clear that

$$\Psi(v) = \bigcup_{G \in \mathcal{G}} G(v) = \bigcup_{G \in \mathcal{G}} \Phi(v) = \Phi(v)$$

for all $v \in D_\Phi$. Therefore, Ψ is also an extension of Φ.

Moreover, since the mappings $y \mapsto -y$ and $y \mapsto n\,y$, where $y \in Y$ and $n \in \mathbb{N}$, are injective, we can easily see that

$$\Psi(-x) = \bigcup_{G \in \mathcal{G}} G(-x) = \bigcup_{G \in \mathcal{G}} -G(x) = -\bigcup_{G \in \mathcal{G}} G(x) = -\Psi(x)$$

and

$$\Psi(n\,x) = \bigcup_{G \in \mathcal{G}} G(n\,x) \subset \bigcup_{G \in \mathcal{G}} n\,G(x) = n \bigcup_{G \in \mathcal{G}} G(x) = n\,\Psi(x)$$

for all $x \in D_\Psi$. Thus, Ψ is also odd and $\mathbb{N}$–semi-subhomogeneous.

On the other hand, if $x, y \in X$ and $z \in \Psi(x)$ and $w \in \Psi(y)$, then by the definition of Ψ there exist $G_1, G_2 \in \mathcal{G}$ such that $z \in G_1(x)$ and $w \in G_2(y)$. Moreover, since $\mathcal{G}$ is linearly ordered, we have either $G_1 \subset G_2$ or $G_2 \subset G_1$. Hence, it is clear that either

$$z + w \in G_1(x) + G_2(y) \subset G_2(x) + G_2(y) = G_2(x+y) \subset \Psi(x+y)$$

or

$$z + w \in G_1(x) + G_2(y) \subset G_1(x) + G_1(y) = G_1(x+y) \subset \Psi(x+y)$$

holds. Therefore,

$$\Psi(x) + \Psi(y) \subset \Psi(x+y).$$

Thus, Ψ is also superadditive.

Thus, we have proved that $\Psi \in \mathcal{F}$. Moreover, by Theorems 3.12, 3.19 and 4.3, we can also state that Ψ is actually $\mathbb{Z} \setminus \{0\}$-semi-homogeneous and quasi-additive. Therefore, it remains only to show that $D_\Psi = X$ also holds. For this, assume on the contrary that there exists $a \in X$ such that $x \notin D_\Psi$, and define

$$Z = U + D_\Psi \qquad \text{with} \qquad U = \mathbb{Z}\,a.$$

Then, it is clear that Z is a subgroup of X that properly contains D_Ψ.

Note that if $U \cap D_\Psi \neq \emptyset$, then by Definition 14.1 there exists $b \in Y$ and $n \in \mathbb{N}$ such that $n\,b \in \Psi\,(\,n\,a\,)$. Moreover, we can note that

$$\begin{aligned}\big(F+\Phi(0)\big)(\,m\,x\,) &= F\,(\,m\,x\,)+\Phi(0) \subset m\,F(x)+\Phi(0) = \\ &= m\,F(x)+m\,\Phi(0) = m\,\big(\,F(x)+\Phi(0)\big) = m\,\big(\,F+\Phi(0)\big)(x)\end{aligned}$$

for all $m \in \mathbb{N}$ and $x \in X$. Therefore, the relation $F+\Phi(0)$ is also $\mathbb{N}$–subhomogeneous. Hence, by Theorem 13.7, we can state that there exists a unique $\mathbb{Z}\setminus\{0\}$–homogeneous additive relation Ω of Z to Y extending Ψ such that $\Omega(a) = b+\Psi(0) = b+\Phi(0)$ and $\Omega \subset F+\Phi(0)$.

