UNIVERSITY OF DEBRECEN

Remarks and problems at the Conference on Inequalities and Applications, Hajdúszoboszló, Hungary, 2016

Árpád Száz

Preprints No. 418
(Technical Reports No. 2016/9)

INSTITUTE OF MATHEMATICS

2016

REMARKS AND PROBLEMS AT THE CONFERENCE ON INEQUALITIES AND APPLICATIONS, HAJDÚSZOBOSZLÓ, HUNGARY, 2016

ÁRPÁD SZÁZ

Abstract. This paper contains improved and enlarged versions of the remarks and problems of the author delivered to the Conference on Inequalities and Applications, Hajdúszoboszló, Hungary, 2016.

In particular, the following closely related subjects are investigated:

1. Upper and lower left-invariant generalized metrics
2. Galois connection between generalized metrics and norms
3. Semimetrics and preseminorms on groups
4. Semi-inner products on groups
5. Further properties of the induced generalized seminorm
6. Two basic properties of the parallelepiped equation
7. Parapreseminorms should also be investigated
8. Sub-quadratic and super-quadratic functions are usually quadratic
9. A generalizations of the quadrilateral inequality
10. Two characterizations of additive functions
11. Problems in connection with the Páles equation

Moreover, to motivate some further similar investigations, a rather complete list of relevant references is included.

1. Upper and lower left-invariant generalized metrics

In our unfinished paper [208], we have proved the following

Theorem 1.1. *If X is a group, then for a function d of X^2 to $\mathbb{R}$, the following assertions are equivalent:*

(1) $d(0, y) = d(x, x+y)$ *for all* $x, y \in X$,

(2) $d(x, y) = d(0, -x+y)$ *for all* $x, y \in X$,

(3) $d(x, y) = d(z+x, z+y)$ *for all* $x, y, z \in X$.

(4) $d(z+x, z+y) \le d(x, y)$ *for all* $x, y, z \in X$,

(5) $d(x, y) \le d(z+x, z+y)$ *for all* $x, y, z \in X$.

2010 *Mathematics Subject Classification.* 06A15, 11E39, 15A63, 20A99, 20K99, 22A99, 39B22, 39B32, 39B52, 39B62, 39B82, 46B99, 46C50, 47A07, 47A30, 54E25.

Key words and phrases. Groups, invariant semimetrics, preseminorms, Galois connections, semi-inner products, paralellogram identity, parallelepiped law, parapreseminorms, quadrilateral inequality, superquadratic functions, inequalities implying additivity, Páles equation.

The work of the author has been supported by the Hungarian Scientific Research Fund (OTKA) Grant 111651.

Hint. Note that if (2) holds, then we have

$$d(z+x,\, z+y) = d\big(0, -(z+x)+z+y\big) = d\big(0, -x-z+z+y\big) = d(0, -x+y) = d(x,\, y)$$

for all $x, y, z \in X$, and thus (3) also holds.

While, if for instance (4) holds, then by writing $-z+x$ in place of x and $-z+y$ in place of y in (4), we obtain

$$d(x,\, y) \leq d(-z+x,\, -z+y)\,.$$

for all $x, y, z \in X$. Hence, by writing $-z$ in place of z, we can already see that

$$d(x,\, y) \leq d(z+x,\, z+y)$$

for all $x, y, z \in X$. Thus, (3) also holds.

Definition 1.2. Now, the function d may be naturally called *upper left-invariant (lower left-invariant)* if

$$d(0,\, y) \leq d(x,\, x+y) \qquad \Big(\, d(x,\, x+y) \leq d(0,\, y) \Big)$$

for all $x, y \in X$.

Remark 1.3. Moreover, the function d may be naturally called *left-invariant* if it is both upper and lower left-invariant.

Clearly, d is lower left-invariant if and only if $-d$ is upper left-invariant. Therefore, d is left-invariant if and only if both d and $-d$ are upper left-invariant.

Concerning upper and lower left-invariant metrics, in our former papers [206] and [209], we have established some straightforward generalizations of the following illustrating examples.

Example 1.4. The usual metric d, defined for any $x, y \in \mathbb{R}$ by

$$d(x,\, y) = |\, x-y\,|\,,$$

is both upper and lower left-invariant.

Example 1.5. The bounded metric d, defined for any $x, y \in \mathbb{R}$ by

$$d(x,\, y) = |\, \varphi(x) - \varphi(y)\,|\,, \qquad \text{where} \qquad \varphi(x) = x/(1+|\,x\,|)\,,$$

is neither upper nor lower left-invariant.

Example 1.6. The postman metric d, defined for any $x, y \in \mathbb{C}$ by

$$d(x,\, y) = 0 \quad \text{if} \quad x = y \qquad \text{and} \qquad d(x,\, y) = |\,x\,| + |\,y\,| \quad \text{if} \quad x \neq y\,.$$

is upper left-invariant, but not lower left-invariant.

Problem 1.7. However, I do not know:

What can be said about the upper and lower left-invariances of a common generalization of the postman, radial and river metrics given in our former paper [200] *?*

Some of the statements and proofs of this paper are too difficult for me. Therefore, their validity should be checked by some other mathematicians.

Remark 1.8. Moreover, the upper and lower left-invariance properties of several other curious metrics could also be investigated.

For instance, in [98], [178] and [147, p. 482], one can find the striking metrics defined by

$$d(x, y) = \frac{|x-y|}{\sqrt{x+y}} \qquad \text{for} \qquad x, y \in \mathbb{R}_+ ,$$

$$d(x, y) = \frac{|x-y|}{|x|+|y|} \qquad \text{for} \qquad x, y \in \mathbb{R} \quad \text{with} \quad |x|+|y| \neq 0$$

and

$$d(x, y) = \frac{|x-y|}{\sqrt{1+|x|^2}\,\sqrt{1+|y|^2}} \qquad \text{for} \qquad x, y \in \mathbb{C} ,$$

The second and the third ones are called the *multiplicative and chordal metrics*, respectively. They were substantially generalized by Klamkin and Meir [111].

However, the most complete account on distance functions can be found in Deza and Deza [38]. Moreover, several curious metrics can be derived from generalized metrics by [11].

Remark 1.9. At this point, it is also worth mentioning that several metrics can also be derived from a metric d on a set X by defining

$$\rho(x, y) = f\big(d(x, y)\big)$$

for all $x, y \in X$, with a suitable function f of $\mathbb{R}_+$ to itself (see Corazza [35]), or

$$\rho(u, v) = d\big(\varphi(u), \varphi(v)\big)$$

for all u and v in a set U, with an injective function φ of U to X.

Remark 1.10. If in particular U is a group, X is a vector space and d is derived from a norm on X, then the lower left-invariance of the latter metric ρ means only that

$$\| -\varphi(u) + \varphi(u+v) \| \leq \| -\varphi(0) + \varphi(v) \|$$

for all $u, v \in U$, or equivalently

$$\| -\varphi(-u) + \varphi(v) \| \leq \| -\varphi(0) + \varphi(u+v) \|$$

for all $u, v \in U$.

Note that if in particular φ is odd, then latter condition is equivalent to the requirement that

$$\| \varphi(u) + \varphi(v) \| \leq \| \varphi(u+v) \|$$

for all $u, v \in U$. This property, even if φ is not assumed to be injective, usually implies not only that φ is odd, but also that φ is additive. (See [141] and [207].) Therefore, ρ is actually left-invariant.

Remark 1.11. Characterizations of norms derivable from inner products can be found in Amir [10], Istrătescu [97] and Alsina, Sikorska and Tomás [9].

However, the most fundamental discoveries were already made by Fréchet [71], Jordan and von Neumann [101], and Ficken [63].

While, metrics derivable from norms have only been explicitly studied by Oikhberg and Rosenthal [155], Šemrl [182, 183] and Chmieliński [30, 31].

Moreover, the equivalence of norms to norms derived from inner products have been studied by Joichi [100] and Chmieliński [29]. However, equivalences of metrics to metrics derived from norms seem not to be investigated.

Remark 1.12. At a future conference, Jacek Chmieliński should be asked to hold a survey talk or a special session on the above mentioned subjects which allow of important applications of functional equations and inequalities.

2. Galois connection between generalized metrics and norms

In our former paper [206], for a group X, we considered the sets

$$\mathcal{N} = \mathcal{N}(X) = \mathbb{R}^X \qquad \text{and} \qquad \mathcal{M} = \mathcal{M}(X) = \mathbb{R}^{X^2},$$

to be equipped with the usual pointwise inequality of real-valued functions.

Moreover, having in mind a well-known connection between norms and metrics in vector spaces, for any $p \in \mathcal{N}$, $d \in \mathcal{M}$ and $x, y \in X$ we defined

$$p_d(x) = d(0, x) \qquad \text{and} \qquad d_p(x, y) = p(-x + y).$$

Thus, it can be easily seen that, for any $p \in \mathcal{N}$ and $d \in \mathcal{M}$,

(1) $d_p \le d \implies p \le p_d$, (2) $p \le p_d \implies d_p \le d_{p_d}$.

Moreover, if in particular d is as in Example 1.5, then $d_{p_d} \not\le d$, despite that $p = p_{d_p}$ for all $p \in \mathcal{N}$.

Therefore, by defining

$$\mathcal{M}^\wedge = \mathcal{M}^\wedge(X) = \left\{ d \in \mathcal{M}(X) : \quad d_{p_d} \le d \right\},$$

we can note that the functions, defined by

$$f(p) = d_p \qquad \text{and} \qquad g(d) = p_d$$

for all $p \in \mathcal{N}$ and $d \in \mathcal{M}^\wedge$, establish an increasing Galois connection between the posets $\mathcal{N}$ and $\mathcal{M}^\wedge$ in the sense that, for any $p \in \mathcal{N}$ and $d \in \mathcal{M}^\wedge$, we have

$$f(p) \le d \iff p \le g(d).$$

Thus, several consequences of the definitions of p_d and d_p can be derived from the corresponding results on increasing Galois connections [199, 201]. However, because of the simplicity of the present definitions, it is frequently more convenient to apply some direct proofs.

To let the reader feel the importance of the above mentioned Galois connection, we note that if in particular $p \in \mathcal{N}$ is a preseminorm on X in the sense that

(1) $p(0) \le 0$, (2) $p(-x) \le p(x)$, (3) $p(x + y) \le p(x) + p(y)$

for all $x, y \in X$, then d_p is a left-invariant semimetric on X such that

$$d(p(x), p(y)) = |p(x) - p(y)| \le d_p(x, y)$$

for all $x, y \in X$.

While, if d is a left-invariant semimetric on X, then p_d is a preseminorm on X such that $d = d_{p_d}$. Therefore, preseminorms and left-invariant semimetrics are equivalent tools in a group.

However, in contrast to the opinions of several authors, the former ones, being a function of only one variable, are certainly more convenient tools than the latter ones. (Curiously enough, norms were even called metrics by Jordan and von Neumann [101] .)

In this respect, it is also worth mentioning that if in particular d is as in Example 1.6, then $d \in \mathcal{M}^{\wedge}(\mathbb{C})$, but $d \neq d_{p_d}$.

Concerning the family $\mathcal{M}^{\wedge}$, in [206] , we have also proved the following

Theorem 2.1. *For any $d \in \mathcal{M}$, the following assertions are equivalent:*

(1) $d \in \mathcal{M}^{\wedge}$,

(2) d *is upper left-invariant,*

(3) $p \leq p_d$ *implies* $d_p \leq d$ *for all* $p \in \mathcal{N}$,

(4) $d(0, -x + y) \leq d(x, y)$ *for all* $x, y \in X$.

Proof. By the definition of $\mathcal{M}^{\wedge}$, (1) means only that

$$d_{p_d}(x, y) \leq d(x, y)$$

for all $x\ y \in X$. Hence, by using that

$$d_{p_d}(x, y) = p_d(-x + y) = d(0, -x + y)$$

for all $x, y \in X$, we can see that (1) and (4) are equivalent. Moreover, from the proof of Theorem 1.1, it is clear that (2) and (4) are also equivalent.

Thus, it remains only to prove that (1) and (4) are also equivalent. For this, note if $p \in \mathcal{N}$ and $p \leq p_d$, then by the definition of d_p we also have $d_p \leq d_{p_d}$. Hence, if (1) holds, we can infer that $d_p \leq d$, and thus (3) also holds. While, if (3) holds, the from the trivial inequality $p_d \leq p_d$ we can already infer that $d_{p_d} \leq d$, and thus (1) also holds.

3. Semimetrics and preseminorms on groups

The following definition of semimetrics, which uses inequalities instead of equalities in its first two axioms too, is certainly more uniform than that of pseudometrics used by Kelley [108, p. 119] and Ansari [12, p. 9] , for instance.

Definition 3.1. For any set X, a function d of X^2 to $\mathbb{R}$ is called *semimetric* on X if

(a) $d(x, x) \leq 0$ for all $x \in X$,

(b) $d(y, x) \leq d(x, y)$ for all $x, y \in X$,

(c) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

The appropriateness of this definition is apparent from the following

Theorem 3.2. *If d is a semimetric on X, then*

(1) $d(x, x) = 0$ *for all* $x \in X$,

(2) $d(x, y) \geq 0$ *for all* $x, y \in X$,

(3) $d(x, y) = d(y, x)$ *for all* $x, y \in X$,

(4) $|\,d\,(x,\,y)-d\,(x,\,z)\,|\leq d\,(y,\,z)$ *for all* $x,\,y,\,z\in X$*,*

(5) $|\,d\,(x,\,y)-d\,(z,\,w)\,|\leq d\,(x,\,z)+d\,(y,\,w)$ *for all* $x,\,y,\,z,\,w\in X$*.*

Hint. Assertion (5) can, most briefly, be proved with the help of (3) and (4).

