UNIVERSITY OF DEBRECEN

Semi-inner products and their induced seminorms and semimetrics on groups

Zoltán Boros and Árpád Száz

Preprints No. 415
(Technical Reports No. 2016/6)

INSTITUTE OF MATHEMATICS

2016

SEMI-INNER PRODUCTS AND THEIR INDUCED SEMINORMS AND SEMIMETRICS ON GROUPS

ZOLTÁN BOROS AND ÁRPÁD SZÁZ

Abstract. By introducing a convenient notion of semi-inner products on groups, we shall show that some basic results on semi-inner product spaces can be naturally extended to semi-inner product groups.

The results obtained can, for instance, be used to extend some basic theorems of Gy. Maksa, P. Volkmann, A. Gilányi, J. Rätz and W. Fechner on characterizations of additive and quadratic functions by inequalities.

1. Introduction

For a group X, a function P of X^2 to $\mathbb{C}$ will be called a *semi-inner product* on X if

$$P(x, x) \geq 0, \qquad P(x, y) = \overline{P(y, x)} \qquad \text{and} \qquad P(x+y, z) = P(x, z) + P(y, z)$$

for all $x, y, z \in X$. In particular, the semi-inner product P will be called an *inner product* if $P(x, x) = 0$ implies $x = 0$ for all $x \in X$.

Thus, for instance, if a is an additive function of X to $\mathbb{C}$ and

$$P(x, y) = a(x)\,\overline{a(x)}$$

for all $x, y \in X$, then P is a semi-inner product on X. Moreover, this P is an inner product if and only if a is injective. Note that, by Kuczma [7, p. 292], even an injective additive function of $\mathbb{R}^n$ to $\mathbb{R}$ may be discontinuous.

If P is a semi-inner product on X, then it can be easily seen that

$$P(x+y, z) = P(y+x, z)$$

for all $x, y, z \in X$. Moreover, P is actually biadditive, and thus

$$P(kx, y) = k\,P(x, y) = P(x, ky)$$

also holds for all $k \in \mathbb{Z}$ and $x \in X$.

Thus, the real and imaginary parts, i.e., the first and second coordinate functions P_1 and P_2 of P inherit most of the above properties of P. In particular, P_1 is also an inner product on X. However, P_2 is not symmetric if $P_2 \neq 0$.

If P is a semi-inner product on X, then we define

$$p(x) = \sqrt{P(x, x)} \qquad \text{and} \qquad d(x, y) = p(-x+y)$$

for all $x, y \in X$. Thus, it can be shown that p is a *seminorm* and d is an *semimetric* on X having several useful additional properties.

2010 *Mathematics Subject Classification.* 20A99, 46C50, 46B99, 54E25.

Key words and phrases. Groups, semi-inner products, seminorms, semimetrics.

The work of the authors has been supported by the Hungarian Scientific Research Fund (OTKA) Grant K-111651.

For instance, we shall show that

(a) $|\,P_1(x,\,y)\,| \le p\,(x)\,p\,(y)\,,$

(b) $p\big(\,k\,(x+y)\big) = p\,(\,k\,x + k\,y)\,,$

(c) $p\,(x+y)^2 = P_1(x+y,\,x) + P_1(x+y,\,y)\,,$

(d) $p\,(x+y)^2 = p\,(x)^2 + p\,(y)^2 + 2\,P_1(x,\,y)\,,$

and moreover

(A) $d\,(\,x+y\,,\ y+x\,) = 0\,,$

(B) $d(\,k\,x,\,k\,y) = |\,k\,|\,d(x,\,y)\,,$

(C) $d\,(\,x+y\,,\ z+w\,) \le d\,(x,\,z) + d\,(y,\,w)\,,$

(D) $d\,(\,x,\,y) = d\,(\,z+x\,,\ z+y\,) = d\,(\,x+z\,,\,y+z\,)\,,$

for all $x,\,y,\,z,\,w \in X$ and $k \in \mathbb{Z}$.

