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ABSTRACT. The ultrapower T^* of an arbitrary ordered set T is introduced as an infinitesimal extension of T. It is obtained as the set of equivalence classes of the sequences in T, where the corresponding relation is generated by an ultrafilter on the set of natural numbers. It is established that T^* always satisfies Cantor's property, while one can give the necessary and sufficient conditions for T so that T^* would be complete or it would fulfill the open completeness property, respectively. Namely, the density of the original set determines the open completeness of the extension, while independently, the completeness of T^* is determined by the cardinality of T.

1. INTRODUCTION

A well known statement from the theory of ordered fields is that an ordered field is complete if and only if it simultaneously fulfills the Archimedean property and Cantor's property. To demonstrate the independence of these properties, one needs to construct an ordered field which fulfills Cantor's property but is not complete. This question is usually treated in the framework of non-standard analysis, for instance in the works of Stroyan and Luxemburg [6], [4].

However in the above cited publications one can also find an idea for a construction that needs only standard tools. This idea is the concept of ultrapowers: Let us choose an adequate family of subsets of the set of natural numbers called ultrafilter, and then use it to define an equivalence relation on the set of all sequences of the elements of a given set R. This provides a partition, and the set of the equivalence classes is called the ultrapower of R. After introducing this concept, the authors proceed using mainly non-standard techniques, also when it comes to show the properties of the ultrapower. We note that the result of this method is an extension of the original set, since the classes of the constant sequences can be considered as representatives of the original elements.

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The aforementioned works contain only the main ideas without the technical parts, but in the recent publication [2] of Corazza, a detailed construction of the ultrapower of $\mathbb{Q}$ (denoted by $\mathbb{Q}^{\mathbb{N}}/U$) is displayed. One of the main objectives of his work is to construct a non-archimedean ordeded field that fulfills Cantor's property (in [2] it is referred as Nested Intervals Property). As mentioned above, such a construction provides an ordered field with the Cantor property which is not complete.

Not surprisingly, only the ordering of the ultrapower plays an important role during the investigation of these two order-related properties: completeness and the Cantor property, while the field operations are irrelevant at that point. Actually this fact motivates us to generalize these constructions, introducing the ultrapower of an arbitrary ordered set T, and investigating Cantor's property and completeness in its extension T^*.

We may note that while ultrafilters play central role in all of the constructions cited before, the definition of them is not completely coherent. In fact the ultrafilter has to fulfill some conditions which ensure that the extension is proper (not trivial), but these conditions do not appear in the classical definition of the ultrafilter, e.g. in the monograph of Jech [3]. Hence in [4] and [6] a so-called free ultrafilter is used, while in [2] a nonprincipal ultrafilter is used. The main inconvenience with these special ultrafilters is to prove their existence – typically it is done by using Zorn's lemma. Therefore we find it useful to revisit this question, and define the concept of an ultrafilter in such a way that it would be suitable for our construction, and its existence would follow relatively easily from Tarski's classical existence theorem for 'ordinary' ultrafilters (which may be found in [3, Theorem 7.5]).

Our process of showing Cantor's property for the ultrapower sometimes resembles Corazza's methods, although at one point the fact that we start from an arbitrary ordered set makes a significant difference. Namely, in [2] it is shown that in $\mathbb{Q}^{\mathbb{N}}/U$ the intersection of a chain of countably many open intervals is nonempty. This property is referred to as open completeness, and it clearly implies both Cantor's property and the lack of completeness. In our more abstract setting it is reasonable to investigate these properties separately. Namely, starting from an ordered set T, we shall prove that T^* always satisfies Cantor's property, while we can give the necessary and sufficient conditions for T so that T^* would be complete or it would fulfill the open completeness property, respectively. Namely, the density of the original set determines the open completeness of the extension, while independently, the completeness of T^* is determined by the cardinality of T.

2. Particular properties of ordered sets

In this section we collect the basic concepts for ordered sets that are in the focus of this paper.

As usual, we call a nonempty set X equipped with a relation $\leq$ (on X) an ordered set if the relation $\leq$ is reflexive, anti-symmetric, transitive, and linear (i.e., $x \leq y$ or $y \leq x$ for all $x, y \in X$).