Remark 3.3. Note that if d is a function of X^2 to $\mathbb{R}$ satisfying only (b) and (c), then we already have $d\,(x,\,y)\geq 0$ for all $x,\,y\in X$.

Therefore, by defining for any $x,\,y\in X$

$$\rho\,(x,\,y)=0 \quad \text{if} \quad x=y \qquad \text{and} \qquad \rho\,(x,\,y)=d\,(x,\,y) \quad \text{if} \quad x\neq y\,,$$

we can obtain a semimetric ρ on X.

Now, by using Definition 3.1, we may also naturally introduce

Definition 3.4. A semimetric d on X is called a *metric* if $d\,(x,\,y)=0$ implies $x=y$ for all $x,\,y\in X$.

Remark 3.5. Thus, by (2) in Theorem 3.2, we can state that a semimetric on X is a metric if and only if $d\,(x,\,y)\leq 0$ implies $x=y$ for all $x,\,y\in X$.

In [208], analogously to [197, Definition 1.1], we have also introduced the following

Definition 3.6. A function p of a group X to $\mathbb{R}$ is called a *preseminorm* on X if

(a) $p\,(0)\leq 0\,,$

(b) $p\,(-x)\leq p\,(x)$ for all $x\in X$,

(c) $p\,(x+y)\leq p\,(x)+p\,(y)$ for all $x,\,y\in X$.

The appropriateness of this definition is apparent from the following

Theorem 3.7. *If* p *is a preseminorm on* X*, then*

(1) $p\,(0)=0\,,$

(2) $p\,(x)\geq 0$ *for all* $x\in X$*,*

(3) $p\,(x)=p\,(-x)$ *for all* $x\in X$*,*

(4) $|\,p\,(x)-p\,(y)\,|\leq p\,(x-y)$ *for all* $x,\,y\in X$*,*

(5) $p\,(\,k\,x\,)\leq |\,k\,|\,p\,(x)$ *for all* $k\in\mathbb{Z}$ *and* $x\in X$*.*

Hint. Assertions (1)–(4) can, most briefly, be proved by using Theorem 3.2 and the results of Section 2.

Remark 3.8. Note that if p is a function of X to $\mathbb{R}$ satisfying only (b) and (c) in Definition 3.6, then we already have $p\,(x)\geq 0$ for all $x\in X$.

Therefore, by defining

$$q\,(x)=0 \quad \text{for} \quad x=0 \qquad \text{and} \qquad q\,(x)=p\,(x) \quad for \quad x\in X\setminus\{0\}\,,$$

we can obtain a preseminorm q on X.

Now, by using Definition 3.6, we may also naturally introduce

Definition 3.9. A preseminorm p on a group X is called a *seminorm* if

$$n\,p\,(x) \leq p\,(\,n\,x\,)$$

for all $n \in \mathbb{N}$ and $x \in X$.

Thus, by using Theorem 3.7 and the fact that $-n\,x = (-n)\,x$ for all $n \in \mathbb{N}$ and $x \in X$, we can easily prove the following

Theorem 3.10. *If p is a seminorm on X, then for any $k \in \mathbb{Z}$ and $x \in X$ we have*

$$p\,(\,k\,x\,) = |\,k\,|\,p\,(x)\,.$$

Now, by using Definitions 3.6 and 3.9, we may also naturally introduce

Definition 3.11. A seminorm (preseminorm) p on a group X is be called a *norm (prenorm)* if $p\,(x) = 0$ implies $x = 0$ for all $x \in X$.

Thus, by using Theorem 3.7, we can easily prove the following

Theorem 3.12. *If p is a nonzero seminorm (preseminorm) on X and $X = \mathbb{Z}\,x$ for all $x \in X$ with $x \neq 0$, then p is a norm (prenorm) on X.*

Proof. If this is not the case, then there exists $x \in X$, with $x \neq 0$, such that $p\,(x) = 0$. Now, by using Theorem 3.7, we can see that

$$0 \leq p\,(k\,x) \leq \,|\,k\,|\,p\,(x) = 0\,,$$

and thus $p\,(k\,x) = 0$ for all $k \in \mathbb{Z}$. Hence, by using that $X \subseteq \{\,k\,x : \; k \in \mathbb{Z}\,\}$, we can infer that $p\,(u) = 0$ for all $u \in X$. This contradiction proves the theorem.

From this theorem, by using Lagrange's theorem, we can immediately derive

Corollary 3.13. *If p is a nonzero seminorm (preseminorm) on X and the order $n = \operatorname{card}(X)$ of X is a prime, then p is a norm (prenorm) on X.*

Proof. If $x \in X$ such that $x \neq 0$, then it is clear that $Y = \mathbb{Z}\,x$ is a subgroup of X such that $1 < k = \operatorname{card}\,(Y) \leq \operatorname{card}\,(X) = n$. Moreover, by Lagrange's theorem [173, p. 11], we can also state that k divides n. Hence, since n is prime, it follows that $k = n$, and thus $Y = X$ also holds. Therefore, by Theorem 3.12, the required assertion is also true.

The following remark, established with the help of Gábor Horváth, shows that the latter two results can only have a very limited range of applicability.

Remark 3.14. For any group X, the following assertions are equivalent:

(1) $\operatorname{card}\,(X)$ is a prime,

(2) $X = \mathbb{Z}\,x$ for all $x \in X$ with $x \neq 0$,

(3) X has no nontrivial proper subgroup,

(4) for any $x \in X$, with $x \neq 0$, the order

$$n_x = \inf\big\{\,n \in \mathbb{N} : \quad n\,x = 0\,\big\}$$

of x is a prime and

$$X = \big\{\,0,\; x,\; 2\,x, \dots,\; (n_x - 1)\,x\,\big\}$$

such that the above elements are pairwise distinct.

Since the implication $(1) \Longrightarrow (2)$ has been established in the proof of Corollary 3.13, and the implications $(2) \Longleftrightarrow (3)$ and $(4) \Longrightarrow (1)$ are quite obvious, we need only show that (2) implies (4).

For this, we first note that there exists $n \in \mathbb{N}$ such that $n\,x = 0$, and thus $n_x \neq +\infty$. Namely, if $2\,x = 0$, then the required assertion is true. While, if $2\,x \neq 0$, then by (2) we have $X = \mathbb{Z}\,(2\,x)$. Thus, in particular, there exists $k \in \mathbb{Z}$ such that $x = k\,(2\,x)$. Hence, we can infer that

$$(\,1 - 2\,k\,)\,x = 0 \qquad\qquad \text{and} \qquad\qquad (\,2\,k - 1\,)\,x = 0\,.$$

Thus, since either $1 - 2\,k > 0$ or $2\,k - 1 > 0$, the required assertion is again true.

Now, since $\mathbb{N}$ is well-ordered, it is clear

$$n_x = \min\big\{\,n \in \mathbb{N} : \quad n\,x = 0\,\big\}\,.$$

Moreover, since $1\,x = x \neq 0$, it is clear that $n_x > 1$. Furthermore, by (2) and a basic theorem of algebra [173, p. 12], we can state that

$$X = \mathbb{Z}\,x = \big\{\,0,\ x,\,2\,x,\,\ldots,\ (n_x - 1)\,x\,\big\}$$

such that the above elements are pairwise distinct.

Thus, it remains only to show that n_x is prime. For this, note that if $k,\,l \in \mathbb{N}$, with $k \neq 1$ and $l \neq 1$, such that $n_x = k\,l$, then $k < n_x$ and $l < n_x$. Thus, by the definition of n_x, we have $k\,x \neq 0$ and $l\,x \neq 0$. Hence, by (2), we can infer that $X = \mathbb{Z}\,(k\,x)$. Thus, in particular there exists $m \in \mathbb{Z}$ such that $x = m\,(k\,x)$. Hence, we can infer that

$$l\,x = l\,\big(\,m\,(k\,x)\big) = m\,\big((k\,l)\,x\big) = m\,(\,n_x\,x\,) = m\,0 = 0\,.$$

However, this contradicts to our former observation that $l\,x \neq 0$.

4. Semi-inner products on groups

The following generalization of the ordinary semi-inner product [198] was first introduced in our technical report [207] to generalize a basic theorem of Maksa and Volkmann [141]. (See also [22, 23, 24] for some further developments.)

This works well also in a groupoid, and can be easily modified according to the corresponding definitions of Rubin and Stone [177], Lumer [135], Giles [84], Nath [151], Bognár [20], Antoine and Grossmann [13] and Drygas [51].

Definition 4.1. Let X be a group. Then a function P of X^2 to $\mathbb{C}$ is called a *semi-inner product* on X if for any $x,\,y,\,z \in X$ we have

(a) $P\,(x,\,x) \geq 0$,

(b) $P\,(y,\,x) = \overline{P\,(x,\,y)}$,

(c) $P\,(x + y,\,z) = P\,(x,\,z) + P\,(y,\,z)$.

Remark 4.2. The above semi-inner product P is called an *inner product* if

(d) $P(x,\,x) = 0$ implies $x = 0$ for all $x \in X$.

The following example was sugggested by Zoltán Boros, having in mind a basic theorem of Maksa [138]. He, together with Jens Schwaiger, became interested in semi-inner products on groups at the Conference on Ulam's Type Stability, Cluj–Napoca, Romania, 2016.

Example 4.3. If a is an additive function of X to an inner product space H and

$$P(x, y) = \langle a(x), a(y) \rangle$$

for all $x, y \in X$, then P is a semi-inner product on X. Moreover, P is an inner product if and only if a is injective.

Note that, despite this, P may be a rather curious function even if $X = \mathbb{R}^n$ and $H = \mathbb{R}$. Namely, by Kuczma [119, p. 292], there exists a discontinuous, injective additive function of $\mathbb{R}^n$ to $\mathbb{R}$.

Moreover, in the $n = 1$ particular case, this function may also be required to have some further striking properties by Makai [137], Kuczma [119, p. 293] and Baron [17].

The most basic properties of semi-inner products can be listed in the next

Theorem 4.4. *If P is a semi-inner product on X, then for any $x, y, z \in X$ and $k \in \mathbb{Z}$ we have*

(1) $P(x+y, z) = P(y+x, z)$,

(2) $P(x, y+z) = P(x, z+y)$,

(3) $P(x, y+z) = P(x, y) + P(x, z)$,

(4) $P(kx, y) = kP(x, y) = P(x, ky)$.

Hint. Assertion (4) can be immediately derived from (c) and (3), by using that an additive function f of one group X to another Y is $\mathbb{Z}$–homogeneous in the sense that $f(kx) = kf(x)$ for all $k \in \mathbb{Z}$ and $x \in X$.

Remark 4.5. Note that, in particular, (4) yields

$$P(0, y) = 0 = P(x, 0) \qquad \text{and} \qquad P(-x, y) = -P(x, y) = P(x, -y)$$

for all $x, y \in X$.

Remark 4.6. Moreover, note that the real and immaginary parts (first and second coordinate functions) P_1 and P_2 of P, defined by

$$P_1(x, y) = 2^{-1}\Big(P(x, y) + \overline{P(x, y)}\Big) = 2^{-1}\left(P(x, y) + P(y, x)\right)$$

and

$$P_2(x, y) = (i\,2)^{-1}\Big(P(x, y) - \overline{P(x, y)}\Big) = i^{-1}\,2^{-1}\big(P(x, y) - P(y, x)\big)$$

for all $x, y \in X$, also have the same bilinearity properties as P.

Furthermore, by properties (a) and (b), for any $x, y \in X$ we have

(1) $P_1(x, x) = P(x, x)$ and $P_2(x, x) = 0$,

(2) $P_1(y, x) = P_1(x, y)$ and $P_2(y, x) = -P_2(x, y)$.

Thus, in particular P_1 is also a semi-inner product on X. However, because of its skew-symmetry, P_2 cannot be a semi-inner product on X whenever $P_2 \neq 0$.

Remark 4.7. Conversely, one can also easily see that if P_1 is a real-valued semi-inner product on X and P_2 is a nonzero, skew-symmetric, biadditive function of X^2 to $\mathbb{R}$, then $P = (P_1, P_2) = P_1 + i\,P_2$ is a already complex-valued semi-inner product on X.

Because of property (a) in Definition 4.1, we may naturally introduce

Definition 4.8. If P is a semi-inner product on the group X, then for any $x \in X$ we define

$$p(x) = \sqrt{P(x,\, x)}\ .$$

(Whenever property (a) is not supposed to hold, we have to use the diagonalization $\Delta_P(x) = P(x,\, x)$ of P.)

Example 4.9. If in particular P is as in Example 4.3, then

$$p(x) = \sqrt{\langle\, a(x),\, a(x)\rangle} = \|\, a(x)\,\|$$

for all $x \in X$.

Concerning the function p, we can easily prove the following

Theorem 4.10. *If P is a semi-inner product on X, then for any $x, y \in X$ and $k \in \mathbb{Z}$, we have*

(1) $p(x) \geq 0$,

(2) $p(k\,x) = |\,k\,|\,p(x)$,

(3) $p(x+y) = p(y+x)$,

(4) $p\big(k\,(x+y)\big) = p(k\,x + k\,y)$,

(5) $p(x+y)^2 = P_1(x+y,\, x) + P_1(x+y,\, y)$,

(6) $p(x+y)^2 = p(x)^2 + p(y)^2 + 2\,P_1(x,\, y)$.

Hint. To prove (5) and (6), note that by the Definition 4.8 and Remark 4.6 we have

$$p(x) = \sqrt{P_1(x,\, x)}$$

and

$$p(x+y)^2 = P_1(x+y,\ x+y) = P_1(x+y,\, x) + P_1(x+y,\, y)$$
$$= P_1(x,\, x) + P_1(y,\, x) + P_1(x,\, y) + P_1(y,\, y) = p(x)^2 + 2\,P_1(x,\, y) + p(y)^2.$$

Hence, by the symmetry of P_1 and the commutativity of the addition in $\mathbb{R}$, it is clear that (3) is also true.