In particular, a semi inner product P on X is an inner product if and only p is a norm or equivalently d is a metric on X. Therefore, if there exists an inner product P on X, then by (A) we have $x+y = y+x$ for all $x,\,y \in X$, and thus X is commutative.

If P is a semi-inner product on X and $x = \sum_{i=1}^{n} x_i$ for some $n \in \mathbb{N}$ with $n > 1$ and $x_i \in X$ with $i = 1,\,2,\ldots,\,n\,,$ then as some natural generalizations of (c) and (d) we can also prove that

(1) $$p\,(x)^2 = \sum_{i=1}^{n} P_1(x,\,x_i)\,,$$

(2) $$p\,(x)^2 = \sum_{i=1}^{n} p\,(x_i)^2 + \sum_{1\le i<j\le n} 2\,P_1(\,x_i + x_j)\,,$$

(3) $$p\,(x)^2 = \sum_{1\le i<j\le n} p\,(\,x_i + x_j)^2 - \sum_{i=1}^{n} (n-2)\,p\,(x_i)^2\,.$$

The $n = 3$ and $n = 4$ particular cases of the above equalities can, for instance, be used to prove some natural generalizations of some basic theorems of Maksa and Volkmann [10], Gilányi [6], Rätz [11] and Fechner [4] on characterizations of additive and quadratic functions by functional inequalities. (See [14] and [1].)

2. Semi-inner products

The most important seminorms on vector spaces are derived from semi-inner products [12]. Therefore, it seems convenient to introduce semi-inner products on groups too.

Notation 2.1. Suppose that X is a group and P is a function of X^2 to $\mathbb{C}$ such that, for all $x,\,y,\,z \in X$, we have

(a) $P\,(x,\,x) \ge 0\,,$

(b) $P\,(x,\,y) = \overline{P\,(y,\,x)}\,,$

(c) $P(x+y, z) = P(x, z) + P(y, z)$.

Remark 2.2. In this case the function P will be called a *semi-inner product* on X. This definition differs from that introduced by G. Lumer in 1961 (see Dragomir [3, p. 19]).

Moreover, the semi-inner product P will be called an *inner product* if

(d) $P(x, x) = 0$ implies $x = 0$ for all $x \in X$.

Thus, if P is an inner (semi-inner) product on the group X, then the ordered pair $X(P) = (X, P)$ may be called an inner (semi-inner) product group.

Example 2.3. Note that if a is an additive function of X to an inner product space H and

$$P(x, y) = \langle a(x), a(y) \rangle$$

for all $x, y \in X$, then P is a semi-inner product on X. Moreover, P is an inner product if and only if a is injective.

However, despite this, P may be a rather curious function even if $X = \mathbb{R}^n$ and $H = \mathbb{R}$. Namely, by Kuczma [7, p. 292], there exists a discontinuous, injective additive of $\mathbb{R}^n$ to $\mathbb{R}$.

Moreover, in the $n = 1$ particular case, this function may also be required to have some further striking properties by Makai [8], Kuczma [7, p. 293] and Baron [2].

The most basic properties of the semi-inner product P can be listed in the next

Theorem 2.4. *For all $x, y, z \in X$ and $k \in \mathbb{Z}$, we have*

(1) $P(x+y, z) = P(y+x, z)$,

(2) $P(x, y+z) = P(x, y) + P(x, z)$,

(3) $P(kx, y) = kP(x, y) = P(x, ky)$.

Proof. By (c) and the commutativity of the addition in $\mathbb{C}$, it is clear that (1) is true.

Moreover, by using (b) and (c), and the additivity of complex conjugation, we can easily see that (2) is also true. Thus, P is actually a biadditve function of X^2 to $\mathbb{C}$.

Hence, by the $\mathbb{Z}$-homogeneity of additive functions of one group to another [16, Sec. 2.1], it is clear that (3) is also true.