Once the relation $\leq$ on X is given, we shall also use the relations $\geq$, $<$ and $>$ in the usual sense.

We shall use the concepts of *lower/upper bound*, *minimum/maximum* (denoted by min and max, respectively), a set being *bounded from below/above*, *least upper bound* (sup) and *greatest lower bound* (inf) in the usual sense as well (cf. [3, Definition 2.2]).

Definition 1. An ordered set $(X, \leq)$ is called *complete* if every nonempty subset of X, that is bounded from above, has a least upper bound.

It is well known that the ordered set $(X, \leq)$ is complete if, and only if, every nonempty subset of X, that is bounded from below, has a greatest lower bound (a proof in an abstract setting can be found, for instance, in [1, Theorem 4.6]).

We will define *intervals* as particular subsets of an ordered set $(X, \leq)$ in the usual way. For example, if $a, b \in X$ such that $a < b$, let $[a, b[= \{\, x \in X \,:\, a \leq x < b \,\}$.

We call a sequence (I_n) of non-empty intervals an *interval chain* if $I_{n+1} \subset I_n$ for every $n \in \mathbb{N}$. We can describe Cantor's property and the open completeness of an ordered set X by the phenomena that the intersection of an arbitrary interval chain of closed, respectively, open intervals is non-empty.

Definition 2. We say that an ordered set X satisfies *Cantor's property* if

$$\bigcap_{n\in\mathbb{N}} [a_n, b_n] \neq \emptyset$$

for any sequences $(a_n), (b_n) : \mathbb{N} \to X$ fulfilling

$$a_k \leq a_{k+1} \leq b_{k+1} \leq b_k$$

for every $k \in \mathbb{N}$.

Definition 3. We say that an ordered set X is *open complete* if

$$\bigcap_{n\in\mathbb{N}}]a_n, b_n[\neq \emptyset$$

for any sequences $(a_n), (b_n) : \mathbb{N} \to X$ fulfilling

$$a_k \leq a_{k+1} < b_{k+1} \leq b_k$$

for every $k \in \mathbb{N}$.

Finally, we introduce the concept of density in ordered sets.

Definition 4. We say that an ordered set X is *dense everywhere* if, for any $a, b \in X$ fulfilling $a < b$, there exists $c \in X$ such that $a < c < b$.

3. An extension of ordered sets

3.1. **Ultrafilter.** We introduce the concept of ultrafilter. For the power set of an arbitrary set X we will use the notation $\mathcal{P}(X)$, i.e. the elements of $\mathcal{P}(X)$ are the subsets of X.

Definition 5. Let $\mathbf{J}$ be an infinite set. The nonempty family of sets $\mathcal{U} \subset \mathcal{P}(\mathbf{J})$ is called a *filter* on $\mathbf{J}$, if

(1) $K \in \mathcal{U}$ and $K \subset L \subset \mathbf{J}$ implies $L \in \mathcal{U}$,
(2) $K, L \in \mathcal{U}$ implies $K \cap L \in \mathcal{U}$,
(3) $K \in \mathcal{U}$ implies that K is infinite.

Moreover, $\mathcal{U}$ is called an *ultrafilter* if it is a filter and

(4) if $K \subset \mathbf{J}$, then $K \in \mathcal{U}$ or $\mathbf{J} \setminus K \in \mathcal{U}$ holds.

Remark 1. In many works (such as [4] or [3]) filters and ultrafilters are defined on arbitrary sets and not particularly infinite ones. In that general case assumption *(3)* is replaced by the weaker condition

(3') $\emptyset \notin \mathcal{U}$.

In that weaker sense it holds that any filter can be extended to an ultrafilter (see [3, Theorem 7.5]). Using this result we prove the following statement.