Remark 4.11. Note that, in particular, (2) yields

$$p(0) = 0 \qquad \text{and} \qquad p(-x) = p(x)$$

for all $x \in X$.

Remark 4.12. Moreover, to feel the importance of (4), note that for any $x, y \in X$, we have $2\,(x+y) = 2\,x + 2\,y$ if and only if $y + x = x + y$.

Therefore, if x and y do not commute, then $2\,(x+y) \neq 2\,x + 2\,y$. However, by (4), we still have $p\big(2\,(x+y)\big) = p(2\,x + 2\,y)$.

Remark 4.13. In addition to Theorem 4.10, we can also note that P is an inner product on X if and only if $p(x)=0$ implies $x=0$ for all $x \in X$.

Now, by using Theorems 4.10 and 4.4, we can also easily establish the following

Theorem 4.14. *If P is a semi-inner product on X, then for any $x, y \in X$ we have*

(1) $p(x-y)^2 = p(x+y)^2 - 4\,P_1(x, y)$,

(2) $p(x-y)^2 = 2\,p(x)^2 + 2\,p(y)^2 - p(x+y)^2$.

Moreover, as an immediate consequence of Theorems 4.14 and 4.10, we can also state

Theorem 4.15. *If P is a semi-inner product on X, then for any $x, y \in X$ we have*

(1) $P_1(x, y) = 4^{-1}\big(p(x+y)^2 - p(x-y)^2\big)$,

(2) $P_1(x, y) = 2^{-1}\big(p(x+y)^2 - p(x)^2 - p(y)^2\big)$.

Remark 4.16. Unfortunately, now similar polar formulas for $P_2(x, y)$ cannot be proved. Therefore, P can be recovered from p only in the real-valued case.

5. Further properties of the induced generalized seminorm

In [207], the author claimed that, in a group, even a weaker form of Schwarz inequality cannot be proved. Therefore, he asked several mathematicians in Debrecen and Cluj-Napoca to construct an example.

However, in contrast to this request, Zoltán Boros has proved that a real-valued Schwarz inequality is still true [21]. Actually, it is somewhat more than is sufficient to prove the desired subadditivity of the function p.

At the Conference, a quite similar argument has been used in the talk [77] of Roman Ger without mentioning [22] and an improved and enlarged version of [207], which had been sent to him before the conference.

In the talk, with reference to [2], he noticed that if f is a function of $\mathbb{R}$ to $\mathbb{R}$ such that f^2 is quadratic, then there exist a unique symmetric, biadditive function A of $\mathbb{R}^2$ to $\mathbb{R}$ such that $f(x)^2 = A(x, x)$ for all $x \in X$. Moreover, he proved, in a direct way, that if f is nonnegative, then f is subadditive.

Furthermore, with reference to [193], he noted that here the domain of f may be an arbitrary abelian group. Later, in a remark to the talk [21] of Z. Boros, he remarked that this abelian group may be replaced by a $\mathcal{G}$–group introduced by Roman Badora in [Arch. Math. (Basel) 86 (2006), 517–528].

Mentime, we have observed that a generalization of Schwarz inequality on groups was already proved by Kurepa [122, 127]. Moreover, we have observed that one half of the following Schwarz inequality can also be proved on groupoids [23, 24]. And Schwarz inequality has actually to be replaced by an equality [208].

Theorem 5.1. *If P is a semi-inner product on X, then any $x, y \in X$, we have*

$$|\,P_1(x, y)\,| \le p(x)\,p(y)\,.$$

Proof. By using Theorems 4.10 and 4.4, for any $n, m \in \mathbb{N}$, we get

$$p\,(n\,x + m\,y\,)^2 = n^2\,p\,(x)^2 + m^2\,p\,(y)^2 + 2\,n\,m\,P_1(x,\,y)\,,$$

and thus

$$-2\,P_1(x,\,y) \,\leq\, (n/m)\,p\,(x)^2 + (m/n)\,p\,(y)^2\,.$$

Therefore, we actually have

$$-2\,P_1(x,\,y) \leq r\,p\,(x)^2 + r^{-1}\,p\,(y)^2$$

for all $r \in \mathbb{Q}$ with $r > 0$, and thus also for all $r \in \mathbb{R}$ with $r > 0$, by the sequential density of $\mathbb{Q}$ in $\mathbb{R}$ and the sequential continuity of the operations in $\mathbb{R}$.

Now, by defining

$$f\,(r) = r\,p\,(x)^2 + r^{-1}\,p\,(y)^2$$

for all $r > 0$, we can state that

$$-2\,P_1(x,\,y) \leq \inf_{r>0}\, f\,(r)\,.$$

Hence, by showing that

$$\inf_{r>0}\, f\,(r) = 2\,p\,(x)\,p\,(y) \quad \text{if} \quad p\,(x) \neq 0 \qquad \text{and} \qquad \inf_{r>0}\, f\,(r) = 0 \quad \text{if} \quad p\,(x) = 0\,,$$

we can infer that $-P_1(x,\,y) \leq p\,(x)\,p\,(y)$, and thus

$$P_1(x,\,y) = -P_1(-x,\,y) \leq p\,(-x)\,p\,(y) = p\,(x)\,p\,(y)\,.$$

Therefore, by the definition of the absolute value, the required inequality is also true.

(A very tricky, but instructive proof of Schwarz's inequality in vector spaces was given by von Neumann in [152]. However, suprisingly enough, this proof has not been mentioned in the subsequent works. The separability axiom C of von Neumann was considered to be superfluous by F. Riesz and H. Löwig in 1930 and 1934, respectively.)

Now, by using Theorems 5.1 and 4.10, we can also prove the following

Theorem 5.2. *If P is a semi-inner product on X, then any $x,\,y \in X$, we have*

(1) $p\,(x+y) \leq p\,(x) + p\,(y)\,,$ *(2)* $|\,p\,(x) - p\,(y)\,| \leq p\,(x-y)\,.$

Proof. By Theorems 4.10 and 5.1, it is clear that

$$p\,(x+y)^2 = P_1(x+y,\,x) + P_1(x+y,\,y) \leq p\,(x+y)\,p\,(x) + p\,(x+y)\,p\,(y)\,.$$

Therefore, by the nonnegativity of p, inequality (1) is also true. Now, inequality (2) can be derived from (1) on the usual way.

Remark 5.3. Theorems 4.10 and 5.2 already show that the function p is actually a seminorm on X.

Moreover, from Remark 4.12, we can see that p is a norm on X if and only if P is an inner product on X.

Assertions (5) and (6) of Theorem 4.10 can be naturally extended to all finite and certain infinite families of elements of X. However, in the sequel, we shall only need the following particular case of a result of [22].

Theorem 5.4. *If P is a semi-inner product on X, then for any $x, y, z \in X$ we have*

(1) $p(x+y+z)^2 = P_1(x+y+z, x) + P_1(x+y+z, y) + P_1(x+y+z, z)$,

(2) $p(x+y+z)^2 = p(x)^2 + p(y)^2 + p(z)^2 + 2P_1(x, y) + 2P_1(x, z) + 2P_1(y, z)$,

(3) $p(x+y+z)^2 = p(x+y)^2 + p(x+z)^2 + p(y+z)^2 - p(x)^2 - p(y)^2 - p(z)^2$.

Proof. To prove (3), note that by Theorem 4.10 we have

$$\begin{aligned}
p(x+y+z)^2 &= p(x+y)^2 + p(z)^2 + 2P_1(x+y, z) \\
&= p(x)^2 + p(y)^2 + 2P_1(x, y) + p(z)^2 + 2P_1(x, z) + 2P_1(y, z) \\
&= p(x)^2 + p(y)^2 + p(z)^2 + 2P_1(x, y) + 2P_1(x, z) + 2P_1(y, z) \\
&= p(x)^2 + p(y)^2 + p(z)^2 + p(x+y)^2 - p(x)^2 - p(y)^2 \\
&\quad + p(x+z)^2 - p(x)^2 - p(z)^2 + p(y+z)^2 - p(y)^2 - p(z)^2 \\
&= p(x+y)^2 + p(x+z)^2 + p(y+z)^2 - p(x)^2 - p(y)^2 - p(z)^2 .
\end{aligned}$$

Remark 5.5. The above parallelepiped law (3) played a similar role in characterization of inner product spaces as the parallelogram identity established in assertion (2) of Theorem 4.14.

Their importance in this context was first recognized by Fréchet [71] and Jordan and von Neumann [101].

Remark 5.6. If f is a function of one group X to another Y, then for any $x, y, z \in X$, it is customary to define the first and second order Cauchy differences

$$\big(C^1 f\big)(x, y) = f(x+y) - f(x) - f(y)$$

and

$$\big(C^2 f\big)(x, y, z) = \big(C^1 f\big)(x+y, z) - \big(C^1 f\big)(x, z) - \big(C^1 f\big)(y, z) .$$

Hence, it can be easily seen that

$$\begin{aligned}
\big(C^2 f\big)(x, y, z) &= f(x+y+z) - f(x+y) - f(z) \\
&\quad - \big(f(x+z) - f(x) - f(z)\big) - \big(f(y+z) - f(y) - f(z)\big) \\
&= f(x+y+z) - f(x+y) - f(z) + f(z) + f(x) - f(x+z) + f(z) + f(y) - f(y+z) \\
&= f(x+y+z) - f(x+y) + f(x) - f(x+z) + f(z) + f(y) - f(y+z) .
\end{aligned}$$

Thus, in particular the parallelepiped law (3) can be briefly reformulated by stating that $\big(C^2 p^2\big)(x, y, z) = 0$ for all $x, y, z \in X$.

Problem 5.7. *In addition to Theorem 5.1 and* [23, Example 5.6], *it would be of some interest to construct a groupoid (or rather a commutative monoid) X and a semi-inner product P on X such that the inequality $P_1(x, y) \leq p(x)\,p(y)$ could fail for some $x, y \in X$.*

Namely, by [23, Theorem 5.2], the inequality $-P_1(x, y) \leq p(x)\,p(y)$ is always true.

6. Two basic properties of the parallelepiped equation

The equivalence of the parallelogram identity and the parallelepiped law for a norm on a vector space has been proved by Marinescu, Monea, Opincariu and Strore in [145] .

Now, by using similar arguments, we shall show that under some natural assumptions the *parallelepiped equation*

(1) $$f(x+y+z) = f(x+y) + f(x+z) + f(y+z) - f(x) - f(y) - f(z),$$

is equivalent either to the *quadratic equation*

(2) $$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

or to the *Cauchy equation*

(3) $$f(x+y) = f(x) + f(y).$$

Theorem 6.1. *If f is an even function of an arbitrary group X to a commutative one Y such that equation (1) holds for all $x, y, z \in X$, then equation (2) also holds for all $x, y \in X$.*

Proof. By taking $x = y = z = 0$ in equation (1), we can see that

$$f(0) = 3f(0) + 3(-f(0)) = 3f(0) - 3f(0) = 0.$$

Now, by writing $-y$ in place of z in equation (1), and using $f(0) = 0$ and $f(-y) = f(y)$, we can also see that

$$\begin{aligned} f(x) &= f(x+y) + f(x-y) + f(0) - f(x) - f(y) - f(-y) \\ &= f(x+y) + f(x-y) - f(x) - 2f(y), \end{aligned}$$

and thus

$$f(x) + 2f(y) + f(x) = f(x+y) + f(x-y).$$

for all $x, y \in X$. Hence, by the commutativity of Y, it is clear that equation (2) also holds for all $x, y \in X$.

Remark 6.2. The substitution $z = -y$ is attributed to Jordan and von Neumann [101] by Istrătescu [97, p. 110] and others.

However, they seemed to stress only that Fréchet's criterium on at most three-dimensional subspaces can be reduced to that on at most two-dimensional ones.

In the following counterpart of Theorem 6.1, we shall need the famous *Kannappann condition* [193, p. 315] that

(4) $$f(x+z+y) = f(x+y+z)$$

for all $x, y, z \in X$.

Theorem 6.3. *If f is an odd, 2–homogeneous function of an arbitrary group X to a commutative, 2–cancellable one Y such that equations (1) and (4) hold for all $x, y, z \in X$, then equation (3) also holds for any $x, y \in X$.*

Proof. By the first part of the proof of Theorem 6.1, it is clear that we now have

$$2f(x) = f(x+y) + f(x-y)$$

for all $x, y \in X$.

Hence, by writing $x+y$ in place of x and $x-y$ in place of y, and using equation (4) and the 2–homogeneity of f, we can see that

$$\begin{aligned} 2f(x+y) &= f(x+y+x-y)+f\big(x+y-(x-y)\big) \\ &= f(x+y+x-y)+f(x+y+y-x) = f(x-y+y+x)+f(x-x+y+y) \\ &= f(2x)+f(2y) = 2f(x)+2f(y) \end{aligned}$$

for all $x, y \in X$. Therefore, the function $2f$ is additive.

Now, since Y is commutative and Y is 2–cancellable, it is clear that

$$2f(x+y) = 2\big(f(x)+f(y)\big),$$

and thus equation (3) also holds for all $x, y \in X$.

Remark 6.4. Conversely, we can easily note that if f is a function of an arbitrary group X to a commutative one Y such that equation (3) holds for all $x, y \in X$, then equation (1) also holds for all $x, y, z \in X$.

However, to prove a certain converse to Theorem 6.1, we shall need a rather complicated calculation.

Theorem 6.5. *If f is an even function of an arbitrary group X to a 2–cancellable, commutative one Y such that equations (2) and (4) hold for any $x, y, z \in X$, then equation (1) also holds for all $x, y, z \in X$.*

Proof. Now, in addition to

$$2f(x)+2f(y) = f(x+y)+f(x-y),$$

we also have

$$\begin{aligned} 2f(x+y+z)+2f(z) &= 2f(x+y+z)+2f(-z) \\ &= f(x+y+z-z)+f(x+y+z+z) = f(x+y)+f(x+y+2z) \end{aligned}$$

for all $x, y, z \in X$.