While, if $\mathbb{N}\,a \cap D_\Psi = \emptyset$, then since D_Ψ is a subgroup of X we can note that $U \cap D_\Psi = \{0\}$. Thus, by Theorem 5.13, $Z = U \oplus D_\Psi$. Moreover, by Definition 14.1, there exists $b \in Y$ such that $n\,b \in (\,F * \Psi\,)(\,n\,a\,)$ for all $n \in \mathbb{N}$. Hence, by Remark 14.3, we can see that

$$n\,b \in \Big(\big(\,F+\Phi(0)\big) * \Psi\Big)\,(\,n\,a\,)$$

also holds for all $n \in \mathbb{N}$. Moreover, we can note that

$$\begin{aligned}\big(\,F+\Phi(0)\big)(-x\,) &= F\,(-x\,)+\Phi\,(0) = \\ &= -F(x)-\Phi(0) = -\big(\,F(x)+\Phi(0)\big) = -\big(\,F+\Phi(0)\big)(x)\end{aligned}$$

for all $x \in X$. Therefore, the relation $F+\Phi(0)$ is also odd. Moreover, now we can also note that $n\,a \neq 0$ for all $n \in \mathbb{N}$. Therefore, by Theorem 13.3 and Remark 13.5, we can state that there exist a unique $\mathbb{Z}\setminus\{0\}$–homogeneous additive relation Ω of Z to Y extending Ψ such that $\Omega(a) = b+\Psi(0) = b+\Phi(0)$ and

$$\Omega \subset F+\Phi(0)+\Psi(0) = F+\Phi(0)+\Phi(0) = F+\Phi(0)\,.$$

Thus, in both cases, there exists an $\Omega \in \mathcal{F}$ such that Ω is a proper extension of Ψ. Hence, we can see that $\mathcal{G} \cup \{\Omega\}$ is a strictly larger linearly ordered subset of $\mathcal{F}$ than $\mathcal{G}$. This contradiction shows that $D_\Psi = X$ also holds.

Remark 14.6. Note that if in particular Φ is a function in the above theorem, then by Theorem 4.3 we have $0 \in \Phi(0)$, and hence $\Phi(0) = \{0\}$. Therefore, $\Psi(0) = \Phi(0) = \{0\}$. Thus, by Corollary 4.13, Ψ is also a function.

15. Direct applications of the main extension theorem

Now, by using Theorem 14.5, we can prove the following

Theorem 15.1. *Suppose that F is an odd $\mathbb{N}$–subhomogeneous superadditive relation of a commutative group X to a uniquely $\mathbb{N}$–divisible commutative group Y. Moreover, assume that Φ is a nonvoid odd $\mathbb{N}$–semi-subhomogeneous superadditive partial selection relation of F. Then Φ can be extended to a total $\mathbb{Z}\setminus\{0\}$–homogeneous additive selection relation Ψ of F.*

Proof. Note that, by Theorems 3.12 and 3.19, F is now $\mathbb{Z}\setminus\{0\}$–homogeneous. Moreover, by Theorem 4.3, F is additive and Φ is quasi-additive. Furthermore, $0 \in \Phi(0) \subset F(0)$. Thus, by Corollary 10.2, $F = F+\Phi(0)$.

Next, we show that Φ is admissible. For this, assume that Ω is a $\mathbb{Z}\setminus\{0\}$–semi-homogeneous quasi-additive partial selection relation of $F+\Phi(0)$ that extends Φ. Then, because of $F+\Phi(0)=F$, Ω is also a partial selection relation of F. Moreover, $0\in\Phi(0)=\Omega(0)$. Thus, by Corollary 10.6,

$$F*\Omega=F+\Omega(0)=F+\Phi(0)=F\,.$$

Now, by taking $x\in X$ and $y\in F(x)$, we can see that

$$k\,y\in k\,F(x)=F\,(k\,x\,)=(\,F*\Omega\,)(\,k\,x\,)$$

for all $k\in\mathbb{Z}\setminus\{0\}$. Therefore, the condition (2) of Definition 14.1, with Ω in place of Ψ, is substantially satisfied.

Moreover, we can also note that if $x\in X$ such that $n\,x\in D_\Omega$ for some $n\in\mathbb{N}$, then there exists $y\in Y$ such that $y\in\Omega\,(\,n\,x\,)$. Furthermore, by the n–divisibility of Y there exists $z\in Y$ such that $y=n\,z$. Thus, we also have $n\,z\in\Omega\,(\,n\,x\,)$. Therefore, the condition (1) of Definition 14.1, with Ω in place of Ψ, is also substantially satisfied.