Theorem 1. *Let $\mathbf{J}$ be an infinite set and let $K \subset \mathbf{J}$ be also infinite. Then there exists an ultrafilter $\mathcal{U} \subset \mathcal{P}(\mathbf{J})$ such that $K \in \mathcal{U}$.*

Proof. Let us define the so-called Fréchet-filter:

$$\mathcal{F} = \{S \subset \mathbf{J} \mid \mathbf{J} \setminus S \text{ is finite.}\}$$

It is easy to see that $\mathcal{F}$ is indeed a filter. Let us define another subset of $\mathcal{P}(\mathbf{J})$:

$$\mathcal{M} = \{M \subset \mathbf{J} \mid \exists L \in \mathcal{F} \ : \ K \cap L \subset M\}.$$

Now we show that $\mathcal{M}$ is a filter (in the weaker sense). Let M, N be arbitrary sets in $\mathcal{M}$, thus there exist $L_M, L_N \in \mathcal{F}$ such that $K \cap L_M \subset M$ and $K \cap L_N \subset N$.

(1) If $M \subset S$ then $K \cap L_M \subset M \subset S$, so $S \in \mathcal{M}$.
(2) $K \cap (L_M \cap L_N) = (K \cap L_M) \cap (K \cap L_N) \subset M \cap N$, and as $L_M \cap L_N \in \mathcal{F}$, it also holds that $M \cap N \in \mathcal{M}$.
(3') Assume $\emptyset \in \mathcal{M}$, which means $K \cap L = \emptyset$ for some $L \in \mathcal{F}$. But this would imply $K \subset \mathbf{J} \setminus L$ and that is impossible, since K is infinite and $\mathbf{J} \setminus L$ is finite.

Notice that $\mathcal{F} \subset \mathcal{M}$ trivially holds. Now $\mathcal{M}$ can be extended to an ultrafilter $\mathcal{U}$ (again in the weaker sense). However $\mathcal{U}$ is an ultrafilter in our restrictive sense, too. Indeed, if $F \in \mathcal{U}$ for some finite subset F, then $\mathbf{J} \setminus F \notin \mathcal{U}$ which contradicts $\mathbf{J} \setminus F \in \mathcal{F}$. □

As a corollary of this statement, we get that there exists an ultrafilter on the set of natural numbers. Finally we emphasize that the existence of an appropriate ultrafilter on $\mathbb{N}$ can be proven in several, slightly different ways. However this often requires the

introduction of further definitions such as *free ultrafilter* (in [4]) or *nonprincipal ultrafilter* (in [2]).

3.2. **Ultrapower of an ordered set.** In the next step we construct a so called *ultrapower* of any ordered set T. The existence of an ultrafilter on the set of natural numbers provides us a way to define an equivalence relation on the set of all sequences of elements of T, in such manner that an adequate order on the equivalence classes would generate an ordered set. As it is common in the literature, we will use an asterisk (*) to denote the operation that assigns its ultrapower to the original ordered set.

In the subsequent sections let T be an ordered set and $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$.

Let $\mathcal{T} = \{(a_n) \mid (a_n) : \mathbb{N} \to T\}$ denote the set of all sequences of elements of T.

Proposition 1. *Let us define the relation $\sim \subset \mathcal{T} \times \mathcal{T}$ in the following way:*

$$(a_n) \sim (b_n) \Longleftrightarrow \{n \in \mathbb{N} : a_n = b_n\} \in \mathcal{U}.$$

Then $\sim$ is an equivalence relation. Furthermore, let us denote the set of the equivalence classes by T^*, while the class of an element $(a_n) \in \mathcal{T}$ be denoted by $\overline{(a_n)}$. The relation $\leq \subset T^* \times T^*$, given by

$$\overline{(a_n)} \leq \overline{(b_n)} \Longleftrightarrow \{n \in \mathbb{N} : a_n \leq b_n\} \in \mathcal{U}\,,$$

is well-defined, and $(T^*, \leq)$ is an ordered set.