Moreover, we also have

$$\begin{aligned} 2f(x+z)+2f(y+z) &= f(x+z+y+z)+f\big(x+z-(y+z)\big) \\ &= f(x+y+z+z)+f(x+z-z-y) = f(x+y+2z)+f(x-y), \end{aligned}$$

and thus

$$f(x+y+2z)+f(x-y) = 2f(x+z)+2f(y+z)$$

for all $x, y, z \in X$.

Now, by adding the corresponding three equalities, we can state that

$$\begin{aligned} &2f(x)+2f(y)+2f(x+y+z)+2f(z)+f(x+y+2z)+f(x-y) \\ &= f(x+y)+f(x-y)+f(x+y)+f(x+y+2z)+2f(x+z)+2f(y+z), \end{aligned}$$

and thus

$$2\big(f(x)+f(y)+f(x+y+z)+f(z)\big) = 2\big(f(x+y)+f(x+z)+f(y+z)\big)$$

for all $x, y, z \in X$.

Hence, we can already infer that

$$f(x)+f(y)+f(x+y+z)+f(z) = f(x+y)+f(x+z)+f(y+z),$$

and thus equation (1) also holds for all $x, y, z \in X$.

Remark 6.6. Note that in the above theorems f may for instance be a suitable function of X to $\mathbb{C}$, or more specially p^2 for a suitable function p of X to $\mathbb{R}$.

However, it is now more important to note that if p is a function of a group X to $\mathbb{R}$ such that for any $x, y, z \in Y$ we have

(1) $p(x+y)^2 = p(y+x)^2$,

(2) $p(x+y+z)^2 = p(x+y)^2 + p(x+z)^2 + p(y+z)^2 - p(x)^2 - p(y)^2 - p(z)^2$,

then by [193, Proposition 13.25] of Stetkaer there exist a unique additive function a of X to $\mathbb{R}$ and a unique symmetric, biadditive function A of X^2 to $\mathbb{R}$ such that

$$p(x)^2 = a(x) + A(x, x)$$

for all $x \in X$.

Hence, if in addition p is even, we can infer that

$$a(x) + A(x, x) = p(x)^2 = p(-x)^2 = a(-x) + A(-x, -x) = -a(x) + A(x, x),$$

and thus $a(x) = 0$ for all $x \in X$. Therefore, we have $p(x)^2 = A(x, x)$ for all $x \in X$. Hence, if in addition p is nonnegative, we can infer that $p(x) = \sqrt{A(x, x)}$ for all $x \in X$.

7. Parapreseminorms should also be investigated

The following definition was first introduced by the present author, in an improved and enlarged version of [207], to prove a natural generalization of a basic theorem of Maksa and Volkmann [141] on additive functions.

Definition 7.1. Let X be a group. Then, a function p of X to $\mathbb{R}$ is called a *parapreseminorm* on X if for any $x, y, z \in X$ we have

(a) $0 \leq p(x)$,

(b) $p(-x) \leq p(x)$,

(c) $p(y+x) \leq p(x+y)$,

(d) $p(x+y+z)^2 \leq p(x+y)^2 + p(x+z)^2 + p(y+z)^2 - p(x)^2 - p(y)^2 - p(z)^2$.

Remark 7.2. The above parapreseminorm p is called a *paraseminorm* if

(e) $p(nx) = n\,p(x)$ for all $n \in \mathbb{N}$ and $x \in X$.

Moreover, a paraseminorm (parapreseminorm) p on X is called a *paranorm (paraprenorm)* if

(f) $p(x) = 0$ implies $x = 0$ for all $x \in X$.

This definition differs from that of Wilansky [225, p. 15]. However, it is in accordance with the definitions introduced in Section 3.

By our former results on seminorms derived from semi-inner products, we can at once state

Theorem 7.3. *If P is a semi-inner product on X and*

$$p(x) = \sqrt{P(x, x)}$$

for all $x \in X$, *then* p *is a paraseminorm on* X.

Remark 7.4. Moreover, we can also state that p is paranorm on X if and only if P is an inner product on X.

Now, by using Definition 7.1, we can also easily prove the following

Theorem 7.5. *If* p *is a parapreseminorm on* X, *then for any* $x, y \in X$ *we have*

(1) $p(0) = 0$,

(2) $p(x) = p(-x)$,

(3) $2\,p(x) \leq p(2\,x)$,

(4) $p(x+y) = p(y+x)$,

(5) $2\,p(x)^2 + 2\,p^2(y) \leq p(x+y)^2 + p(x-y)^2$.

Proof. Assertions (2) and (4) can be immediately derived from properties (b) and (c), respectively, by writing $-x$ in place of x in (b) and changing the roles x and y in (c), respectively.

Moreover, by taking $z = 0$ in (d), we can easily see that (1) also hold. Now, from (d), by taking $z = -y$ and using (1) and (2), we can also easily see that (5) is also true. Moreover, from (5), by taking $x = y$ and using (1) and (a), we can easily see that (3) is also true.

Remark 7.6. If in particular p is paraseminorm on X, then by using the corresponding definitions and assertion (2) of Theorem 7.5 we can also easily see that

(6) $p(k\,x) = |\,k\,|\,p(x)$ for all $k \in \mathbb{Z}$ and $x \in X$.

The following example shows that even some very simple norms need not be paranorms.

Example 7.7. For any $x = (x_1, x_2) \in \mathbb{R}^2$, define

$$p_1(x) = |\,x_1\,| + |\,x_2\,|, \qquad p_2(x) = \sqrt{x_1^2 + x_2^2} \qquad \text{and} \qquad p_\infty(x) = \max\big\{\,|\,x_1\,|,\ |\,x_2\,|\,\big\}\,.$$

Then, p_2 is a paranorm, but p_1 and p_∞ are not paranorms on $\mathbb{R}^2$.

Since p_2 can be derived from an inner product on $\mathbb{R}^2$, by Theorem 7.3 and Remark 7.4, it is clear that p_2 is a paranorm on $\mathbb{R}^2$.

Moreover, by taking

$$a = (1,\,0)\,, \qquad b = (0,\,1)\,, \qquad c = (1,\,1)\,, \qquad d = (-1,\,1)\,,$$

we can see that

$$2\,p_1(c)^2 + 2\,p_1(d)^2 = 2^4, \qquad \text{but} \qquad p_1(c+d)^2 + p_1(c-d)^2 = 2^3,$$

and

$$2\,p_\infty(a)^2 + 2\,p_\infty(b)^2 = 2^2, \qquad \text{but} \qquad p_\infty(a+b)^2 + p_\infty(a-d)^2 = 2\,.$$

Therefore, by the assertion (5) of Theorem 7.5, p_1 and p_∞ cannot be paranorms on $\mathbb{R}^2$.

Problem 7.8. *At this point, we should also give some natural paranorms which are not norms.*

Moreover, in addition to Theorem 7.5, we should find some further interesting properties parapreseminorms.

And, we should give some reasonable conditions in order that a preseminorm (parapreseminorm) could be a parapreseminorm (preseminorm).

8. Sub-quadratic and super-quadratic functions are usually quadratic

The following theorem has been proved at the Conference, with the help of Gyula Maksa and Zoltán Boros, to correct a quick answer of Zsolt Páles to a question of the present author concerning the talk of Roman Ger.

Theorem 8.1. *If f is a super-quadratic function of a group X to $\mathbb{R}$ such that*

(1) $f(2x) \le 4f(x)$, *(2)* $f(x+y+z) \le f(x+z+y)$,

for all $x, y, z \in X$, then f is quadratic.

Proof. Since f is super-quadratic, for any $x, y \in X$, we have

$$2f(x) + 2f(y) \le f(x+y) + f(x-y).$$

Hence, it is clear that, for any $u, v \in X$, we have

$$2f(u+v) + 2f(u-v) \le f(u+v+u-v) + f\big(u+v-(u-v)\big).$$

Moreover, by using assumptions (2) and (1), we can see that

$$\begin{aligned} f(u+v+u-v) &= f\big(u+(v+u)-v\big) \\ &\le f\big(u-v+(v+u)\big) = f(2u) \le 4f(u) \end{aligned}$$

and

$$\begin{aligned} f\big(u+v-(u-v)\big) &= f(u+v+v-u) \\ &= f(u+2v-u) = f(u-u+2v) = f(2v) \le 4f(v). \end{aligned}$$

Therefore,

$$2f(u+v) + 2f(u-v) \le 4f(u) + 4f(v),$$

and thus

$$f(u+v) + f(u-v) \le 2f(u) + 2f(v)$$

also holds. This shows that we actually have

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

for all $x, y \in X$, and thus f is quadratic.

Remark 8.2. Note that if f is a subquadratic function of a group X to $\mathbb{R}$, then for any $x, y \in X$ we have

$$f(x+y) + f(x-y) \le 2f(x) + 2f(y).$$

Hence, by putting $y = 0$ we can infer that $0 \le f(0)$. Moreover, by putting $x = y$ and using $0 \le f(0)$, we can infer that $f(2x) \le 4f(x)$.

Therefore, assumption (1) of the theorem follows from the assertion of the theorem. To see its necessity, Zoltán Boros noticed that if $c < 0$, X is a group and

$f(x)=c$ for all $x \in X$, then f is a super-quadratic, but not quadratic, function of X to $\mathbb{R}$ such that the Kannappann inequality (2) trivially holds for f.

The necessity of condition (2) could certainly be demonstrated only with the help of a much more difficult example. In this respect, it is noteworthy that, by Stetkaer [193, Example 13.20 and Lemma B.4] there exists an integer-valued quadratic function f of a free group X, with three generators, such that the Kannappann condition does not hold for f.

From Theorem 8.1, by noticing that a function f is subquadratic if $-f$ is super-quadratic, we can immediately derive the following

Corollary 8.3. *If f is a subquadratic function of a group X to $\mathbb{R}$ such that*

(1) $4\,f(x) \leq f(2\,x)$, *(2)* $f(x+y+z) \leq f(x+z+y)$,

for all $x, y, z \in X$, then f is quadratic.

However, it is now more important to note that, as an immediate consequence of Theorem 7.5 and 8.1, we can also state the following

Theorem 8.4. *If p is a parapreseminorm on a group X such that*

(1) $p(2\,x) \leq 2\,p(x)$, *(2)* $p(x+y+z) \leq p(x+z+y)$,

for all $x, y, z \in X$, then the function p^2 is quadratic.

Proof. From Theorem 7.5 and assumptions (1) and (2), we can see that

$$2\,p^2(x)+2\,p^2(y) \leq p^2(x+y)+p^2(x-y)$$

and

$$p^2(2\,x) \leq 4\,p^2(x) \qquad \text{and} \qquad p^2(x+y+z) \leq p^2(x+z+y)$$

for all $x, y, z \in X$. Therefore, by Theorem 8.1, the required assertion is also true.

Hence, by using an improvement of a part of an argument of Ger [77], we can easily derive the following

Corollary 8.5. *If p is as in Theorem 8.4, then for any $x, y \in X$, the following assertions are equivalent:*

(1) $p(x+y) \leq p(x)+p(y)$, *(2)* $p(x+y)^2 - p(x-y)^2 \leq 4\,p(x)\,p(y)$.

Proof. By the nonnegativity of p and Theorem 8.4, it is clear that the following inequalities are equivalent:

$$p(x+y) \leq p(x)+p(y),$$

$$p(x+y)^2 \leq \big(p(x)+p(y)\big)^2,$$

$$p(x+y)^2 \leq p(x)^2+p(y)^2+2\,p(x)\,p(y),$$

$$2\,p(x+y)^2 \leq 2\,p(x)^2+2\,p(y)^2+4\,p(x)\,p(y),$$

$$2\,p(x+y)^2 \leq p(x+y)^2+p(x-y)^2+4\,p(x)\,p(y),$$

$$p(x+y)^2 - p(x-y)^2 \leq 4\,p(x)\,p(y).$$

Remark 8.6. At the Conference, in connection with the talk of Roman Ger, I have suggested to consider first functional inequalities instead of the corresponding equalities, having in mind the various papers on functional inequalities.

However, I could know only from a recent paper of Fechner [62] that even subquadratic and superquadratic functions have already been intensively studied by several mathematicians. (See, for instance, [115, 216, 116, 82, 83] .)

For instance, Kominek and Troczka [115] proved that if φ is a subquadratic function of a real linear space X such that $\varphi(kx) = k^2\varphi(x)$ for some integer $k > 1$ and all $x \in X$, then φ is a quadratic function.

Remark 8.7. At a future conference, Wlodzimierz Fechner should be asked to hold a survey talk or a special session on the most important functional inequalities.

9. A generalization of the quadrilateral inequality

To extendend a result of Edmund Hlawka, presented first by Hornich [93, p. 274] , for a function p of a group X to $\mathbb{R}$ we shall look for some sufficient conditions in order that, for some $x, y, z \in X$, the *parallelepiped inequality*

$$(5) \qquad p(x+y)^2 + p(x+z)^2 + p(y+z)^2 - p(x)^2 - p(y)^2 - p(z)^2 \leq p(x+y+z)^2,$$

could imply the *quadrilateral inequality*

$$(6) \qquad p(x+y) + p(x+z) + p(y+z) - p(x) - p(y) - p(z) \leq p(x+y+z).$$

For this, by following an argument of Smiley and Smiley [191] , rather than that of Hlawka [93] and Djoković [42] , we shall prove the following

Theorem 9.1. *If p is a preseminorm on a group X, then for any $x, y, z \in X$ satisfying the Kannappann inequality*

$$(7) \qquad p(x+z+y) \leq p(x+y+z),$$

the parallelepiped inequality (5) implies the quadrilateral inequality (6).