Remark 2.5. Note that, in particular, (3) yields

$$P(0, y) = 0 = P(x, 0) \qquad \text{and} \qquad P(-x, y) = -P(x, y) = P(x, -y)$$

for all $x, y \in X$.

Remark 2.6. Moreover, the first and second coordinate functions P_1 and P_2 of P also have the same commutativity and bilinearity properties as P.

Furthermore, by properties (a) and (b), for any $x, y \in X$ we have

(1) $P_1(x, x) = P(x, x) \geq 0$ and $P_2(x, x) = 0$,

(2) $P_1(y, x) = P_1(x, y)$ and $P_2(y, x) = -P_2(x, y)$.

Thus, in particular P_1 is also a semi-inner product on X. However, because of its skew-symmetry, P_2 cannot be a semi-inner product on X whenever $P_2 \neq 0$.

3. The induced seminorms

Definition 3.1. For all $x \in X$, we define

$$p(x) = \sqrt{P(x, x)}.$$

Example 3.2. Note that if in particular P is as in Example 2.3, then

$$p(x) = \| a(x) \|$$

for all $x \in X$.

The most immediate properties of the function p can be listed in the following

Theorem 3.3. *For all $x, y \in X$ and $k \in \mathbb{Z}$, we have*

(1) $p(x) \geq 0$,

(2) $p(kx) = |k| \, p(x)$,

(3) $p(x+y) = p(y+x)$,

(4) $p\big(k(x+y)\big) = p(kx + ky)$,

(5) $p(x+y)^2 = P_1(x+y, x) + P_1(x+y, y)$,

(6) $p(x+y)^2 = p(x)^2 + p(y)^2 + 2 P_1(x, y)$.

Proof. To prove (5) and (6), note that by the Definition 3.1 and Remark 2.6 we have

$$p(x) = \sqrt{P_1(x, x)}$$

and

$$p(x+y)^2 = P_1(x+y, \ x+y) = P_1(x+y, x) + P_1(x+y, y)$$
$$= P_1(x, x) + P_1(y, x) + P_1(x, y) + P_1(y, y) = p(x)^2 + 2 P_1(x, y) + p(y)^2.$$

Hence, by the symmetry of P_1 and the commutativity of the addition in $\mathbb{R}$, it is clear that (3) is also true.

Moreover, by using Theorems 3.3 and 2.4, we can see that

$$p\big(k(x+y)\big)^2 = k^2 p(x+y)^2 = k^2 p(x)^2 + k^2 p(y)^2 + 2k^2 P_1(x, y)$$

and

$$p(kx + ky)^2 = p(kx)^2 + p(kx)^2 + 2 P_1(kx, ky)$$
$$= k^2 p(x)^2 + k^2 p(y)^2 + 2k^2 P_1(x, y).$$

Therefore, $p\big(k(x+y)\big)^2 = p(kx+ky)^2$, and thus by the nonnegativity of p (4) also holds.

Remark 3.4. Note that, in particular, (2) yields

$$p(0) = 0 \qquad \text{and} \qquad p(-x) = p(x)$$

for all $x \in X$.

Remark 3.5. Moreover, to feel the importance of (4), note that for any $x, y \in X$, we have $2(x+y) = 2x + 2y$ if and only if $y + x = x + y$.

Therefore, if x and y do not commute, then $2(x+y) \neq 2x + 2y$. However, by (4), we still have $p\big(2(x+y)\big) = p(2x + 2y)$.

Remark 3.6. Moreover, we can also note that P is an inner product on X if and only if $p(x) = 0$ implies $x = 0$ for all $x \in X$.

Now, by using Theorems 3.3 and 2.4, we can also easily prove the following

Theorem 3.7. *For all $x, y \in X$ we have*

(1) $p(x-y)^2 = p(x+y)^2 - 4P_1(x, y)$,

(2) $p(x-y)^2 = 2p(x)^2 + 2p(y)^2 - p(x+y)^2$.

