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A GENERALIZATION OF A THEOREM MAKSA AND VOLKMANN ON ADDITIVE FUNCTIONS

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Abstract. By introducing inner products on groups, we generalize a famous theorem of Gyula Maksa and Peter Volkmann on additive functions of a group to an inner product space.

1. Introduction

In this paper, by introducing inner products on groups, we shall generalize the following famous theorem of Maksa and Volkmann [7].

Theorem 1.1. *For functions $f : G \to E$ from a group G to a real or complex inner product space E, the inequality*

(1) $$\| f(xy) \| \geq \| f(x) + f(y) \| \qquad \big(x, y \in G \big)$$

implies

(2) $$f(xy) = f(x) + f(y) \qquad \big(x, y \in G \big).$$

Remark 1.2. For the origins of this theorem and the long history of the alternative Cauchy equation

(3) $$\| f(x+y) \| = \| f(x) + f(y) \|,$$

see Hosszú [5], Fischer and Muszély [3], Kurepa [6], Skof [11, 12], Schöpf [10], and Ger and Koclega [4].

2. Semi-inner products on groups

The most useful seminorms on vector spaces are derived from semi-inner products [13]. Therefore, it seems convenient to introduce the following

Definition 2.1. For any group X, a function P of X^2 to $\mathbb{C}$ will be called a *semi-inner product* on X if

(a) $P(x, x) \geq 0$ for all $x \in X$,

(b) $P(y, x) = \overline{P(x, y)}$ for all $x, y \in X$,

(c) $P(x+y, z) = P(x, z) + P(y, z)$ for all $x, y, z \in X$.

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Remark 2.2. By using (b) and (c) and the additivity of complex conjugation, we can see

$$P(x,\,y+z)=\overline{P(y+z,\,x)}=\overline{P(y,\,x)+P(z,\,x)}$$
$$=\overline{P(y,\,x)}+\overline{P(z,\,x)}=P(x,\,y)+P(x,\,z)$$

for all $x,\,y,\,z\in X$. Thus, the semi-inner product P is in particular a biadditve function of X^2 to $\mathbb{C}$.

Therefore, as an immediate consequence of some basic facts on additive functions of one group to another [16, Sec. 2.1], we can at once state the following

Theorem 2.3. *If P is a semi-inner product on a group X, then*

(1) $P(0,\,x)=0$ *and* $P(x,\,0)=0$ *for all* $x\in X$,

(2) $P(-x,\,y)=-P(x,\,y)$ *and* $P(x,\,-y)=-P(x,\,y)$ *for all* $x\in X$,

(3) $P(kx,\,y)=k\,P(x,\,y)$ *and* $P(x,\,ky)=k\,P(x,\,y)$ *for all* $k\in\mathbb{Z}$ *and* $x\in X$.

Remark 2.4. Note that the first and second coordinate functions P_1 and P_2 also have the corresponding bilinearity properties.

Moreover, because of properties (a) and (b), we have

(1) $P_1(x,\,x)=P(x,\,x)\geq 0$ and $P_2(x,\,x)=0$ for all $x\in X$,

(2) $P_1(y,\,x)=P_1(x,\,y)$ and $P_2(y,\,x)=-P_2(x,\,y)$ for all $x,\,y\in X$.

Thus, in particular P_1 is also a semi-inner product on X. However, because of its skew-symmetry, P_2 cannot be a semi-inner product on X whenever it is not identically zero.

The importance of semi-inner products lies mainly in the following

Theorem 2.5. *If P is a semi-inner product on a group X and*

$$p(x)=\sqrt{P(x,\,x)}$$

for all $x\in X$, *then*

(1) $p(0)=0$,

(2) $p(-x)=p(x)$ *for all* $x\in X$,

(3) $p(kx)=|k|\,p(x)$ *for all* $k\in\mathbb{Z}$ *and* $x\in X$,

(4) $p(x+y)^2=p(x)^2+p(y)^2+2\,P_1(x,\,y)$ *for all* $x,\,y\in X$.

Proof. To check (4), note that by the corresponding definitions and the biadditivity of P we have

$$p(x+y)^2=P(x+y,\;x+y)=P(x,\;x+y)+P(y,\;x+y)$$
$$=P(x,\,x)+P(x,\,y)+P(y,\,x)+P(y,\,y)=p(x)^2+P(x,\,y)+\overline{P(x,\,y)}+p(y)^2$$
$$=p(x)^2+2\,P(x,\,y)_1+p(y)^2=p(x)^2+2\,P_1(x,\,y)+p(y)^2$$

for all $x,\,y\in X$. Therefore, the required equality is true.

Now, by using Theorems 2.5 and 2.3, we can easily establish the following two useful corollaries.

Corollary 2.6. *If* p *is as in Theorem 2.5, then for any* $x, y \in X$ *the following assertions are equivalent:*

(1) $p(x+y) \le p(x) + p(y)$, *(2)* $P_1(x, y) \le p(x)\, p(y)$.

Proof. To prove that (1) implies (2), note that if (1) holds, then by Theorem 2.5 we have

$$2\, P_1(x, , y) = p(x+y)^2 - p(x)^2 - p(y)^2$$
$$\le \big(p(x) + p(y)\big)^2 - p(x)^2 - p(y)^2 = 2\, p(x)\, p(y).$$

Therefore, (3) also holds.