Proof. By using the notations

$$S_1 = p(x)+p(y)+p(z), \qquad S_2 = p(x+y)+p(x+z)+p(y+z), \qquad S_3 = p(x+y+z),$$

we have to prove that

$$S_2 - S_1 \leq S_3, \qquad \text{or equivalently} \qquad S_1 - S_3 \leq 2S_1 - S_2.$$

For this, we can note that

$$S_1^2 - S_3^2 = p(x)^2 + p(y)^2 + p(z)^2 + 2p(x)p(y) + 2p(x)p(z) + 2p(y)p(z) - p(x+y+z)^2.$$

Moreover, if the parallelepiped inequality (5) holds, then we have

$$-p(x+y+z)^2 \leq p(x)^2 + p(y)^2 + p(z)^2 - p(x+y)^2 - p(x+z)^2 - p(y+z)^2.$$

Therefore,

$$S_1^2 - S_3^2 \leq 2p(x)^2 + 2p(y)^2 + 2p(z)^2$$
$$+ 2p(x)p(y) + 2p(x)p(z) + 2p(y)p(z) - p(x+y)^2 - p(x+z)^2 - p(y+z)^2.$$

On the other hand, by defining

$$C(x, y) = p(x) + p(y) - p(x+y) \qquad \text{and} \qquad D(x, y) = p(x) + p(y) + p(x+y),$$

we can see that

$$C(x, y)\, D(x, y) = \big(p(x) + p(y) \big)^2 - p(x+y)^2$$
$$= p(x)^2 + p(y)^2 + 2\, p(x)\, p(y) - p(x+y)^2 .$$

Therefore,

$$C(x, y)\, D(x, y) + C(x, z)\, D(x, z) + C(y, z)\, D(y, z)$$
$$= p(x)^2 + p(y)^2 + 2\, p(x)\, p(y) - p(x+y)^2$$
$$+ p(x)^2 + p(z)^2 + 2\, p(x)\, p(z) - p(x+z)^2$$
$$+ p(y)^2 + p(z)^2 + 2\, p(y)\, p(x) - p(y+z)^2$$
$$= 2\, p(x)^2 + 2\, p(y)^2 + 2\, p(z)^2$$
$$+ 2\, p(x)\, p(y) + 2\, p(x)\, p(z) + 2\, p(y)\, p(z) - p(x+y)^2 - p(x+z)^2 - p(y+z)^2$$

Hence, we can see that

$$S_1^2 - S_3^2 \leq C(x, y)\, D(x, y) + C(x, z)\, D(x, z) + C(y, z)\, D(y, z) .$$

Moreover, by noticing that

$$D(x, y) = p(x) + p(y) + p(x+y) = S_1 + p(x+y) - p(z) \leq S_1 + p(x+y+z) = S_1 + S_3 ,$$

$$D(y, z) = p(y) + p(z) + p(y+z) = S_1 - p(x) + p(y+z) \leq S_1 + p(x+y+z) = S_1 + S_3 ,$$

$$D(x, z) = p(x) + p(z) + p(x+z) = S_1 + p(x+z) - p(y) \leq S_1 + p(x+z+y)$$
$$\leq S_1 + p(x+y+z) = S_1 + S_3 ,$$

and $C(x, y)$, $C(x, z)$ and $C(y, z)$ are nonnegative, we can see that

$$(S_1 - S_3)(S_1 + S_3) = S_1^2 - S_3^2 \leq C(x, y)(S_1 + S_3) + C(x, z)(S_1 + S_3) + C(y, z)(S_1 + S_3)$$
$$= \big(C(x, y) + C(x, z) + C(y, z) \big) = (2\, S_1 - S_2)(S_1 + S_3) .$$

Hence, since $S_1 + S_3 \geq 0$, and moreover $S_1 + S_3 = 0$ implies $S_1 = 0$ and $S_3 = 0$, we can already infer that

$$S_1 - S_3 \leq 2\, S_1 - S_2 , \qquad \text{and thus} \qquad S_2 - S_1 \leq S_3 .$$

Therefore, the quadrilateral inequality (6) also holds.

From this theorem, by using Theorems 4.10, 5.1 and 5.3, we can immediately derive

Corollary 9.2. *If P is a semi-inner product on a group X and*

$$p(x) = \sqrt{P(x, x)}$$

for all $x \in X$, then the quadrilateral inequality (6) holds for all $x, y, z \in X$.

Remark 9.3. If p is only a preseminorm on a group X satisfying the Kannappann inequality (7), then by using Theorem 3.7 we can only state that

$$p(x+y) - p(z) \leq p(x+y+z) ,$$
$$-p(x) + p(y+z) \leq p(x+y+z) ,$$
$$p(x+z) - p(y) \leq p(x+z+y) \leq p(x+y+z) ,$$

and thus

$$p(x+y)+p(x+z)+p(y+z)-p(x)-p(y)-p(z) \leq 3\,p(x+y+z).$$

Remark 9.4. In [191], Smiley and Smiley noticed that if p is a function of a commutative group X to $\mathbb{R}$ such that the quadrilateral inequality (2) holds for all $x,\, y,\, z \in X$, then by putting $(2x,\ -x+y,\ -x+y)$ in place of $(x,\, y,\, z)$ in (6) we get

$$2\,p(x+y)+p\big(2(-x+y)\big)-p(2x)-2\,p(-x+y) \leq p(2y)$$

for all $x,\, y \in X$. Hence, if in addition p is 2-homogeneous, we can infer that

$$p(x+y) \leq p(x)+p(y)$$

for all $x,\, y \in X$. That is, p is subbadditive.

Remark 9.5. Kelly, Smiley and Smiley [109] and Sudbery [194] proved that if p is a norm on a vector space X, then for any three linearly dependent elements $x,\, y$ and z of X the quadrilateral inequality (6) holds.

Therefore, to provide a counterexample to the quadrilateral inequality (6), we have to start with at least a three dimensional normed spaces whose norm cannot be derived from an inner product.

The following example of Fechner [61] shows that the supremum norm p on $\mathbb{R}^n$, with $n>2$, fails to satisfy the quadrilateral inequality (6) for all $x,\, y,\, z \in \mathbb{R}^n$.

Example 9.6. For any $x=(x_1,\, x_2,\, x_3) \in \mathbb{R}^3$, define

$$p(x)=\max\big\{\,|x_1|,\ |x_2|,\ |x_3|\,\big\}.$$

Then, p is a norm on $\mathbb{R}^3$.

Moreover, by taking

$$x=(1,\,1,-1), \qquad x=(1,\,-1,1), \qquad x=(-1,\,1,\,1),$$

we can see that

$$x+y=(2,\,0,\,0), \quad x+z=(0,\,2,\,0), \quad y+z=(0,\,0,\,2), \quad x+y+z=(1,\,1,\,1),$$

and

$$p(x)+p(y)+p(z)=3, \qquad p(x+y)+p(x+z)+(y+z)=6, \qquad p(x+y+z)=1.$$

Therefore,

$$p(x+y)+p(x+z)+p(y+z)-p_1(x)-p(y)-p(z)=3 \not\leq 1=p(x+y+z).$$

Remark 9.7. Finally, we note that if p is a function of a group X to $\mathbb{R}$ and $x,\, y,\, z \in X$ such that the parallelepiped inequality (5) holds then by using our former notations and the identity

$$a^2+b^2+c^2=(a+b+c)^2-2ab-2ac-2bc,$$

we can see that

$$\begin{aligned} S_3^2 \geq S_2^2 &- 2\,p(x+y)\,p(x+z)-2\,p(x+y)\,p(y+z)-2\,p(x+z)\,p(y+z) \\ &- S_1^2+2\,p(x)\,p(y)+2\,p(x)\,p(z)+2\,p(y)\,p(z). \end{aligned}$$

Moreover, if in particular p is a preseminorm on X, then we can also see that

$$\begin{aligned} -2\,p(x+y)\,p(x+z) &\geq -2\big(p(x)+p(y)\big)\big(p(x)+p(z)\big) \\ &= -2\,p(x)^2 - 2\,p(x)\,p(y) - 2\,p(x)\,p(z) - 2\,p(y)\,p(z)\,, \end{aligned}$$

$$\begin{aligned} -2\,p(x+y)\,p(y+z) &\geq -2\big(p(x)+p(y)\big)\big(p(y)+p(z)\big) \\ &= -2\,p(y)^2 - 2\,p(x)\,p(y) - 2\,p(x)\,p(z) - 2\,p(y)\,p(z)\,, \end{aligned}$$

$$\begin{aligned} -2\,p(x+z)\,p(y+z) &\geq -2\big(p(x)+p(z)\big)\big(p(y)+p(z)\big) \\ &= -2\,p(z)^2 - 2\,p(x)\,p(y) - 2\,p(x)\,p(z) - 2\,p(y)\,p(z)\,. \end{aligned}$$

Therefore,

$$\begin{aligned} S_3^2 \geq\ & S_2^2 - S_1^2 - 2\,p(x)^2 - 2\,p(y)^2 - 2\,p(z)^2 \\ & - 4\,p(x)\,p(y) - 4\,p(x)\,p(z) - 4\,p(y)\,p(z) \\ & = S_2^2 - S_1^2 - 2\,S_1^2 = S_2^2 - 3\,S_1^2\,, \end{aligned}$$

and thus

$$S_3^2 - S_1^2 \geq S_2^2 - (\,2\,S_1)^2\,.$$

Hence, we can infer that

$$(\,S_1 - S_3)\,(\,S_1 + S_3) = S_1^2 - S_3^2 \leq (\,2\,S_1)^2 - S_2^2 = (\,2\,S_1 - S_2)\,(\,2\,S_1 + S_2)\,.$$

However, this rough estimation cannot be used to derive the required inequality $S_1 - S_3 \leq 2\,S_1 - S_2$, i.e., $S_2 - S_1 \leq S_3$. Moreover, we cannot prove of a counterpart of Theorem 9.1 for parapreseminorms.

10. Two characterizations of additive functions

By using paraseminorms and inner products on groups, we can quite easily prove a natural generalization of a basic theorem of Maksa and Volkmann [141].

This theorem greatly improves several former results on norm Cauchy equations and inequalities. (See [94, 219, 118, 68, 128].) However, for instance, it is not cited in [130].

Theorem 10.1. *If f is a function of one group X to another Y, q is a paraprenorm on Y, then the following assertions are equivalent:*

(1) f *is additive,*

(2) $q\big(f(x)+f(y)\big) = q\big(f(x+y)\big)$ *for all* $x, y \in X$,

(3) $q\big(f(x)+f(y)\big) \leq q\big(f(x+y)\big)$ *for all* $x, y \in X$.

Hint. From (3), by using (3) in Theorem 7.5 and a quite similar argument as in [141], it is easy to see that $f(0)=0$ and f is odd.

Hence, by using (2) in Theorem 7.5, we can also easily see that

$$q\big(f(-x)\big) = q\big(-f(x)\big) = q\big(f(x)\big)\,,$$

and thus

$$q\big(f(-y-x)\big) = q\,\big(f\big(-(x+y)\big)\big) = q\big(f(x+y)\big)$$

for all $x, y \in X$.

Now, by using the above facts, (c) in Definition 7.1, (4) in Theorem 7.5 and assertion (3), we can also see that

$$q\big(f(x+y)-f(y)-f(x)\big)^2 = q\big(f(x+y)+f(-y)+f(-x)\big)^2$$
$$\leq q\big(f(x+y)+f(-y)\big)^2 + q\big(f(x+y)+f(-x)\big)^2 + q\big(f(-y)+f(-x)\big)^2$$
$$- q\big(f(x+y)\big)^2 - q\big(f(-y)\big)^2 - q\big(f(-x)\big)^2 =$$
$$q\big(f(x+y)+f(-y)\big)^2 + q\big(f(-x)+f(x+y)\big)^2 + q\big(f(-y)+f(-x)\big)^2$$
$$- q\big(f(x+y)\big)^2 - q\big(f(y)\big)^2 - q\big(f(x)\big)^2 \leq$$
$$q\big(f(x)\big)^2 + q\big(f(y)\big)^2 + q\big(f(x+y)\big)^2$$
$$- q\big(f(x+y)\big)^2 - q\big(f(y)\big)^2 - q\big(f(x)\big)^2 = 0$$

for all $x, y \in X$. Therefore, we necessarily have

$$f(x+y)-f(y)-f(x)=0\,,$$

for all $x, y \in X$, and thus (1) also holds.

Now, in addition to this theorem, we can also easily prove the following

Theorem 10.2. *If f is a function of one group X to another Y, Q is an inner product on Y and*

$$q(y)=\sqrt{Q(y,\,y)}$$

for all $y \in Y$, then the following assertions are equivalent:

(1) f is additive,

(2) $2\,Q_1\big(f(x),\ f(y)\big) = q\big(f(x+y)\big)^2 - q\big(f(x)\big)^2 - q\big(f(y)\big)^2$ for all $x, y \in X$,

(3) $2\,Q_1\big(f(x),\ f(y)\big) \leq q\big(f(x+y)\big)^2 - q\big(f(x)\big)^2 - q\big(f(y)\big)^2$ for all $x, y \in X$.

Hint. If (3) holds, then by Theorem 4.9 and assertion (3) we have

$$q\big(f(x)+f(y)\big)^2 = q\big(f(x)\big)^2 + q\big(f(y)\big)^2 + 2\,Q_1\big(f(x),\ f(y)\big) \leq$$
$$q\big(f(x)\big)^2 + q\big(f(y)\big)^2 + q\big(f(x+y)\big)^2 - q\big(f(x)\big)^2 - q\big(f(y)\big)^2 = q\big(f(x+y)\big)^2$$

for all $x, y \in X$. Therefore, by the nonnegativity of q, we also have

$$q\big(f(x)+f(y)\big) \leq q\big(f(x+y)\big)$$

for all $x, y \in X$. Hence, by Theorems 7.3 and 10.1, we can see that (1) also holds.