Now, by Theorem 14.5, we can state that Φ can be extended to a total $\mathbb{Z}\setminus\{0\}$–homogeneous additive selection relation Ψ of $F+\Phi(0)$. Thus, since $F+\Phi(0)=F$, the required assertion is also true.

Now, as an immediate consequence of this theorem, we can also state

Corollary 15.2. *If F is as in Theorem 15.1, then for each $\mathbb{N}$–divisible subgroup Z of $F(0)$ there exists a $\mathbb{Z}\setminus\{0\}$–homogeneous additive selection relation Ψ of F such that $\Psi(0)=Z$.*

Proof. Define $\Phi=\{0\}\times Z$. Then, it is clear that Φ is a nonvoid odd $\mathbb{N}$–semi-subhomogeneous superadditive partial selection relation of F. Thus, by Theorem 15.1, Φ can be extended to a total $\mathbb{Z}\setminus\{0\}$–homogeneous additive selection relation Ψ of F. Therefore, since $\Psi(0)=\Phi(0)=Z$, the required assertion is also true.

Hence, it is clear that in particular we also have

Corollary 15.3. *If F is as in Theorem 15.1, then there exists an additive selection function ψ of F.*

Proof. By Theorem 4.3, we have $0\in F(0)$. Thus, $\{0\}$ is $\mathbb{N}$–divisible subgroup of $F(0)$. Therefore, by Corollary 15.2, there exists a $\mathbb{Z}\setminus\{0\}$–homogeneous additive selection relation ψ of F such that $\psi(0)=\{0\}$. Moreover, by Corollary 4.13, ψ is a function. Therefore, the required assertion is true.

Remark 15.4. If in particular F is a linear relation of one vector space X over $\mathbb{K}$ to another Y, then we can similarly see that F has a linear selection function ψ.

Remark 15.5. This fact was first proved by Géza Száz in a work prepared for a student competition in Hungary in 1971. By giving a remarkable example, he also proved that an additive relation F of $\mathbb{R}$ to itself, with $0\in F(0)$, need not have an additive selection function.

Later, these results, which had not been appreciated by the referees of the competition, were included in [73]. However, the corresponding example is usually attributed to Godini [28] in the extensive literature on set-valued functions and the stability of functional equations. (See, for instance, [55, p. 182].)

Now, by using Corollary 15.3 and Theorem 14.5, we can also prove the following

Theorem 15.6. *Suppose that F is an odd $\mathbb{N}$–subhomogeneous relation of a commutative group X to a uniquely $\mathbb{N}$–divisible commutative group Y such that each nonvoid odd quasi-additive partial selection function φ of F is admissible. Then, each nonvoid odd $\mathbb{N}$–semi-subhomogeneous superadditive partial selection relation Φ of F can be extended to a total $\mathbb{Z}\setminus\{0\}$–homogeneous additive selection relation Ψ of $F+\Phi(0)$.*

Proof. If Φ is as above, then D_Φ is a subgroup of X. Thus, by Corollary 15.3, there exists an additive selection function φ of Φ. Clearly, φ is a nonvoid odd quasi-additive partial selection function of F. Therefore, by the assumption of the theorem, φ is admissible. Hence, by using Theorem 14.5 and Remark 14.6, we can see that φ can be extended to a total additive selection function ψ of F. Moreover, by Corollary 3.22, we can also note that ψ is $\mathbb{Z}$–homogeneous.

Define $\Psi=\psi+\Phi(0)$. Then, by using Remark 14.2, we can easily see that Ψ is a $\mathbb{Z}\setminus\{0\}$–homogeneous additive relation of X to Y such that $\Psi=\psi+\Phi(0)\subset F+\Phi(0)$. Moreover, by using Remark 4.6 and Theorem 4.11, we can also see that

$$\Psi(x)=\big(\psi+\Phi(0)\big)(x)=\psi(x)+\Phi(0)=\varphi(x)+\Phi(0)=\Phi(x)$$

for all $x\in D_\Phi$. Therefore, Ψ is an extension of Φ. Thus, the required assertion is true.