Proof. By Theorem 3.3, we have $p(x+y)^2 = p(x)^2 + p(y)^2 + 2P_1(x, y)$. Hence, by Theorems 3.3 and 2.4, we can see that

$$p(x-y)^2 = p(x)^2 + p(-y)^2 + 2P_1(x, -y) = p(x)^2 + p(y)^2 - 2P_1(x, y).$$

Therefore,

$$p(x+y)^2 + p(x-y)^2 = 2p(x)^2 + 2p(y)^2 \quad \text{and} \quad p(x+y)^2 - p(x-y)^2 = 4P_1(x, y).$$

Thus, the required equalities are also true.

Moreover, as an immediate consequence of Theorems 3.7 and 3.3, we can state

Theorem 3.8. *For all $x, y \in X$ we have*

(1) $P_1(x, y) = 4^{-1}\big(p(x+y)^2 - p(x-y)^2\big)$,

(2) $P_1(x, y) = 2^{-1}\big(p(x+y)^2 - p(x)^2 - p(y)^2\big)$.

Remark 3.9. Unfortunately, now similar polar formulas for $P(x, y)$ cannot be proved. Therefore, P can be recovered from p only in the real-valued case.

Moreover, in the present generality, the usual Schwarz's inequality cannot also be proved. We can only prove a weakened form of it. This will, however, be sufficient to prove the subadditivity of p.

Lemma 3.10. *For any $x, y \in X$, we have*

$$|P_1(x, y)| \leq p(x)\,p(y).$$

Proof. By Theorems 3.3 and 2.4, for any $n \in \mathbb{N}$ and $k \in \mathbb{Z}$, we have

$$\begin{aligned} 0 \leq p(nx + ky)^2 &= p(nx)^2 + p(ky)^2 + 2P_1(nx, ky) \\ &= n^2 p(x)^2 + k^2 p(y)^2 + 2nk\,P_1(x, y). \end{aligned}$$

Hence, we can see that

$$0 \leq p(x)^2 + (k/n)^2 p(y)^2 + 2(k/n)\,P_1(x, y).$$

Therefore, we actually have

$$0 \leq p(x)^2 + r^2 p(y)^2 + 2r\,P_1(x, y).$$

for all $r \in \mathbb{Q}$. Hence, by using that each real number is a limit of a sequence of rational numbers, we can already infer that

$$0 \leq p(x)^2 + \lambda^2 p(y)^2 + 2\lambda\,P_1(x, y),$$

and thus

$$0 \le p(x)^2 + \lambda P_1(x, y) + \lambda \left(\lambda p(y)^2 + P_1(x, y) \right)$$

also holds for all $\lambda \in \mathbb{R}$.

Hence, if $p(y) \neq 0$, then by taking $\lambda = -P_1(x, y)/p(y)^2$ we can see that

$$0 \le p(x)^2 - P_1(x, y)^2/p(y)^2, \qquad \text{and thus} \qquad P_1(x, y)^2 \le \left(p(x)\, p(y) \right)^2.$$

Therefore, by the nonnegativity of p, the required inequality is also true.

While, if $p(y) = 0$, then by taking $n \in \mathbb{N}$ and $\lambda = -n P_1(x, y)$, we can see that

$$0 \le p(x)^2 - 2n P_1(x, y)^2, \qquad \text{and thus} \qquad P_1(x, y)^2 \le p(x)^2/2n.$$

Hence, by taking the limit $n \to \infty$, we can infer that $P_1(x, y) = 0$. Therefore, the required inequality trivially holds.

Now, by using assertion (5) or (6) of Theorem 3.3 and a consequence of the above lemma, we can easily prove the first statement of the following

Theorem 3.11. *For any* $x, y \in X$, *we have*

(1) $p(x+y) \le p(x) + p(y)$, *(2)* $|\, p(x) - p(y)\, | \le p(x-y)$.