Remark 10.3. Note that the above proofs do not require any particular trick. Therefore, they are more simple than the one given by Maksa and Volkmann [141] which was actually based on assertion (1) of Theorem 5.4.

In [207], we have used assertion (2) of Theorem 5.4 to prove the implication (3) $\Longrightarrow$ (1) of Theorem 9.2. However, assertion (3) of Theorem 5.4 seems to be a more convenient mean than (1) and (2). Namely, in the proof of the implication (3) $\Longrightarrow$ (1) of Theorem 10.1, we have only used some properties of q and q^2.

Remark 10.4. Moreover, assertion (3) of Theorem 5.4, and its possible generalizations [22], can certainly be also applied to some other important functional inequalities. A first such one is the quadratic functional equality studied by Gillányi [80] and Rätz [171]. (See also Fechner [60] for some further developments.)

However, note that if for instance f is an even, subadditive function of group X to $\mathbb{R}$, then f is nonnegative, and thus

$$|\, f\,(x+y)\,| = f\,(x+y) \leq f\,(x) + f\,(y) = |\, f\,(x) + f\,(y)\,|$$

for all $x,\, y \in \mathbb{R}$.

Moreover, if f is an odd isometry of an arbitrary preseminormed group X to a 2–cancellable one Y, then $f(0) = 0$, and thus

$$\begin{aligned} \|\, f\,(x+y)\,\| &= \|\, f\,(x+y) - f(0)\,\| = \|\, x+y-0\,\| \\ &= \|\, x-(-y)\,\| = \|\, f(x) - f\,(-y)\,\| = \|\, f\,(x) + f\,(y)\,\| \end{aligned}$$

for all $x,\, y \in X$.

According to the easier part of a famous characterization theorem of Ger [76], the composition of an odd isometry and an additive function is also a solution of the corresponding equality.

Therefore, to prove certain counterparts of Theorems 10.1 and 10.2 some additional requirements will be needed. For some ideas in this respect, see [128, Theorem 2] which should also be proved with the help of assertion (3) of Theorem 5.4.

Remark 10.5. Finally, to justify the appropriateness of our present treatment, we also note that an application of Theorem 10.1, to the proof of the left-invariance of some lower left-invariant generalized metrics, will be given in [209].

11. Problems in connection with the Páles equation

In [157] and [140], Páles and Maksa investigated a multiplicative form of the equation

$$F\,(x,\, y) + \frac{1}{n} \sum_{i=1}^{n} F\,\big(\, x + \varphi_i(y)\,,\, z\,\big) = \frac{1}{n} \sum_{i=1}^{n} F\,\big(\, x\,,\; y + \varphi_i(z)\big) + F\,(\, y,\, z\,)\,,$$

where F is a function of a product semigroup X^2 to $\mathbb{C}$ or a normed space Y, and the functions φ_i are pairwise distinct additive functions of X to itself, which form a group with respect to composition.

First of all, they have proved that, for an arbitrary function f of X to Y, the difference function F_f, defined by

$$F_f\,(x,\, y) = f\,(x) + f\,(y) - \frac{1}{n} \sum_{i=1}^{n} f\,\big(\, x + \varphi_i\,(y)\big)$$

for all $x,\, y \in X$, is a solution of the above functional equation.

The importance of these considerations lie mainly in the fact that in the $n = 1$ and $\varphi_1(x) = x$ particular case we get the *cocycle equation*

$$F(x,\, y) + F(x+y,\, z) = F(x,\, y+z) + F(y,\, z)$$

and the *Cauchy difference*

$$F_f(x, y) = f(x) + f(y) - f(x+y).$$

While, if X is a commutative group, then in the $n = 2$, $\varphi_1(x) = x$ and $\varphi_2(x) = -x$ particular case we get the *Székelyhidi equation* [210]

$$F(x, y) + 2^{-1}\big(F(x+y, z) + F(x-y, z)\big) = 2^{-1}\big(F(x, y+z) + F(x, y-z)\big) + F(y, z).$$

and the *quadratic difference*

$$F_f(x, y) = f(x) + f(y) - 2^{-1}\big(f(x+y) - f(x-y).$$

In the light of the above observations, it is an interesting problem that:

Problem 11.1. *How can the several results on the cocycle and Székelyhidi equations be extended to the Páles equation?*

Páles and Maksa have proved four such results. However, I am mainly interested in some more elementary ones.

In particular, I do not know that:

Problem 11.2. *How my former two generalizations of the cocycle equation* [203]

$$\begin{aligned} &F(x, y) + F(u, y+v) + F(x+y, u+v) \\ &\qquad = F(x, u) + F(y, u+v) + F(x+u, y+v), \end{aligned}$$

and

$$\begin{aligned} &F(x, y) + F(x-u, u) + F(y-v, u) + F(y-v, v) \\ &\qquad = F(u, v) + F(u, y-v) + F(x-u, y-v) + F(x+y-u-v, u+v) \end{aligned}$$

can be extended to the Páles equation?

After solving the latter problem:

Problem 11.3. *It would also be of some interest to extend the results of our former papers* [204] *and* [205] *to the corresponding generalizations of the Páles equation.*

Finally, we remark that cocycle equation can also be written in the difference forms

$$F(x+y, z) - F(y, z) = F(x, y+z) - F(x, y)$$

and

$$F(x+y, z) - F(x, z) - F(y, z) = F(x, y+z) - F(x, y) - F(x, z)$$

which may also lead to some generalizations.

Collecting and reading most of the items of the subsequent extensive References have required a lot more efforts and energy than finding out and writing down the contents of all the former eleven sections. Fortunately, only a very few items have not been available in our Library and on the Internet. However, papers written in French or Italian were not readable for me.

References

[1] J. Aczél, *The general solution of two functional equations by reduction to functions additive in two variables and with the aid of Hamel bases*, Glasnik Mat. **20** (1965), 65–72.

[2] J. Aczél and J. Dhombres, *Functional Equations in Several Variables*, Cambridge University Press, Cambridge, 1989.

[3] D. D. Adamović, *Généralisation d'une identité de Hlawka et de l'inégalité correspondante*, Mat. Vesnik **1** (1964), 39–43.

[4] M. Almahalebi, *Approximate Drygas mappings on a set of measure zero*, Proyecciones J. Math. **35** (2016), 225–233.

[5] C. Alsina, *On a generalized Lagrange's identity characterizing inner product spaces*, Publ. Math. Debrecen **52** (1998), 275–280.

[6] C. Alsina and J. L. Garcia-Rog, *On a functional equation related to the Ptolemaic inequality*, Aequationes Math. **34** (1987), 298–303.

[7] C. Alsina and J. L. Garcia-Rog, *On a functional equation characterizing inner product spaces*, Publ. Math. Debrecen **39** (1991), 299–304.

[8] C. Alsina, J. L. Garcia-Rog and R. Ger, *How to keep the quadrilateral incribed*, Abh. Math. Sem. Univ. Hamburg **65** (1995), 123–133.

[9] C. Alsina, J. Sikorska and M. S. Tomás, *Norm Derivatives and Characterizations of Inner Product Spaces*, World Scientific, New Yersey, 2010.

[10] D. Amir, *Characterizations of Inner Product Spaces*, Birkhäuser, Besel, 1986.

[11] T. V. An, N. V. Dung, Z. Kadelburg and S. Radenovič, *Various generalizations of metric spaces and fixed point theorems*, RACSAM **109** (2015), 175–198.

[12] Q. H. Ansari, *Metric Spaces*, Alpha Science International, Oxford, 2010.

[13] J.-P. Antoine and A. Grossmann, *Partial inner product spaces. I. General properties*, J. Funct. Anal. **23** (1976), 369–378.

[14] T. M. Apostol, *Ptolemy's inequality and the chordal metric*, Math. Magazine **40** (1967), 233–235.

[15] A. Bahyrycz, Zs. Páles and M. Piszczek, *Asymptotic stability of the Cauchy and Jensen functional equations*, Acta Math. Hungar., **150** (2016), 131–141.

[16] J. A. Baker, *On quadratic functionals continuous along rays*, Glasnik. Mat. **3** (1968), 215–229.

[17] K. Baron, *On additive involutions and Hamel bases*, Aquationes Math. **87** (2014), 159–163.

[18] B. Batko and J. Tabor, *Stability of an alternative Cauchy equation on a restricted domain*, Aequationes Math. **57** (1999), 221–232.

[19] B. Belaid, E. Elqorachi and Th. M. Rassias, *On the generalized Hyers-Ulam Stability of Swiatak's functional equation*, J. Math. Ineq. **1** (2007), 291–300.

[20] J. Bognár, *Indefinite Inner Product Spaces* , Springer, Berlin, 1974.

[21] Z. Boros, *Schwarz inequality over groups*, Talk held at the Conference on Inequalities and Applications, dedicated to the 75th birthday of L. Losonczi and the 60th birthday of Zs. Páles, Hajdúszoboszló, Hungary, 2016.

[22] Z. Boros and Á. Száz, *Semi-inner products and their induced seminorms and semimetrics on groups*, Tech. Rep., Inst. Math., Univ. Debrecen 2016/6, 11 pp.

[23] Z. Boros and Á. Száz, *A weak Schwarz inequality for semi-inner products on groupoids*, Tech. Rep., Inst. Math., Univ. Debrecen 2016/7, 10 pp.

[24] Z. Boros and Á. Száz, *A most general Schwarz inequality for generalized semi-inner products on groupoids*, Tech. Rep., Inst. Math., Univ. Debrecen 2016/8, 9 pp.

[25] Z. Boros and Á. Száz, *A generalization of a theorem of Gilányi on quadratic functions*, in preparation.

[26] G. Brown, *Convexity and Minkowski's inequality*, Amer. Math. Monthly **112** (2005), 740–742.

[27] M. L. Buzano, *Generalizzazione della diseguaglianza di Cauchy–Schwarz*, Rend. Semin. Mat. Torino **31** (1971/73), 405–409.

[28] J. X. Chen and Z. L. Chen, *On supra-additive and supra-multiplicative maps*, J. Funct. Spaces and Appl. 2013, Art. ID 108535, 3 pp.

[29] J. Chmieliński, *Normed spaces equivalent to inner product spaces and stability of functional equations*, Aequationes Math. **87** (2014), 147–157.

[30] J. Chmieliński, *A note on a characterization of metrics generated by norms*, Rocky Mt. J. Math. **45** (2015), 1801–1805.

[31] J. Chmieliński, *On functional equations related to additive mappings and isometries*, Aequationes Math. **89** (2015), 97–105.

[32] J. Chung, *On the Drygas functional equation in restricted omains*, Aequationes Math. **90** (2016), 799–808.

[33] J. Chung and J. M. Rassias, *Quadratic functional equations in a set of Lebesgue measure of zero*, J. Math. Anal. Appl. **419** (2014), 1065–1075.

[34] V. Coibotariu-Boer, *Generalizations of some properties of convex functions*, J. Math. Ineq. **3** (2009), 107–113.

[35] P. Corazza, *Introduction to metric-preserving functions*, Amer. Math. Monthly **104** (1999), 309–323.

[36] M. M. Day, *Some characterizations of inner-product spaces*, Trans. Amer. Math. Soc. **62** (1947), 320–337.

[37] H. Dehghan, *A characterization of inner oroduct spaces related to the skew-angular distance*, Math. Notes **93** (2013), 556–560.

[38] M.-M. Deza and E. Deza, *Dictionary of Distances*, Springer, Heidelberg, 2009.

[39] J. Dhombres, *Some Aspects of Functional Equations*, Chulalongkorn Univ., Bangkok, 1979.

[40] J. Dhombres, *Sur les fonctions simultanément suradditives et surmultiplicatives*, C. R. Math. Acad. Sci. Canada **5** (1983), 207–210.

[41] J. B. Diaz and F. T. Metcalf, *A complementary triangle inequality in Hilbert and Banach spaces*, Proc. Amer. Math. Soc. **17** (1966), 88–97.

[42] D. Ž. Doković, *Generalizations of Hlawka's inequality*, Glasnik Mat. **18** (1963), 169–175.

[43] Y. Dong, *Generalized stabilities of two functional equations*, Aequationes Math. **86** (2013), 269–277.

[44] Y. Dong and L. Chen, *On generalized Hyers-Ulam stability of additive mappings on restricted domains of Banach spaces*, Aequationes Math. **90** (2016), 871–878.

[45] S. S. Dragomir, *Some Grüss type inequalities in inner product spaces*, J. Inequal. Pure Appl. Math. **4** (2003).

[46] S. S. Dragomir, *Semi-Inner Products and Applications*, Nova Science Publishers, Hauppauge, NY, 2004.

[47] S. S. Dragomir, *Generalization of the Pečarič-Rajič inequality in normed linear spaces*, Math. Ineq. Appl. **12** (2009), 53–65.

[48] S. S. Dragomir, *Inequalities for the p-angular distance in normed linear spaces*, Math. Ineq. Appl. **12** (2009), 391–401.

[49] S. S. Dragomir and J. J. Koliha, *The mapping $\delta_{x,y}$ in normed linear spaces and refinements of the Cauchy Schwarz inequality*, Nonlinear Analysis: Theory, Methods and Applications **41** (2000), 205–220.

[50] S. S. Dragomir, Y. J. Cho and S. S. Kim, *Some inequalities in inner product spaces related to generalized triangle inequality*, Appl. Math. Comput. **217** (2011), 7462–7468.

[51] H. Drygas, *Quasi-inner products and their applications*, In: A. K. Gupta (Ed.), Advances in Multivariate Statistical Analysis, Theory Decis. Lib. Ser. B, Math. Statis. Methods, Reidel, Dorrecht, 1987, 13–30.