Now, as a certain converse to the above theorem, we can also prove the following

Theorem 15.7. *Suppose that F is a relation of one group X to another Y such that each nonvoid odd $\mathbb{N}$–semi-homogeneous quasi-additive partial selection relation Ω of F can be extended to a total $\mathbb{Z}\setminus\{0\}$–homogeneous additive selection relation Ψ of $F+\Omega(0)$. Then, every nonvoid odd $\mathbb{N}$–semi-subhomogeneous superadditive partial selection relation Φ of F, with $F+\Phi(0)=F$, is admissible.*

Proof. Suppose that Φ is as above, and moreover assume that Ω is a $\mathbb{Z}\setminus\{0\}$–semi-homogeneous quasi-additive partial selection relation of $F+\Phi(0)=F$ that extends Φ. Then, by the assumption of the theorem, Ω can be extended to a total $\mathbb{Z}\setminus\{0\}$–homogeneous additive selection relation Ψ of $F+\Omega(0)=F+\Phi(0)=F$.

Hence, by taking $x\in X$ and $y\in\Psi(x)$, we can see that

$$k\,y\in k\,\Psi(x)=\Psi\,(\,k\,x\,)=\Omega\,(\,k\,x\,)$$

for all $k\in\mathbb{Z}$ with $k\,x\in D_\Omega$. Moreover, by using Theorems 9.8 and 9.6, we can also see that

$$\Psi\subset F*\Psi\subset F*\Omega\,.$$

Hence, by taking $x\in X$ and $y\in\Psi(x)$, we can see that

$$k\,y\in k\,\Psi(x)=\Psi\,(\,k\,x\,)\in(\,F*\Omega\,)(\,k\,x\,)$$

for all $k\in\mathbb{Z}$. Thus, the conditions (1) and (2) of Definition 4.11, with Ω in place of Ψ, are substantially satisfied. Therefore, Φ is admissible.

Remark 15.8. Note that if in particular Φ is a function then because of $\Phi(0)=\{0\}$, we have $F+\Phi(0)=F$. While, if in particular F is right-zero-superadditive, then by Corollary 10.2 we also have $F+\Phi(0)=F$.

16. A strong totality properties of the intersection convolution

Definition 16.1. A family $\mathcal{A}$ of sets is said to have the binary intersection property if $A \cap B \neq \emptyset$ for all $A, B \in \mathcal{A}$.

Remark 16.2. This terminology differs from that of Nachbin [41] and his close followers.

But, it is in accordance with the usual definition of the finite intersection property [39, p. 135].

Now, by extending an argument of Z. Gajda, A. Smajdor and W. Smajdor [22], we can prove the following counterpart of [67, Theorem 9.3].

Theorem 16.3. *Suppose that F and G are relations on a commutative group X to a vector space Y over $\mathbb{K}$ such that:*

(1) D_F and D_G are subgroups of X;

(2) $F(x) \cap G(x) \neq \emptyset$ for all $x \in D_F \cap D_G$;

(3) F and G are odd $\mathbb{N}$–semi-subhomogeneous and semi-subadditive.

Then, the family

$$\Big\{ n^{-1}\big(F(n x - v) + G(v) \big) : \qquad n \in \mathbb{N}, \quad v \in (-D_F + n x) \cap D_G \Big\}$$

has the binary intersection property for all $x \in X$.

Proof. Suppose that $x \in X$, $n, m \in \mathbb{N}$, and

$$v \in (-D_F + n x) \cap D_G \qquad \text{and} \qquad t \in (-D_F + m x) \cap D_G .$$

Then, $v \in -D_F + n x$ and $t \in -D_F + m x$, and $v, t \in D_G$. Hence, by using (1) and the commutativity of X, we can infer that

$$n t - m v \in n D_G - m D_G \subset D_G - D_G = D_G$$

and

$$\begin{aligned} n t - m v &\in n\big(-D_F + m x\big) - m\big(-D_F + n x\big) = \\ &= -n D_F + n m x + m D_F - m n x = -n D_F + m D_F \subset -D_F + D_F = D_F . \end{aligned}$$

