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COMPOSITION ITERATES, CAUCHY EQUATIONS, TRANSLATION EQUATIONS, AND SINCOV INCLUSIONS

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Abstract. Improving and extending some ideas of Gottlob Frege (1874) (on a generalization of the notion of the composition iterates of a function), we consider the composition iterates φ^n of a relation φ on a set X, defined by

$$\varphi^0 = \Delta_X \,, \qquad \varphi^n = \varphi \circ \varphi^{n-1} \quad if \quad n \in \mathbb{N} \,, \qquad \text{and} \qquad \varphi^\infty = \bigcup_{n=0}^{\infty} \varphi^n \,.$$

In particular, by using the relational equation $\varphi^{n+m} = \varphi^n \circ \varphi^m$ only for $n,\, m \in \mathbb{N}_0 = \{0\} \cup \mathbb{N}$, we show that the function α, defined by

$$\alpha\,(n) = \varphi^n \quad \text{for} \quad n \in \mathbb{N}_0 \,,$$

satisfies the Cauchy problem

$$\alpha\,(n+m) = \alpha(n) \circ \alpha\,(m) \,, \qquad \alpha\,(0) = \Delta_X \,.$$

Moreover, the function f, defined by

$$f\,(n,\, A) = \alpha\,(n)\,[\,A\,] \qquad \text{for} \qquad n \in \mathbb{N}_0 \quad \text{and} \quad A \subseteq X,$$

satisfies the translation problem

$$f\,(n+m,\, A) = f\,(n,\, f\,\big(m,\, A)\big) \,, \qquad f\,(0,\, A) = A \,.$$

Furthermore, the function F, defined by

$$F\,(A,\, B) = \big\{\, n \in \mathbb{N}_0 : \ \ f\,(n,\, B) = A\, \big\} \qquad \text{for} \qquad A,\, B \subseteq X,$$

satisfies the Sincov problem

$$F\,(A,\, B) + F\,(B,\, C) \subseteq F\,(A,\, C) \,, \qquad 0 \in F\,(A,\, A) \,.$$

Motivated by these observations, we systematically investigate a function F on a product set X^2 to the power groupoid $\mathcal{P}\,(U)$ of an additive groupoid U which is supertriangular in the sense that

$$F\,(x,\, y) + F\,(y,\, z) \subseteq F\,(x,\, z)$$

for all $x,\, y,\, z \in X$. For this, we introduce the convenient notations

$$R(x,\, y) = F\,(y,\, x) \qquad \text{and} \qquad S\,(x,\, y) = F\,(x,\, y) + R\,(x,\, y) \,,$$

and

$$\Phi\,(x) = F\,(x,\, x) \qquad \text{and} \qquad \Psi\,(x) = \bigcup_{y \in X} S\,(x,\, y) \,.$$

Moreover, we gradually assume that U and F have some useful additional properties. For instance, U has a zero, U is a group, U is commutative, U is cancellative, or U has a suitable distance function. And, F is nonpartial, F is symmetric, skew symmetric, or single-valued.

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