Proof. The reflexivity and symmetry of $\sim$ is obvious. To check the transitivity, assume $(a_n) \sim (b_n)$ and $(b_n) \sim (c_n)$. Then

$$\{n \in \mathbb{N} : a_n = c_n\} \supset \{n \in \mathbb{N} : a_n = b_n\} \cap \{n \in \mathbb{N} : b_n = c_n\} \in \mathcal{U} \qquad (\star)$$

implies $(a_n) \sim (c_n)$, so $\sim$ is indeed an equivalence relation. Similarly, if $(a_n) \sim (\tilde{a}_n)$, $(b_n) \sim (\tilde{b}_n)$ and $\overline{(a_n)} \leq \overline{(b_n)}$, then

$$\{n \in \mathbb{N} : \tilde{a}_n \leq \tilde{b}_n\} \supset \{n \in \mathbb{N} : a_n \leq b_n\} \cap \{n \in \mathbb{N} : a_n = \tilde{a}_n\} \cap \{n \in \mathbb{N} : b_n = \tilde{b}_n\} \in \mathcal{U}$$

ensures that $\overline{(\tilde{a}_n)} \leq \overline{(\tilde{b}_n)}$, hence $\leq$ is independent of the choice of representatives, i.e. it is a well-defined relation on T^*.

Clearly $\leq$ is reflexive, and also notice that if we replace the equalities with inequalities in $(\star)$, we get the transitivity of $\leq$. Furthermore,

$$\{n \in \mathbb{N} : a_n = b_n\} \supset \{n \in \mathbb{N} : a_n \leq b_n\} \cap \{n \in \mathbb{N} : b_n \leq a_n\} \in \mathcal{U}$$

shows that $\leq$ is antisymmetric. Finally, since the sets $\{n \in \mathbb{N} : a_n \leq b_n\}$ and $\{a_n > b_n\}$ give a disjoint partition of $\mathbb{N}$, exactly one of them is in $\mathcal{U}$. These properties together provide that $(T^*, \leq)$ is an ordered set. □

3.3. **Cantor's property for the extension.** In this section we will show that the operation * always produces an ordered set that satisfies Cantor's property.

Theorem 2. *If T is an ordered set then its extension T^* satisfies Cantor's property, i.e. if $a_k = ((a_k)_n) \in \mathcal{T}$ and $b_k = ((b_k)_n) \in \mathcal{T}$ $(k \in \mathbb{N})$ such that for every $k \in \mathbb{N}$*

$$\overline{a_k} \leq \overline{a_{k+1}} \leq \overline{b_{k+1}} \leq \overline{b_k}\,,$$

then

$$\bigcap_{k\in\mathbb{N}} \left[\overline{a_k}\,,\, \overline{b_k}\right] \neq \emptyset\,.$$

Proof. We define the following sets:

$$A_i = \{n \in \mathbb{N} : (a_i)_n \leq (a_{i+1})_n\}, \qquad B_i = \{n \in \mathbb{N} : (b_i)_n \geq (b_{i+1})_n\},$$
$$\text{and } C_i = \{n \in \mathbb{N} : (a_i)_n \leq (b_i)_n\} \quad \text{for every } i \in \mathbb{N}.$$

Using these we construct the following sets:

$$\mathcal{A}_k = \bigcap_{i=1}^{k-1} A_i, \qquad \mathcal{B}_k = \bigcap_{i=1}^{k-1} B_i, \qquad \mathcal{C}_k = \bigcap_{i=1}^{k} C_i \qquad (k \in \mathbb{N} \setminus \{1\}).$$

Obviously $\mathcal{A}_k$, $\mathcal{B}_k$, $\mathcal{C}_k$ belong to $\mathcal{U}$, as they are intersections of finitely many sets from $\mathcal{U}$. For the same reason $\mathcal{A}_k \cap \mathcal{B}_k \cap \mathcal{C}_k = \mathcal{D}_k \in \mathcal{U}$.