Therefore, $n t - m v \in D_F \cap D_G$, and thus by (2)

$$F(n t - n v) \cap G(n t - m v) \neq \emptyset .$$

Moreover, we can also note that

$$n x - v \in n x - (-D_F + n x) = D_F \qquad \text{and} \qquad m x - t \in m x - (-D_F + m x) = D_F .$$

Now, by using (3) and the commutativity of X and Y, we can already see that

$$\begin{aligned}
0 \in \big(F(nt - mv) - G(nt - mv) \big) &= \\
&= F(nt + nmx - nmx - mv) - G(nt - mv) = \\
&= F\big(m(nx - v) - n(mx - t) \big) - G(nt - mv) \subset \\
&\subset mF(nx - v) - nF(mx - t) - nG(t) + mG(v) = \\
&= m\big(F(nx - v) + G(v) \big) - n\big(F(mx - t) + G(t) \big).
\end{aligned}$$

Hence, it is clear that

$$0 \in n^{-1}\big(F(nx - v) + G(v) \big) - m^{-1}\big(F(mx - t) + G(t) \big),$$

and thus

$$n^{-1}\big(F(nx - v) + G(v) \big) \cap m^{-1}\big(F(mx - t) + G(t) \big) \neq \emptyset.$$

Therefore, the required assertion is also true.

Definition 16.4. A family $\mathcal{A}$ of subsets of a set X is called a Nachbin system in X if for every subfamily $\mathcal{B}$ of $\mathcal{A}$, having the binary intersection property, we have $\bigcap \mathcal{B} \neq \emptyset$.

Remark 16.5. Quite similarly a family of subsets of a set may be called a Riesz system if every subfamily of it having the finite intersection property has a nonvoid intersection.

Moreover, a family of subsets of a uniform space may be called a Cantor system if every subfamily of it containing small sets and having the finite intersection property has a nonvoid intersection.

Namely, according to Kelley [39, pp. 136 and 193], this terminology allows us to briefly state that a topological (uniform) space is compact (complete) if and only if the family of its closed subsets forms a Riesz (Cantor) system.

Example 16.6. It can be easily seen that the family of all closed balls in $\mathbb{R}$ is a Nachbin system. While, the family all closed balls in $\mathbb{R}^n$, with $n > 1$, is not a Nachbin system.

Example 16.7. More generally, it can be shown that if Γ is a nonvoid set, the family of closed balls in the supremum normed space of all bounded functions of Γ to $\mathbb{R}$ is also a Nachbin system.

Now, as an immediate consequence of Theorems 16.3 and 9.5, we can also state the following

Theorem 16.8. *If F is a and G are as in Theorem 16.3 and there exists a Nachbin system $\mathcal{A}$ in Y such that:*

(4) $n^{-1}\big(F(nx - v) + G(v) \big) \in \mathcal{A}$ *for all* $n \in \mathbb{N}$, $x \in X$ *and* $v \in (-D_F + nx) \cap D_G$;

then

$$\bigcap_{n=1}^{\infty} n^{-1}(F * G)(nx) \neq \emptyset$$

for all $x \in X$.

Proof. If $x \in X$, then by Theorem 16.3 the family

$$\Big\{ n^{-1}\big(F(n x - v) + G(v)\big) : \quad n \in \mathbb{N}, \quad v \in (-D_F + n x) \cap D_G \Big\}$$

has the binary intersection property. Hence, by (4) and Theorem 9.5, it is clear that

$$\bigcap_{n=1}^{\infty} n^{-1}(F * G)(n x) =$$
$$= \bigcap_{n=1}^{\infty} n^{-1} \bigcap \big\{ F(n x - v) + G(v) : \quad v \in (-D_F + n x) \cap D_G \big\} =$$
$$= \bigcap_{n=1}^{\infty} \bigcap \Big\{ n^{-1}\big(F(n x - v) + G(v)\big) : \quad v \in (-D_F + n x) \cap D_G \Big\} =$$
$$= \bigcap \Big\{ n^{-1}\big(F(n x - v) + G(v)\big) : \quad n \in \mathbb{N}, \quad v \in (-D_F + n x) \cap D_G \Big\} \neq \emptyset .$$

Therefore, the required assertion is also true.