[52] C. F. Dunkl and K. S. Williams, *A simple norm inequality*, Amer. Math. Monthly **71** (1964), 53–54.

[53] B. R. Ebanks, Pl. Kannappan and P. K. Sahoo, *A common generalization of functional equations characterizing normed and quasi-inner product spaces*, Canad. Math. Bull. **35** (1992), 321–327.

[54] B. R. Ebanks and C.T. Ng, *Characterizations of quadratic differences*, Publ. Math. Debrecen **48** (1996), 89–102.

[55] E. Elqorachi, Y. Manar and Th. M. Rassias, Hyers-Ulam stability of the quadratic functional equation, In: Th. M. Rassias and J. Brdek (Eds.), Functional Equations in Mathematical Analysis, Springer Optimization and its Applications **52** (2012), 97–105.

[56] V. A. Faĭziev and P. K. Sahoo, *On the stability of the Drygas functional equation on groups*, Banach J. Math. **1** (2007), 43–55.

[57] D. Fearnley-Sander and J. S. V. Symons, *Apollonius and inner products*, Amer. Math. Montly **81** (1974), 990–993.

[58] W. Fechner, *On functions with the Cauchy difference bounded by a functional II*, Int. J. Math. Math. Sci. **12** (2005), 1889–1898.
[59] W. Fechner, *On the Hyers-Ulam stability of functional equations connected with additive and quadratic mappings*, J. Math. Anal. Appl. **322** (2006), 774–786.
[60] W. Fechner, *Stability of a functional inequality associated with the Jordan-von Neumann functional equation*, Aequationes Math. **71** (2006), 149–161.
[61] W. Fechner, *Hlawka's functional inequality*, Aequationes Math. **87** (2014), 71–87.
[62] W. Fechner, *Hlawka's functional inequality on topological groups*, manuscript.
[63] F. A. Ficken, *Note on the existence of clalar products in normed linear spaces*, Ann. Math. **45** (1944), 362–366.
[64] I. Fidytek, *On a functional equation connected with Ptolemaic inequality*, Ann. Math. Sil. **15** (2001), 27–44.
[65] E. Fischer and P. Scherk, *A remark on Schwarz's inequality*, C. R. Math. Rep. Acad. Sci. Canada **2** (1980), 171–174.
[66] P. Fischer, *Sur l'équivalence des équations fonctionnelles* $f\,(x+y) = f(x) + f(y)^2$ *et* $f^2(x+y) = [\,f(x)+f(y)\,]^2$, Ann. Fac. Sci. Univ. Toulouse 1968, 71–73. (By Kuczma [118] , this paper contains some errors.)
[67] P. Fischer and Gy. Muszély, *Some generalizations of Cauchy's functional equations*, Mat. Lapok **16** (1965), 67–75. (Hungarian.)
[68] P. Fischer and Gy. Muszély, *On some new generalizations of the functional equation of Cauchy*, Canadian Math. Bull. **10** (1967), 197–205.
[69] M. Fochi, *An alternative functional equation on a restricted domain*, Aequationes Math. **70** (2005), 201–212.
[70] I. Franić, *A remark of the strict convexity for seminorms* Glasnik Mat. **22** (1987), 99–101.
[71] M. Fréchet, *Sur lá definition axiomatique d'une classe d'espaces vectoriels distanciés applicables vectoriellement sur l'espace de Hilbert*, Ann. Math. **36** (1935), 705–718.
[72] P. de P. Friis and H. Stetkaer, *On the quadratic functional equation on groups*, Publ. Math. Debrecen **69** (2006), 65–93.
[73] P. Găvruta, *On the Hyers-Ulam-Rassias stability of the quadratic mappings*, Nonlin. Funct. Anal. Appl. **9** (2004), 415–428.
[74] R. Ger, *On a characterization of strictly convex spaces*, Atti Acad. Sci. Torino Cl. Sci. Fis. Mat. Natur. **127** (1993), 131–138.
[75] R. Ger, *A Pexider-Type equation in normed linear spaces*, Sitzungsber. Abt. II **206** (1997), 291–303.
[76] R. Ger, *Fischer–Muszély additivity on Abelian groups*, Comment Math., Tomus Specialis in Honorem Juliani Musielak, 2004, 83–96.
[77] R. Ger, *On a problem of Navid Safaei*, Talk held at the Conference on Inequalities and Applications, dedicated to the 75th birthday of L. Losonczi and the 60th birthday of Zs. Páles, Hajdúszoboszló, Hungary, 2016.
[78] R. Ger and B. Koclega, *Isometries and a generalized Cauchy equation*, Aequationes Math. **60** (2000), 72–79.
[79] A. Gilányi, *On the stability of the square-norm equation*, Publ. Math. Debrecen **52** (1998), 419-427.
[80] A. Gilányi, *Eine zur Parallelogrammgleichung äquivalente Ungleichung* , Aequationes Math. **62** (2001), 303-309.
[81] A. Gilányi, *On a problem by K. Nikodem*, Math. Ineq. Appl. **5** (2002), 707-710.
[82] A. Gilányi and K. Troczka-Pawelec, *Regularity of weakly subquadratic functions*, J. Math. Anal. Appl. **382** (2011), 814-821.
[83] A. Gilányi, Cs. Kézi and K. Troczka-Pawelec, *On two different concepts of subquadraticity*, Inequalities and Applications 2010, Internat Ser Numer. Math. **161**, Birkhäuser, Basel, 2012, pp. 209–215.
[84] J. R. Giles, *Classes of semi-inner-product spaces*, Trans. Amer. Math, Soc. **129** (1967), 436–446.
[85] T. Glavosits and Á. Száz, *Divisible and cancellable subsets of groupoids*, Ann. Math. Inform. **43** (2014), 67-91.
[86] A. M. Gleason, *The definition of quadratic form*, Amer. Math. Monthly **73** (1966), 1049-1056.
[87] S. Golab, *Über eine Verallgemeinerung der Schwarschen Ungleichung*, Publ. Math. Debrecen **21** (1974), 155-158.

[88] S. Golab, *Über das Carnotsche Scalarproduct in schwach normierten Vectorrräumen*, Aequationes Math. **13** (1975), 9-13.

[89] S. Golab and H. Światak, *Note on inner products in vector spaces*, Aequationes Math. **8** (1972), 74-75.

[90] M. E. Gordji and H. Khodaei, *On the generalized Hyers-Ulam-Rassias stbility of quadratic functional equations*, Abst. Apll. Anal. Vol. 2009, Art. ID 923476, 11 pp.

[91] I. Gusić, *A note on certain maps between ordered fields*, J. Math. Ineq. **3** (2009), 657–661.

[92] K. J. Heuvers, *On the types of functions which can serve as scalar products in a complex linear space*, Lin. Alg. Appl. **6** (1973), 83-96.

[93] H. Hornich, *Eine Ungleichung für Vektorlängen*, Math. Zeitschrift **48** (1942), 268–274.

[94] M. Hosszú, *On an alternative functional equation*, Mat. Lapok **14** (1963), 98–102. (Hungarian.)

[95] M. Hosszú, *A remark on the square norm*, Aequationes Math. **2** (1969), 190–193.

[96] M. Hosszú and M. Csikós, *Normenquadrat über Gruppen*, Recueil des travaux de l'Institut Mathématique **1** (1976), 35–39.

[97] V. I. Istrătescu, *Inner Product Structures, Theory and Applications*, Reidel, Dorrecht, 1987.

[98] W. Janous, *Eine Metric auf* $\mathbb{R}_+$, Wissensch. Nacht. **83** (1990), 29–30.

[99] G. G. Johson, *Inner products characterized by difference equations*, Proc. Amer. Math. Soc. **37** (1973), 535–536.

[100] J. T. Joichi, *Normed linear spaces equivalent to inner product spaces*, Proc. Amer. Math. Soc. **17** (1966), 423–426.

[101] P. Jordan and J. von Neumann, *On inner products in linear metric spaces*, Ann. Math. **36** (1935), 719–723.

[102] S.-M. Jung, *On the Hyers-Ulam stability of the functional equations that have the quadratic property*, J. Math. Anal. Appl. **222** (1935), 126–137.

[103] S.-M. Jung and P. Sahoo, *Stability of a functional equation of Drygas*, Aequatones Math. **64** (1935), 263–273.

[104] S.-M. Jung, D. Popa and M. Th. Rassias, *On the stability of the linear functional equation in a single variable on complete metric groups*, J. Glob. Optim. **59** (2014), 165–171.

[105] Pl. Kannappan, *Quadratic fuctional equation and inner product spaces*, Results Math. **27** (1995), 368–372.

[106] Pl. Kannappan, *On quadratic fuctional equation*, Int. J. Math. Statist. Sci. **9** (2000), 10–35.

[107] M. Kato, K.-S. Saito and T. Tamura, *Scharp triangle inequality and its reverse in Banach spaces*, Math. Ineq. Appl. **10** (2007), 451–460.

[108] J. L. Kelley, *General Topology*, Van Nostrand Reinhold, New York, 1955.

[109] L. M. Kelly, D. M. Smiley and M. F. Smiley, *Two dimensional spaces are quadrilateral spaces*, Amer. Math. Monthly **72** (1965), 753-754.

[110] W. A. Kirk and M. F. Smiley, *Another characterization of inner product spaces*, Amer. Math. Mothly **71** (1964), 890-891.

[111] M. S. Klamkin and A. Meir, *Ptolemy's inequality, chordal metric, multiplicative metric*, Pacific J. Math. **101** (1982), 389-392.

[112] D. Kobal, *Inner product space and circle power*, Publ. Math. Debrecen **81** (2012), 1–9.

[113] Z. Kominek, *Stability of a quadratic functional equation on semigroups*, Publ. Math. Debrecen **75** (2009), 173–178.

[114] Z. Kominek, *On a Dygras inequality*, Tatra Mt. Math. Publ. **52** (2012), 65–70.

[115] Z. Kominek and K. Troczka, *Some Remarks on subquadratic functions*, Demontsratio Math. **39** (2006), 751–757.

[116] Z. Kominek and K. Troczka, *Continuity of real valued subquaratic functions*, Comment. Math. **51** (2011), 71–75.

[117] L. Kosmák and L. Winterová, *Some aspects of the C.-B.-S. inequality*, Acta Math. Univ. Comen. **48-49** (1986), 341–348.

[118] M. Kuczma, *On some alternative functional equations*, Aequationes Math. **17** (1978), 182–198.

[119] M. Kuczma, *An Introduction to the Theory of Functional Equations and Inequalities*, Państwowe Wydawnictwo Naukowe, Warszawa, 1985.

[120] S. Kurepa, *On the quadratic functional*, Publ. Inst. Math. Acad. Serbe Sci. Beograd **13** (1959), 57–72.

[121] S. Kurepa, *The Cauchy functional equation and scalar product in vector spaces*, Glasnik Mat. **19** (1964), 23–35.
[122] S. Kurepa, *Quadratic and sesquilinear functionals*, Glasnik Mat. **20** (1965), 79–91.
[123] S. Kurepa, *On the Buniakowsky-Cauchy-Schwarz inequality*, Glasnik Mat. **1** (1966), 147–158.
[124] S. Kurepa, *On an inequality*, Glasnik Mat. **3** (1968), 193–196.
[125] S. Kurepa, *Note on inequalities associated with hermitian functionals*, Glasnik Mat. **3** (1968), 197–206.
[126] S. Kurepa, *Quadratic functionals conditioned on an algebraic basic set*, Glasnik. Mat. **6** (1971), 265–275.
[127] S. Kurepa, *On bimorphisms quadratic forms on groups*, Aequationes Math. **9** (1973), 30–45.
[128] S. Kurepa, *On P. Volkmann's paper*, Glasnik Mat. **22** (1987), 371–374.
[129] S. Kurepa, *On quadratic forms*, Aequationes Math. **34** (1987), 125–138.
[130] Y. H. Kwon, H. M. Lee, J. S. Sim, J. Yang and C. Park, *Jordan–von Neumann type functional inequalities*, J. Chungcheong Math. Soc. **20** (2007), 269–277.
[131] D. Laugwitz, *On characterizations of inner product spaces*, Proc. Amer. Math. Soc. **50** (1975), 184–188.
[132] J. R. Lee, C. Park and D. Y. Shin, *On the stability of generalized additive functional inequalities in Banach spaces*, J. Ineq. Appl. 2008, Art. ID 210626, 13 pp.
[133] Y.-H. Lee, S.-M. Jung and M. Th. Rassias, *On an n-dimensional mixed type additive and quadratic functional equation*, Appl. Math. Comput. **228** (2014), 13–16.
[134] L. Losonczi, *On the stability of Hossú's functional equation*, Results Math. **29** (1996), 305–310.
[135] G. Lumer, *Semi-inner-product spaces*, Trans. Amer. Math, Soc. **100** (1961), 29–43.
[136] H. A. McLean, *The triangle inequality*, Amer. Math. Mothly **85** (1978), 105–106.
[137] I. Makai, *Über invertierbare Lösungen der additive Cauchy-Functionalgleichung*, Publ. Math. Debrecen **16** (1969), 239–243.
[138] Gy. Maksa, *A remark of symmetric biadditive functions having nonnegative diagonalization*, Glasnik Mat. **15** (1980), 279–282.
[139] Gy. Maksa and Zs. Páles, *On Hosszú's functional inequality*, Publ. Math. Debrecen **36** (1989), 187–189.
[140] Gy. Maksa and Zs. Páles, *Hyperstability of a class of linear functional equations*, Acta Math, Acad. Paedagog. Nyházi (N.S.) **17** (2001), 107–112.
[141] Gy. Maksa and P. Volkmann, *Characterizations of group homomorphisms having values in an inner product space*, Publ. Math. Debrecen **56** (2000), 197–200.
[142] L. Maligranda, *Simple norm inequalities*, Amer. Math. Monthly **113** (2006), 256–260.
[143] L. Maligranda, *Some remarks on the triangle inequality for norms*, Banach J. Math. Anal. **2** (2008), 31–41.
[144] Y. Manar and E. Elqorachi, *On functional inequalities associated with Drygas functional equation*, Tbilisi Math. J. **7** (2014).
[145] D. S. Marinescu, M. Monea, M. Opincariu and M. Stroe, *Some equivalent characterizations of inner product spaces and their consequences*, Filomat **29** (2015), 1587–1599.
[146] P.R. Mercer, *The Dunkl-Williams inequality in an inner product space*, Math. Ineq. Appl. **10** (2007), 447–450.
[147] D. S. Mitrinović, J. E. Pečarić and A. M. Fink, *Classical and New Inequalities in Analysis*, Kluwer, Dordrecht, 1993.
[148] M. H. Moore, *An inner product inequality*, SIAM J. Math. Anal. **4** (1973), 514–518.
[149] F. Moradlou, *Additive functional inequalities and derivations on Hilbert C^*-modules*, Glasgow Math. J. **55** (2013), 341–348.
[150] A. Najati, J.-R. Lee and C. Park, *On a Cauchy-Jensen functional inequality*, Bull. Malays. Math. Sci. Soc. **33** (2010), 253–263.
[151] B. Nath, *On a generalization of semi-inner product spaces*, Math. J. Okayama Univ. **15** (1971), 1–6.
[152] J. von Neumann, *Allgemeine Eigenwerttheorie Hermitischer Funktionoperatoren* Ann. Math **102** (1929), 49–131.
[153] K. Nikodem and Zs. Páles, *Characterizations of inner product saces by strongly convex functions*, Banach J. Math. **5** (2011), 83–87.