Let $\mathcal{D}_1 = \mathcal{C}_1 = C_1$, thus the set $\mathcal{D}_k$ is now defined for every $k \in \mathbb{N}$, and it consists of the natural numbers n for which the following inequalities hold:

$$(a_1)_n \leq \ldots \leq (a_k)_n \leq (b_k)_n \leq \ldots \leq (b_1)_n.$$

In the next step, for every $n \in \mathbb{N}$, we define another set of natural numbers I_n as follows: $I_n = \{k \in \mathbb{N} : n \in \mathcal{D}_k\}$. It is easy to see from the definition of the sets $\mathcal{D}_k$ that if $k \in I_n$ then $l \in I_n$ is also true for every natural number $l \leq k$. Using these sets we assign a non-negative integer to every $n \in \mathbb{N}$ as follows: let

$$\alpha_n = \begin{cases} 0, & \text{if } I_n = \emptyset\,, \\ n, & \text{if } I_n \text{ has no upper bound,} \\ \min\{n, \max I_n\} & \text{if } I_n \text{ is non-empty and bounded from above.} \end{cases}$$

We should note that if n is an element of $\mathcal{D}_k$ and $k \leq n$ then $\alpha_n \geq k$ $(k, n \in \mathbb{N}\,)$. This also means that $n \notin \mathcal{D}_1$ holds if and only if $I_n = \emptyset$. Hence $n \in \mathcal{D}_k \setminus \{m \in \mathbb{N} : m < k\}$ implies

$$(a_1)_n \leq \ldots \leq (a_k)_n \leq (a_{\alpha_n})_n \leq (b_{\alpha_n})_n \leq (b_k)_n \leq \ldots \leq (b_1)_n.$$

It is trivial that $\mathcal{D}_k \setminus \{m \in \mathbb{N} : m < k\}$ is in the ultrafilter.

After these remarks it is rather easy to construct a common point of the interval chain. We define the sequence $c = (c_n) : \mathbb{N} \longrightarrow \mathbb{R}$ as follows:

$$c_n = \begin{cases} (a_{\alpha_n})_n, & \text{if } n \in \mathcal{D}_1 \\ (a_1)_1, & \text{if } n \notin \mathcal{D}_1. \end{cases}$$

We will show that $\overline{a_k} \leq \overline{c} \leq \overline{b_k}$ for any $k \in \mathbb{N}$. This is a straightforward corollary of our previous remark, namely that

$$(a_1)_n \leq \ldots \leq (a_k)_n \leq (a_{\alpha_n})_n = c_n \leq (b_k)_n \leq \ldots \leq (b_1)_n$$

holds if $n \in \mathcal{D}_k \setminus \{m \in \mathbb{N} : m < k\}$. Since $\mathcal{D}_k \setminus \{m \in \mathbb{N} : m < k\} \in \mathcal{U}$, the sets

$$\{n \in \mathbb{N} : (a_k)_n \leq c_n\} \text{ and } \{n \in \mathbb{N} : c_n \leq (b_k)_n\}$$

are also elements of $\mathcal{U}$ (obviously they are supersets of $\mathcal{D}_k \setminus \{m \in \mathbb{N} : m < k\}$).

The final step of the proof is to use the definition of the ordering relation on T^*, so we obtain

$$\overline{c} \in \bigcap_{k \in \mathbb{N}} [\overline{a_k}, \overline{b_k}]$$

□

Remark 2. It seems reasonable to make a similar proposition and replace the closed intervals by open intervals. However if we do so then we must require the T ordered set to be dense everywhere. Otherwise a trivial counterexample can be made as an empty open interval exists.

On the other hand, the criterion concerning the density of T is sufficient to prove the alternate form of the previous theorem (i.e. open completeness). We sum up these perceptions in the following theorem:

Theorem 3. *Let T be an ordered set. The following statements are equivalent:*

(a) *T is dense everywhere.*

(b) *if $a_k = ((a_k)_n) \in \mathcal{T}$ and $b_k = ((b_k)_n) \in \mathcal{T}$ $(k \in \mathbb{N})$ such that for every $k \in \mathbb{N}$*

$$\overline{a_k} \leq \overline{a_{k+1}} < \overline{b_{k+1}} \leq \overline{b_k}\,,$$

then

$$\bigcap_{k \in \mathbb{N}}]\overline{a_k}\,,\, \overline{b_k}[\neq \emptyset\,.$$

Proof. To show (b) $\Longrightarrow$ (a) we explain the counterexample which was mentioned in Remark 2. Let $p, q \in T$ such that $p < q$ and there is no element of T in the open interval $]p, q[$.