17. A general Hahn-Banach type extension theorem

Since the family of all closed balls in a normed space X is closed under translations by vectors and multiplications by scalars, we may also naturally introduce the following

Definition 17.1. A family $\mathcal{A}$ of subsets of a vector space X over $\mathbb{K}$ will be called admissible if

(1) $n^{-1} A \in \mathcal{A}$ for all $n \in \mathbb{N}$ and $A \in \mathcal{A}$;

(2) $x + A \in \mathcal{A}$ for all $x \in X$ and $A \in \mathcal{A}$.

Remark 17.2. By using our former conventions, the above properties can be briefly expressed by writing that:

(1) $n^{-1}\mathcal{A} \subset \mathcal{A}$, or equivalently $\mathcal{A} \subset n\,\mathcal{A}$ for all $n \in \mathbb{N}$;

(2) $x + \mathcal{A} \subset \mathcal{A}$ for all $x \in X$, or equivalently $x + \mathcal{A} = \mathcal{A}$ for all $x \in X$.

Therefore, (1) and (2) are certain $\mathbb{N}$–divisibility and translation-invariance properties of the family $\mathcal{A}$ in the space $\mathcal{P}(X)$ of all subsets of X.

Now, as a useful consequence of Theorem 16.8, we can also state

Theorem 17.3. *Suppose that* F *is a relation and* g *is a function on a commutative group* X *to a vector space* Y *over* $\mathbb{K}$ *and* $\mathcal{A}$ *is an admissible Nachbin system in* Y *such that:*

(1) $F(x) \in \mathcal{A}$ *for all* $x \in D_F$;

(2) D_F *and* D_g *are subgroups of* X;

(3) $g(x) \in F(x)$ *for all* $x \in D_F \cap D_g$ *;*

(4) F *is odd* $\mathbb{N}$*–semi-subhomogeneous and semi-subadditive and* g *is semi-additive.*

Then

$$\bigcap_{n=1}^{\infty} n^{-1}(F * g)(nx) \neq \emptyset$$

for all $x \in X$*.*

Proof. If $n \in \mathbb{N}$, $x \in X$ and $v \in (-D_F + nx) \cap D_g$, then $nx - v \in D_F$ and $v \in D_g$. Thus, $F(nx - v) \in \mathcal{A}$ and $g(v) \in Y$. Hence, by Definition 17.1, it is clear that

$$n^{-1}\big(F(nx - v) + g(v)\big) = n^{-1}F(nx - v) + n^{-1}g(v) \in \mathcal{A}.$$

Thus, Theorem 16.8 can be applied to get the required assertion. Namely, by Corollary 3.22, g is $\mathbb{Z}$-semi-homogeneous.

From the above theorem, it is clear that in particular we also have

Corollary 17.4. *If* F *is an odd* $\mathbb{N}$*-subhomogeneous subadditive relation of a commutative group* X *to a vector space* Y *over* $\mathbb{K}$ *and there exists an admissible Nachbin system* $\mathcal{A}$ *in* Y *such that* $F(x) \in \mathcal{A}$ *for all* $x \in X$*, then*

$$\bigcap_{n=1}^{\infty} n^{-1}(F * \varphi)(nx) \neq \emptyset$$

for all $x \in X$ *and odd semi-additive partial selection function* φ *of* F*.*

Now, as an important consequence of Theorems 15.6 and 17.3, we can easily establish the following straightforward generalization of [22, Theorem 1] of Z. Gajda, A. Smajdor and W. Smajdor.