Proof. By using Theorem 3.3 and the inequality $P_1(x, y) \le p(x)\, p(y)$, we can see that

$$p(x+y)^2 = P_1(x+y, x) + P_1(x+y, y) \le p(x+y)\, p(x) + p(x+y)\, p(y).$$

Therefore, by the nonnegativity of p, (1) also holds.

Now, by using (1), we can also see that

$$p(x) = p(x - y + y) \le p(x-y) + p(y),$$

and thus $p(x) - p(y) \le p(x-y)$. Hence, by changing the roles of x and y, and using Theorem 3.3, we can infer that

$$-\left(p(x) - p(y) \right) = p(y) - p(x) \le p(y-x) = p\left(-(x-y) \right) = p(x-y).$$

Therefore, by the definition of the absolute value, (2) also holds.

Remark 3.12. On the other hand, by using Theorem 3.3, we can easily see that $p(x+y) \le p(x) + p(y)$ implies $P_1(x, y) \le p(x)\, p(y)$ for all $x, y \in X$.

Moreover, if $P_1(x, y) \le p(x)\, p(y)$ holds for all $x, y \in X$, then by using Remarks 2.5 and 3.4 we can see that

$$-P_1(x, y) = P_1(x, -y) \le p(x)\, p(-y) = p(x)\, p(y),$$

and thus $|\, P_1(x, y)\, | \le p(x)\, p(y)$ also holds for all $x, y \in X$.

Remark 3.13. Theorems 3.3 and 3.11 show that the function p is a *seminorm* on X by a terminology of [15].

Moreover, by Remark 3.6, we can see that p is a *norm* on X if and only if P is an inner product on X.

4. Some further results on the seminorms p

Theorem 4.1. *For any* $n \in \mathbb{N}$ *with* $n > 1$ *and*

$$x = \sum_{i=1}^{n} x_i \qquad \textit{with} \qquad x_i \in X,$$

we have

(1) $p(x)^2 = \sum_{i=1}^{n} P_1(x, x_i)$,

(2) $p(x)^2 = \sum_{i=1}^{n} p(x_i)^2 + \sum_{1 \leq i < j \leq n} 2 P_1(x_i + x_j)$,

(3) $p(x)^2 = \sum_{1 \leq i < j \leq n} p(x_i + x_j)^2 - \sum_{i=1}^{n} (n-2) p(x_i)^2$.

Proof. The precise proofs of these equalities, by induction on n, requires lengthy computations.

For instance, to prove (3), we must note that the $n = 2$ particular case of (3) trivially holds. Moreover, if (3) holds and $x_{n+1} \in X$, then by Theorem 3.3, the corresponding additivity property of P_1 and assertion (3), we have

$$p(x + x_{n+1})^2 = p(x)^2 + p(x_{n+1})^2 + 2 P_1(x, x_{n+1})$$

$$= p(x)^2 + p(x_{n+1})^2 + \sum_{i=1}^{n} 2 P_1(x_i, x_{n+1})$$

$$= p(x)^2 + p(x_{n+1})^2 + \sum_{i=1}^{n} \big(p(x_i + x_{n+1})^2 - p(x_i)^2 - p(x_{n+1})^2 \big)$$

$$= \sum_{1 \leq i < j \leq n} p(x_i + x_j)^2 - \sum_{i=1}^{n} (n-2) p(x_i)^2$$

$$+ p(x_{n+1})^2 + \sum_{i=1}^{n} p(x_i + x_{n+1})^2 - \sum_{i=1}^{n} p(x_i)^2 - n p(x_{n+1})^2$$

$$= \sum_{1 \leq i < j \leq n} p(x_i + x_j)^2 + \sum_{i=1}^{n} p(x_i + x_{n+1})^2$$

$$- \sum_{i=1}^{n} (n-1) p(x_i)^2 - (n-1) p(x_{n+1})^2$$

$$= \sum_{1 \leq i < j \leq n} p(x_i + x_j)^2 + \sum_{i=1}^{n} p(x_i + x_{n+1})^2 - \sum_{i=1}^{n+1} (n-1) p(x_i)^2 .$$