This means that the open interval $]\overline{p}, \overline{q}[$ is also empty, where $\overline{p}$ and $\overline{q}$ are the classes of the constant sequences (p_n) and (q_n) defined by $p_n = p$ and $q_n = q$ for every $n \in \mathbb{N}$.

Thus if $p_k = (p_n) \in \mathcal{T}$ and $q_k = (q_n) \in \mathcal{T}$ for every $k \in \mathbb{N}$, then

$$\bigcap_{k \in \mathbb{N}}]\overline{p_k}\,,\, \overline{q_k}[=]\overline{p}, \overline{q}[= \emptyset\,.$$

To show the reverse implication, we can take the same process as we did in the proof of Theorem 2. The only adjustments to be made are that we define

$$C_i = \{\, n \in \mathbb{N} \,:\, (a_i)_n < (b_i)_n \,\} \quad (i \in \mathbb{N})$$

and c_n has to be an element from the interior of the 'α_n-th' interval (obviously, it cannot be an endpoint as it initially was), that is,

$$c_n \in \,](a_{\alpha_n})_n \,,\, (b_{\alpha_n})_n[\quad \text{if} \quad n \in \mathcal{D}_1 \,, \quad \text{while} \quad c_n = (a_1)_1 \quad \text{if} \quad n \notin \mathcal{D}_1 \,.$$

Clearly, the required element c_n exists as T is dense everywhere. We will not repeat the entire proof since every remaining step is analogous. □

3.4. **Completeness of the extension.** Finally we will show that the operation * does not preserve completeness in general. Moreover the completeness of the ultrapower depends only on the cardinality of the initial ordered set.

Lemma 1. *Let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$ and $A_j \subset \mathbb{N}$ $(j = 1, \dots, n)$. If*

$$\bigcup_{j=1}^{n} A_j = \mathbb{N}\,, \text{ then } \exists k \in \{1, \dots, n\} \text{ such that } A_k \in \mathcal{U}$$

Proof. Assume that for all indices $j \in \{1, \dots, n\} : A_j \notin \mathcal{U}$. From the definition of $\mathcal{U}$ we get $\mathbb{N} \setminus A_j \in \mathcal{U}$ for every $j \in \{1, \dots, n\}$. This means

$$\emptyset = \mathbb{N} \setminus \mathbb{N} = \mathbb{N} \setminus \bigcup_{j=1}^{n} A_j = \bigcap_{j=1}^{n} (\mathbb{N} \setminus A_j) \in \mathcal{U}$$

which is an obvious contradiction as $\emptyset$ is not infinite. Therefore some $k \in \{1, \dots, n\}$ must exist for which $A_k \in \mathcal{U}$. □

Theorem 4. *Let T be an ordered set. T^* is complete if and only if T is finite.*

Proof. In the first place we prove that if T is infinite then T^* is not complete. We will use the following basic fact: in an infinite ordered set there exists a strictly monotone sequence of elements. In order to prove this, we may consider an obviously existing injective sequence $(x_n)\colon \mathbb{N} \to T$ (i.e., $x_n \neq x_m$ if $n \neq m$). It is a well-known fact that every sequence in an ordered set contains a monotone subsequence (we can apply the proof for real sequences [5] in this more general context as well). Clearly, such a monotone subsequence of (x_n) is strictly monotone.

We give the details of the proof only for the case of a strictly increasing sequence.

Let $t_1 < t_2 < t_3 < \dots$ be a strictly increasing sequence of elements in T. It is easy to see that the equivalence classes of the constant sequences

$$(s_k)_n = t_k \qquad (n \in \mathbb{N})\ (k \in \mathbb{N})$$

generate a subset

$$S = \{\, \overline{s_k} \mid k \in \mathbb{N} \,\}$$

of T^* which is bounded from above. Indeed, one can easily check that $\overline{(t_n)}$ is an upper bound of S. Now we demonstrate that S has no least upper bound. Let $(b_n) \in \mathcal{T}$ such that $\overline{(b_n)}$ is an upper bound of S. We define some sets in a similar manner as we did in the proof of Theorem 2: let

$$\mathcal{D}_k = \{n \in \mathbb{N} \,:\, b_n \geq t_k\} \in \mathcal{U}, \qquad I_n = \{k \in \mathbb{N} \,:\, b_n \geq t_k\} \qquad (k, n \in \mathbb{N}).$$

We should note that if $k \in I_n$ then $l \in I_n$ for every natural number $l \leq k$. Another easy observation is that, for any $m, k \in \mathbb{N}$, $m \in \mathcal{D}_k$ if and only if $k \in I_m$. (we will use these two remarks later on).