Theorem 17.5. *If* F *is an odd* $\mathbb{N}$*-subhomogeneous subadditive relation of a commutative group* X *to a vector space* Y *over* $\mathbb{K}$ *and there exists an admissible Nachbin system* $\mathcal{A}$ *in* Y *such that* $F(x) \in \mathcal{A}$ *for all* $x \in X$*, then each nonvoid odd* $\mathbb{N}$*–semi-subhomogeneous superadditive partial selection relation* Φ *of* F *can be extended to a total* $\mathbb{Z} \setminus \{0\}$*–homogeneous additive selection relation* Ψ *of* $F + \Phi(0)$*.*

Proof. By Theorem 15.6, it is enough to show only that each odd quasi-additive partial selection function φ of F is admissible. For this, assume that Ω is a $\mathbb{Z} \setminus \{0\}$–semi-homogeneous quasi-additive partial selection relation of $F + \varphi(0)$ that extends φ. Then, because of $\varphi(0) = \{0\}$, we have $F + \varphi(0) = F$. Moreover, by Corollary 4.13, Ω is also a function. Thus, Ω is also $\mathbb{Z} \setminus \{0\}$–semi-homogeneous quasi-additive partial selection function of F. Hence, by Corollary 17.4, we can see that

$$\bigcap_{n=1}^{\infty} n^{-1}(F * \Omega)(nx) \neq \emptyset$$

for all $x \in X$. Thus, for each $x \in X$, there exists $y \in Y$ such that

$$y \in n^{-1}(F * \Omega)(nx), \qquad \text{and thus} \qquad ny \in (F * \Omega)(nx)$$

for all $n \in \mathbb{N}$. Therefore, the condition (2) of Definition 14.1, with Ω in place of Ψ, is substantially satisfied.

Moreover, we can also note that if $x \in X$ such that $n\,x \in D_\Psi$ for some $n \in \mathbb{N}$, then there exists $y \in Y$ such that

$$y \in n^{-1}\Psi(n\,x), \qquad \text{and thus} \qquad n\,y \in (\Psi(n\,x).$$

Therefore, the condition (1) of Definition 14.1, with Ω in place of Ψ, is also substantially satisfied.

Now, as some immediate consequences of the above theorem, we can also state the following counterparts of Corollaries 15.2 and 15.3.

Corollary 17.6. *If F is as in Theorem 17.5, then for each linear subspace Z of $F(0)$ there exists a $\mathbb{Z}\setminus\{0\}$–homogeneous additive selection relation Ψ of F such that $\Psi(0) = Z$.*

Corollary 17.7. *If F is as in Theorem 17.5, then there exists an additive selection function ψ of F.*

Remar 17.8. Conditions for the existence of additive selection functions have been given by Nikodem [43], Gajda and Ger [20], A. Smajdor [56], Gajda [19], Badora [2], Badora, Ger and Páles [4], Popa [51] and Száz [65, 69].

Remark 17.9. If F is a relation on a groupoid X to a vector space Y to a vector space Y over $\mathbb{K}$, then by the above results it seems convenient to define a sequence $\big(F_n\big)_{n=1}^{\infty}$ of relations on X to Y such that

$$F_n(x) = n^{-1}F(n\,x)$$

for all $n \in \mathbb{N}$ and $x \in X$.

We can note that thus $\big(F_{2^n}\big)_{n=0}^{\infty}$ is just the Hyers sequence associated with F in [65]. Therefore, if in particular X is a semigroup and F is 2–subhomogeneous, then by [65, Theorem 3.6], $\big(F_{2^n}\big)_{n=0}^{\infty}$ is a decresing sequence of subsets of F. Moreover, by [65, Theorem 3.9], $\bigcap_{n=0}^{\infty} F_{2^n}$ is a 2–homogeneous relation on X to Y.

Remark 17.10. Note that if in particular F and G are as in Corollary 11.2, then we already have the recursive formula

$$\begin{aligned}(n+1)(F*G)_{n+1}(x) &= (F*G)\big((n+1)x\big) = \\ &= (F*G)(n\,x+x) = (F*G)(n\,x) + G(x) = n\,(F*G)_n(x) + G(x),\end{aligned}$$

and hence

$$(F*G)_{n+1}(x) = n\,(n+1)^{-1}(F*G)_n(x) + (n+1)^{-1}G(x)$$

for all $n \in \mathbb{N}$ and $x \in D_G$. However, despite this, we cannot say any reasonable sufficient condition for the decreasingness of the sequence $\big((F*G)_n\big)_{n=1}^{\infty}$.

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