Therefore, to obtain

$$p(x + x_{n+1})^2 = \sum_{1 \leq i < j \leq n+1} p(x_i + x_j)^2 + \sum_{i=1}^{n+1} (n+1-2) p(x_i)^2 ,$$

it remains only to prove that

$$\sum_{1\leq i<j\leq n} p\,(\,x_i + x_j\,)^2 + \sum_{i=1}^{n} p\,(\,x_i + x_{n+1}\,)^2 = \sum_{1\leq i<j\leq n+1} p\,(\,x_i + x_j\,)^2 \,.$$

The latter assertion should also be proved by induction on n.

Remark 4.2. Fortunately, in papers [14] and [1], we only need the $n = 3$ and $n = 4$ particular cases of the above theorem.

In these particular cases of (3), we must only prove that

(a) $$p\,(\,x_1 + x_2 + x_3\,)^2 = p\,(\,x_1 + x_2\,)^2 + p\,(\,x_1 + x_3\,)^2 + p\,(\,x_2 + x_3\,)^2 - p\,(x_1)^2 - p\,(x_2)^2 - p\,(x_3)^2 \,,$$

(b) $$p\,(\,x_1 + x_2 + x_3 + x_4\,)^2 = p\,(\,x_1 + x_2\,)^2 + p\,(\,x_1 + x_3\,)^2 + p\,(\,x_1 + x_4\,)^2 + p\,(\,x_2 + x_3\,)^2 + p\,(\,x_2 + x_4\,)^2 + p\,(\,x_3 + x_4\,)^2 - 2\,p\,(x_1)^2 - 2\,p\,(x_2)^2 - 2\,p\,(x_3)^2 - 2\,p\,(x_4)^2 \,,$$

for any $x_1,\, x_2,\, x_3,\, x_4 \in X$.

Remark 4.3. By Stetkaer [16, p. 248], the above parallelepiped law (a) plays a similar role in characterizations of inner product spaces as the parallelogram identity established in assertion (2) of Theorem 3.7. Moreover, the letter one can be derived from the former one by taking $x_3 = -x_2$.

Remark 4.4. In this respect, it is also worth noticing that if f is a function of a group X to $\mathbb{C}$ such that for any $x,\, y,\, z \in Y$ we have

(A) $f\,(x+y)^2 = f\,(y+x)^2$,

(B) $f\,(x{+}y{+}z\,)^2 = f\,(x{+}y)^2 {+} f\,(x{+}z)^2 {+} f\,(y{+}z)^2 {-} f\,(x)^2 {-} f\,(y)^2 {-} f\,(z)^2$,

then by [16, Proposition 13.25] of Stetkaer there exist a unique additive function a of X to $\mathbb{C}$ and a unique symmetric, biadditive function A of X^2 to $\mathbb{C}$ such that

$$f\,(x)^2 = a\,(x) + A\,(x,\, x)$$

for all $x \in X$.

Hence, if in addition f is even, we can infer that

$$a\,(x) + A\,(x,\, x) = f\,(x)^2 = f\,(-x)^2 = a\,(-x) + A\,(-x,\, -x) = -a\,(x) + A\,(x,\, x)\,,$$

and thus $a\,(x) = 0$ for all $x \in X$. Therefore, we have $f\,(x)^2 = A\,(x,\, x)$ for all $x \in X$. Hence, if in addition f is nonnegative, we can infer that $f\,(x) = \sqrt{A\,(x,\, x\,)}$ for all $x \in X$.