Now we can define a mapping $\alpha : \mathbb{N} \to \mathbb{N} \cup \{0\}$ as follows: let

$$\alpha_n = \begin{cases} 0, & \text{if } I_n = \emptyset, \\ n, & \text{if } I_n \text{ has no upper bound}, \\ \min\{n, \max I_n\}, & \text{otherwise.} \end{cases}$$

With the notation $\beta_n = \left\lfloor \dfrac{\alpha_n}{2} \right\rfloor$ it is possible to construct an upper bound for S which is smaller than $\overline{b}$ (here $\lfloor\ \rfloor$ denotes the floor, i.e., $\lfloor x \rfloor = \max\{\, z \in \mathbb{Z} \,:\, z \leq x \,\}$).

We define $(c_n) \in \mathcal{T}$ as follows:

$$c_n = \begin{cases} b_n, & \text{if } \alpha_n < 2 \\ t_{\beta_n}, & \text{if } \alpha_n \geq 2 \end{cases}$$

For any natural number k the following argumentation can be made: if $\alpha_n \geq 2k$ then $\beta_n \geq k$ and therefore $c_n \geq t_k$. Since

$$\{n \in \mathbb{N} \,:\, c_n \geq t_k\} \supset \{n \in \mathbb{N} \,:\, \alpha_n \geq 2k\} = \mathcal{D}_{2k} \setminus \{m \in \mathbb{N} \,:\, m < 2k\} \in \mathcal{U}$$

follows from the two simple remarks that were stated earlier, we have obtained that $\overline{(c_n)}$ is an upper bound of S. On the other hand, for every $m \in \mathcal{D}_2 \setminus \{1\}$, the value c_m is indeed smaller than b_m, because $2 \leq \alpha_m \in I_m$, and thus

$$c_m = t_{\beta_m} < t_{\alpha_m} \leq b_m,$$

so $\overline{(c_n)} < \overline{(b_n)}$. Therefore S has no least upper bound.

With some obvious adjustments it can be shown that if $u_1 > u_2 > u_3 > \dots$ is a strictly decreasing sequence of elements in T and $v_k \in \mathcal{T}$ such that $(v_k)_n = u_k$ for all $n, k \in \mathbb{N}$, then the set

$$V = \{\ \overline{v_k} \mid k \in \mathbb{N}\} \subset T^*$$

does not have a greatest lower bound.

In the second part of the proof we will verify the reverse implication, namely that if T is finite then T^* is complete. Since a finite ordered set is always complete, it is sufficient to show that, for any finite ordered set T, T^* is finite as well.

Let $k \in \mathbb{N}$, $T = \{\, t_1, \dots, t_k \,\}$, and for each $j \in \{1, \dots, k\}$, let $s_j \in \mathcal{T}$ such that $(s_j)_n = t_j$ for all $n \in \mathbb{N}$ (a constant sequence). Now let us consider an arbitrary sequence

$(a_n) \in \mathcal{T}$. For every $j \in \{1, \ldots, k\}$ we define the sets $A_j = \{\, n \in \mathbb{N} : a_n = t_j \,\}$. Obviously, $\bigcup_{j=1}^{k} A_j = \mathbb{N}$. According to Lemma 1, there exists an index $m \in \{1, \ldots, k\}$ such that $A_m \in \mathcal{U}$ and therefore $\overline{(a_n)} = \overline{s_m}$ (i.e., (a_n) is the equivalent with the constant t_m sequence). So we may conclude that T^* contains only the equivalence classes of finitely many constant sequences, which implies that T^* is complete as well. □

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