[154] C. T. Ng and H. Y. Zhao, *Kernel of the second order Cauchy difference on groups*, Aequationes Math. **86** (2013), 155–170.
[155] T. Oikhberg and H. Rosenthal, *A metic characterization of normed linear spaces*, Rocky Mt. J. Math. **37** (2007), 597–608.
[156] Zs. Páles, *On two variable functional inequalities*, C. R. Math. Rep. Acad. Sci. Canada **10** (1988), 25–28.
[157] Zs. Páles, *Bounded solutions and stability of functional equations for two variable functions*, Results Math. **26** (1994), 360–365.
[158] C. Park and A. Najati, *Generalized additive functional inequalities in Banach algebras*, Int. J. Nonlin. Anal. Appl. **1** (2010), 54–62.
[159] C. Park, *Additive ρ–functional inequalities*, J. Nonlinear Sci. Appl. **7** (2014), 296–310.
[160] C. Park, *Additive ρ-functional inequalities and equations*, J. Math. Ineq. **9** (2015), 17–26.
[161] C. Park, *Additive ρ–functional inequalities in normed spaces*, J. Nonlinear Sci. Appl. **9** (2016), 247–253.
[162] C. Park, Y. S. Cho and M.-H. Han, *Functional inequalities associated with Jordan–von Neumann-type additive functional equations*, J. Ineq. Appl. 2007, Art. ID 41820, 13 pp.
[163] C. Park, J. R. Lee and Th. M. Rassias, *Functional inequalities in Banach spaces and fuzzy Banach spaces*, In: Th. M. Rassias and P. M. Pardalos (Eds.), Essays in Mathematics and its Applications: In Honor of Vladimir Arnold, Springer, Switzerland, 2016, 263–310.
[164] J. Pečarič and R. Rajič, *The Dunkl-Williams inequality with n elements in normed linear spaces*, Math. Ineq. Appl. **10** (2007), 461–470.
[165] K. Piejko, *Note on Robinson's Functional equation*, Demonstratio Math. **32** (1999), 713–715.
[166] T. Precupanu, *On a generalization of Cauchy-Buniakowski-Schwarz inequality*, An. Stiint. Univ. Al. I. Cuza Iasi **23** (1976), 173–175.
[167] J. Röhmel, *Eine Characterisierung quadratischer Formen durch eine Functionalglechung*, Aequationes Math. **15** (1977), 163–168.
[168] J. M. Rassias, *On approximation of approximately quadratic mappings by quadratic mappings*, Ann. Math. Sil. **15** (2001), 67–78.
[169] J. Rätz, *On isometries of generalized inner product spaces*, SIAM J. Appl. Math. **18** (1970), 6–9.
[170] J. Rätz, *Additive mappings on inner product spaces*, Glasnik Mat. **13** (1978), 63-68.
[171] J. Rätz, *On inequalities associated with the Jordan-von Neumann functional equation*, Aequationes Math. **66** (2003), 191-200.
[172] P. Ressel, *The Hornich–Hlawka inequality and Berstein functions*, J. Math. Ineq. **9** (2015), 883–888.
[173] D. Robinson, *A Course in the Theory of Groups*, Springer, New York, 1982.
[174] J. Roh and I.-S. Chang, *Functional inequalities associated with additive mappings*, Abst. Appl. Anal. 2008, Art. ID 136592, 11 pp.
[175] J. Röhmel, *Eine Charakterisierung quadratischer Formen durch Funktionalgleichungen*, Aequationes Math. **15** (1977), 163–168.
[176] D. D. Rothmann, *A nearly discrete metric*, Amer. Math. Monthly **81** (1974), 1018–119.
[177] H. Rubin and M. H. Stone, *Postulates for generalizations of Hilbert space*, Proc. Amer. Math. Soc. **4** (1953), 611–616.
[178] D. J. Schattschneider, *A multiplicative metric*, Math. Mag. **49** (1976), 203–205.
[179] I. J. Schoenberg, *A remark on M. M. Day's characterization of inner-product spaces and a conjecture of L. M. Blumenthal*, Proc. Amer. Math. Soc. **3** (1953), 961-964.
[180] P. Schöpf, *Solutions of* $\| f(\xi+\eta) \| = \| f(\xi) + f(\eta) \|$, Math. Pannon. **8** (1997), 117–127.
[181] P. Šemrl, *Additive functions and two characterization of inner-product spaces*, Glasnik Mat. **25** (1990), 309–307.
[182] P. Šemrl, *A characterization of normed spaces*, J. Math. Anal. Appl. **343** (2008), 1047–1051.
[183] P. Šemrl, *A characterization of normed spaces among metric spaces*, Rocky Mt. J. Math. **41** (2011), 293–298.
[184] J. Sikorska, *Stability of the preservation of the equality of distance*, J. Math. Anal. Appl. **311** (2005), 209–217.
[185] J. Sikorska, *On a direct method for proving the Hyers-Ulam stability of functional equations*, J. Math. Anal Appl. **372** (2010), 99–109.

[186] A. Simon and P. Volkmann, *On two geometric inequalities*, Ann. Math. Sil. **9** (1995), 137–140.

[187] J. St.-C. L. Sinnadurai, *A proof of Cauchy's inequality and a new generalization*, Math. Gaz. **47** (1963), 36-37.

[188] D. Sitaru and L. Giugiuc, *Hlawka's inequalities for a class of functions*, avaliable on the Internet.

[189] F. Skof, *On the functional equation* $\| f(x+y) - f(x) \| = \| f(y) \|$, Atti Acad. Sci. Torino Cl. Sci. Fis. Mat. Natur. **127** (1993), 229–237.

[190] F. Skof, *On two conditional forms of the equation* $\| f(x+y) \| = \| f(x) + f(y) \|$, Aequationes Math. **45** (1993), 167–178.

[191] D. M. Smiley and M. F. Smiley, *The polygonal inequalities*, Amer. Math. Monthly **71** (1964), 755–760.

[192] J. M. Steele, *The Cauchy-Scharz Master Class (An Introduction to the Art of Mathematical Inequalities)*, Cambridge University Press, Cambridge, 2004.

[193] H. Stetkaer, *Functional Equations on Groups*, World Scientific, New Jersey, 2013.

[194] A. Sudbery, *The quadrilateral inequality in two dimensions*, Amer. Math. Mothly **82** (1975), 629–632.

[195] H. Swiatak, *On the functional equation* $f(x+y)^2 = [f(x) + f(y)]^2$, Publ. Techn. Univ. Miskolc **30** (1970), 307–308.

[196] H. Swiatak and M. Hosszú, *Remarks on the functional equation* $e(x, y) f(x+y) = f(x) + f(y)$, Publ. Techn. Univ. Miskolc **30** (1970), 323–325.

[197] Á. Száz, *Preseminormed spaces*, Publ. Math. Debrecen **30** (1983), 217–224.

[198] Á. Száz, *An instructive treatment of convergence, closure and orthogonality in semi-inner product spaces*, Tech. Rep., Inst. Math., Univ. Debrecen 2006/2, 29 pp.

[199] Á. Száz, *Galois type connections and closure operations on preordered sets*, Acta Math. Univ. Comen. **78** (2009), 1–21.

[200] Á. Száz, *A common generalization of the postman, radial, and river metrics*, Rostock. Math. Kolloq. **67** (2012), 89–125.

[201] Á. Száz, *Galois and Pataki connections on generalized ordered sets*, Tech. Rep., Inst. Math., Univ. Debrecen 2014/3, 27 pp.

[202] Á. Száz, *Remarks and Problems at the Conference on Inequalities and Applications, Hajdúszoboszló, Hungary, 2014*, Tech. Rep., Inst. Math., Univ. Debrecen, 2014/5, 12 pp.

[203] Á. Száz, *Two natural generalizations of cocycles*, J. Int. Math. Virtual Inst. **6** (2016), 66–86.

[204] Á. Száz, *Generalizations of an asymptotic stability theorem of Bahyrycz, Páles and Piszczek on Cauchy differences to generalized cocycles*, Tech. Rep., Inst. Math., Univ. Debrecen 20016/2, 12 pp.

[205] Á. Száz, *Generalizations of a restricted stability theorem of Losonczi on Cauchy differences to generalized cocycles*, Tech. Rep., Inst. Math., Univ. Debrecen 20016/3, 13 pp. (The publication of this work in the Acta Math. Hungar. was probably prevented by L. Székelyhidi with the approval of the former and present leaders of our Analysis Department.)

[206] Á. Száz, *A natural Galois connection between generalized norms and metrics*, Tech. Rep., Inst. Math., Univ. Debrecen 2016/4, 9 pp.

[207] Á. Száz, *A generalization of a theorem of Maksa and Volkmann on additive functions*, Tech. Rep., Inst. Math., Univ. Debrecen 2016/5, 6. pp. (The publication of an improved and enlarged version of this work in the Anal. Math. was probably prevented by a close colleague of R. Ger.)

[208] Á. Száz, *Schwarz inequality has to be replaced by an equality*, Lecture at the Seminar on Analysis, Debrecen, 2016.

[209] Á. Száz, *Semimetrics and preseminorms on groups*, in preparation.

[210] L. Székelyhidi, *Stability properties of functional equations in several variables*, Publ. Math. Debrecen **47** (1995), 95–100.

[211] Jacek Tabor, *Stability of the Fisher–Muszély functional equation*, Publ. Math. Debrecen **62** (2003), 205–211.

[212] Jacek Tabor and Józef Tabor, *19. Remark* (Solution of the 7. Problem posed by K. Nikodem.), Aequationes Math. **61** (2001), 307–309.

[213] S.-E. Takahashi, Y. Takahashi and S. Wada, *An extension of Hlawka's inequality*, Mat. Ineq. Appl. **3** (2000), 63–67.

[214] R. A. Tapia, *A characterization of inner product space*, Proc. Amer. Math. Soc. **41** (1973), 569–574.
[215] T. Trif, *A sharp inequality for vector norms*, Octogon Math. Mag. **10** (2002), 108–110.
[216] K. Troczka-Pawelec, *Some inequalities connected with a quadratic functional equation*, Pr. Nauk. Akad. Jana Dlugosza Czest. Mat. **13** (2008), 73–79.
[217] I. Vidav, *Zur Axiomatik des Hilbertschen Raumes*, Math. Z. **71** (1959), 458–460.
[218] J. Vigestrand, *Inequalities in Hilbert spaces*, Master Thesis, Norwegian University of Science and Technology, Tronheim, 2008.
[219] E. Vincze, *On solutions of alternative functional equations*, Mat. Lapok **15** (1964), 179–195. (Hungarian.)
[220] E. Vincze, *Über eine Verallgemeinerung der Cauchyschen Functionalgleichung*, Funkc. Ekvacioj **6** (1964), 55–62.
[221] E. Vincze, *Beitrag zur Theorie der Cauchyschen Funktionalgleichungen*, Arch. Math. **15** (1964), 132–135.
[222] P. Volkmann, *Eine Characterizsierung der positive definiten quadratischen Formen*, Aequationes Math. **11** (1974), 174–181.
[223] P. Vrbová, *Quadratic functionals and bilinear forms*, Časopis Pěst. Mat. **98** (1973), 159–161.
[224] D. Watson, *On quadrilateral spaces*, Amer. Math. Monthly **74** (1967), 1213–1215.
[225] A. Wilansky, *Modern Methods in Topological Vector Spaces*, McGraw-Hill, New York, 1978.
[226] D. Yang, *Remarks on the stability of Drygas's equation and the Pexidered-Quadratic equation*, Aequationes Math. **68** (2045), 108–116.
[227] D.-L. Yang, *The quadratic functional equation on groups*, Publ. Math. Debrecen **66** (2005), 327–348.
[228] E. H. Zarantonello, *The meaning of the Cauchy-Schwarz-Buniakovski inequality*, Proc. Amer. Math. Soc. **59** (1976), 133–137.

Árpád Száz, Department of Mathematics, University of Debrecen, H–4002 Debrecen, Pf. 400, Hungary
E-mail address: szaz@science.unideb.hu