Now, by using Theorem 4.1, one can also prove the following

Theorem 4.5. *Under the notations of Theorem 4.1, for any injective function* ν *of the set* $\{\,1,\, 2, \ldots,\, n\,\}$ *onto itself, we have*

$$p\,(x) = p\,\Big(\sum_{i=1}^{n} x_{\nu(i)}\Big)\,.$$

Remark 4.6. Fortunately, in the sequel, we shall only need the $n = 4$, and

$$\nu(1) = 1, \qquad \nu(2) = 4, \qquad \nu(3) = 1, \qquad \nu(4) = 2,$$

and

$$\nu(1) = 2, \qquad \nu(2) = 3, \qquad \nu(3) = 1, \qquad \nu(4) = 4,$$

particular cases of the above theorem.

That is, the consequences of Remark 4.2 and Theorem 3.3 that

(a) $p(x_1 + x_2 + x_3 + x_4) = p(x_1 + x_4 + x_3 + x_2)$,

(b) $p(x_1 + x_2 + x_3 + x_4) = p(x_2 + x_3 + x_1 + x_4)$,

hold true for all $x_1, x_2, x_3, x_4 \in X$.

5. The induced semimetrics

Definition 5.1. For any $x, y \in X$, we define

$$d(x, y) = p(-x + y).$$

Example 5.2. Note that if in particular p is as in Example 3.2, then

$$d(x, y) = \| a(x) - a(y) \|$$

for all $x, y \in X$.

The most basic properties of the function d can be listed in the following

Theorem 5.3. *For any $x, y, z, w \in X$, we have*

(1) $d(x, y) \geq 0$,

(2) $d(x, y) = d(y, x)$,

(3) $d(x + y, y + x) = 0$,

(4) $d(kx, ky) = |k| d(x, y)$,

(5) $d(x, y) = d(z + x, z + y)$,

(6) $d(x, y) = d(x + z, y + z)$,

(7) $d(x, z) \leq d(x, y) + d(y, z)$,

(8) $|d(x, y) - d(x, z)| \leq d(y, z)$,

(9) $d(x + y, z + w) \leq d(x, z) + d(y, w)$,

(10) $|d(x, y) - d(z, w)| \leq d(x, z) + d(y, w)$.

Proof. By Definition 5.1 and Theorems 3.3 and 3.11, we have

$$d(y, x) = p(-y + x) = p\big(-(-x + y)\big) = p(-x + y) = d(x, y),$$

and

$$\begin{aligned} d(x, z) = p(-x + z) &= p(-x + y - y + z) \\ &\leq p(-x + y) + p(-y + z) = d(x, y) + d(y, z). \end{aligned}$$

Therefore, assertions (2) and (7) are.

Moreover, by using Theorem 3.3, we can also see that

$$d(z+x,\ z+y) = p\big(-(z+x)+z+y\big) = p(-x-z+z+y) = p(-x+y) = d(x,\,y),$$

and

$$\begin{aligned} d(x+z,\ y+z) &= p\big(-(x+z)+y+z\big) = p\big(y+z-(x+z)\big) \\ &= p(y+z-z-x) = p(y-x) = p(-x+y) = d(x,\,y). \end{aligned}$$

Therefore, assertions (5) and (6) are also true.

Furthermore, by using Theorem 3.3 and Remark 4.8, we can see that

$$\begin{aligned} d(kx,\,ky) &= p(-kx+kx) = p\big(k(-x)+kx\big) \\ &= p\big(k(-x+y)\big) = |k|\,p(-x+y) = |k|\,d(x,\,y) \end{aligned}$$

$$\begin{aligned} d(x+y,\ y+x) &= p\big(-(x+y)+y+x\big) \\ &= p(-y-x+y+x) = p(-x+y-y+x) = p(0) = 0. \end{aligned}$$

Therefore, assertions (4) and (3) are also true.

In a more general setting, assertion (8) can be derived from (7), and (10) can be derived from (8), by using (2). However, in our present setting, we can apply more direct proofs.

For instance, by using Theorem 3.11 and Remark 4.8, we can easily see that

$$\begin{aligned} &|\,d(x,\,y)-d(z,\,w)\,| = |\,p(-x+y)-p(-z+w)\,| \\ &\quad \le p\big(-x+y-(-z+w)\big) = p(-x+y-w+z) = p(-x+z-w+y) \\ &\quad\quad \le p(-x+z)+p(-w+y) = d(x,\,y)+d(w,\,y) = d(x,\,y)+d(y,\,w). \end{aligned}$$

Therefore, assertion (10) is also true. Now, by by taking x in place of z and z in place of w in (10), we can see that (8) is also true.

Remark 5.4. The above theorem shows that d is a translation-invariant semimetric on X.

Moreover, we can also state that d is a metric on X if and only if P is an inner product on X.

Therefore, as an immediate consequence of assertion (3) of Theorem 5.3, we can also state

Corollary 5.5. *If in particular P is an inner product on X, then X is necessarily commutative.*

Remark 5.6. Note that the rectangle inequality (10) assures some continuity property of the semimetric d with respect to itself.

While, assertions (4) and (9) assure some continuity properties of two algebraic operations in X with respect to d.

References

[1] Z. Boros and Á. Száz, *A generalization of a theorem of Gilányi on quadratic functions*, in preparation.

[2] K. Baron, *On additive involutions and Hamel bases*, Aquationes Math. **87** (2014), 159–163.

[3] S. S. Dragomir, *Semi-Inner Products and Applications*, Nova Science Publishers, Hauppauge, NY, 2004.

[4] W. Fechner, *Stability of a functional inequality associated with the Jordan-von Neumann functional equation*, Aequationes Math. **71** (2006), 149–161.

[5] W. Fechner, *Hlawka's functional inequality*, Aequationes Math. **87** (2014), 71–87.

[6] A. Gilányi, *Eine zur Parallelogrammgleichung äquivalente Ungleichung* , Aequationes Math. **62** (2001), 303-309.

[7] M. Kuczma, *An Introduction to the Theory of Functional Equations and Inequalities*, Państwowe Wydawnictwo Naukowe, Warszawa, 1985.

[8] I. Makai, *Über invertierbare Lösungen der additive Cauchy-Functionalgleichung*, Publ. Math. Debrecen **16** (1969), 239–243.

[9] Gy. Maksa, *A remark of symmetric biadditive functions having nonnegative diagonalization*, Glasnik Mat. **15** (1980), 279–282.

[10] Gy. Maksa and P. Volkmann, *Characterizations of group homomorphisms having values in an inner product space*, Publ. Math. Debrecen **56** (2000), 197–200.

[11] J. Rätz, *On inequalities associated with the Jordan-von Neumann functional equation*, Aequationes Math. **66** (2003), 191-200.

[12] Á. Száz, *An instructive treatment of convergence, closure and orthogonality in semi-inner product spaces*, Tech. Rep., Inst. Math., Univ. Debrecen 2006/2, 29 pp.

[13] Á. Száz, *A natural Galois connection between generalized norms and metrics*, Tech. Rep., Inst. Math., Univ. Debrecen 2016/4, 9 pp.

[14] Á. Száz, *A generalization of a Maksa and Volkmann on additive functions*, Tech. Rep., Inst. Math., Univ. Debrecen 1016/5, 6 pp.

[15] Á. Száz, *Semimetrics and preseminorms on groups*, in preparation.

[16] H. Stetkaer, *Functional Equations on Groups*, World Scientific, New Jersey, 2013.

Árpád Száz, Department of Mathematics, University of Debrecen, H–4002 Debrecen, Pf. 400, Hungary

E-mail address: szaz@science.unideb.hu

Zoltán Boros, Department of Mathematics, University of Debrecen, H–4002 Debrecen, Pf. 400, Hungary

E-mail address: boros@science.unideb.hu