Remark 2.7. Unfortunately, now even the weakened Schwarz inequality (2) cannot be proved. Therefore, the function p considered in Theorem 2.5 need not be subadditive, and thus a seminorm.

Corollary 2.8. *If* p *is as in Theorem 2.5, then for any* $x, y \in X$ *we have*

(1) $4\, P_1(x, y) = p(x+y)^2 - p(x-y)^2$,

(2) $p(x+y)^2 + p(x-y)^2 = 2\, p(x)^2 + 2\, p(y)^2$.

Proof. To check this, note that by Theorems 2.5 and 2.3, we have

$$p(x+y)^2 = p(x)^2 + p(y)^2 + 2\, P_1(x, y)$$

and

$$p(x-y)^2 = p(x)^2 + p(-y)^2 + 2\, P_1(x, -y) = p(x)^2 + p(y)^2 - 2\, P_1(x, y).$$

Remark 2.9. Unfortunately, now a similar polar formula for $P_2(x, y)$ cannot be proved. Therefore, P can be recovered from p only in the real-valued case.

Assertion (4) of Theorem 2.5 can be extended to certain families of elements of X. However, in the sequel, we shall only need the following very particular result.

Theorem 2.10. *If* p *is as in Theorem 2.5, then for any* $x, y, z \in X$, *we have*

$$p(x+y+z)^2 = p(x)^2 + p(y)^2 + p(z)^2 + 2\, P_1(x, y) + 2\, P_1(x, z) + 2\, P_1(y, z).$$

Proof. By using Theorem 2.5, we can easily see that

$$p(x+y+z)^2 = p\big((x+y)+z\big)^2 = p(x+y)^2 + p(z)^2 + 2\, P_1(x+y, z)$$
$$= p(x)^2 + p(y)^2 + 2\, P_1(x, y) + p(z)^2 + 2\, P_1(x, z) + 2\, P_1(y, z).$$

Therefore, the required equality is also true.

Now, by using the Theorems 2.5 and 2.9, we can also easily prove the following

Corollary 2.11. *If* p *is as in Theorem 2.5, then for any* $x, y, z \in X$, *we have*

(1) $p(x)^2 + p(y)^2 + p(z)^2 + p(x+y+z)^2 = p(x+y)^2 + p(x+z)^2 + p(y+z)^2$,

(2) $p(x+y+z)^2 = P_1(x+y+z, x) + P_1(x+y+z, y) + P_1(x+y+z, z)$.

Proof. To prove (2), note that by Theorem 2.9 we have

$$\begin{aligned}
&P_1(x+y+z,\, x)+P_1(x+y+z,\, y)+P_1(x+y+z,\, z)\\
&\quad = P_1(x,\, x)+P_1(y,\, x)+P_1(z,\, x)\,+\,P_1(x,\, y)+P_1(y,\, y)+P_1(z,\, y)\\
&\qquad\qquad + P_1(x,\, z)+P_1(y,\, z)+P_1(z,\, z)\\
&\quad = p(x)^2+P_1(x,\, y)+P_1(x,\, z)\,+\,P_1(x,\, y)+p(y)^2+P_1(y,\, z)\\
&\qquad\qquad + P_1(x,\, z)+P_1(y,\, z)+p(z)^2\\
&\quad = p(x)^2+p(y)^2+p(z)^2+2\,P_1(x,\, y)+2\,P_1(x,\, z)+2\,P_1(y,\, z)\\
&\qquad\qquad\qquad\qquad = p(x+y+z)^2\,.
\end{aligned}$$

Remark 2.12. By Stetkaer [16, p. 248], the above parallelepiped law plays the same role in characterizations of inner product spaces as the parallelogram identity established in Corollary 2.8.

3. Two characterizations of additive functions

Definition 3.1. A semi-inner product P on a group X will be called an *inner product* if $P(x,\, x)=0$ implies $x=0$ for all $x\in X$.

Remark 3.2. Thus, the semi-inner product P is an inner product if and only if, under the notation of Theorem 2.5, $p(x)=0$ implies $x=0$ for all $x\in X$.

Now, we are ready to prove the following straightforward generalization of Theorem 1.1 of Maksa and Volkmann.

Theorem 3.3. *If f is a function of one group X to another Y, Q is an inner product on Y and*

$$q(y)=\sqrt{Q(y,\, y)}$$

for all $y\in Y$, then the following assertions are equivalent:

(1) f is additive,

(2) $q\big(f(x)+f(y)\big)\le q\big(f(x+y)\big)$ for all $x,\, y\in X$,

(3) f is odd and

$$2\,Q_1\big(f(x),\ f(y)\big)\,\le\, q\big(f(x+y)\big)^2-q\big(f(x)\big)^2-q\big(f(y)\big)^2$$

for all $x,\, y\in X$.

Proof. Since, (1) implies that

$$f(x)+f(y)=f(x+y)\,,\quad \text{and thus}\quad q\big(f(x)+f(y)\big)=q\big(f(x+y)\big)$$

for all $x,\, y\in X$, we need only show that (2) implies (3) implies (1).

From (2), by using Theorem 2.5, we can see that

$$2\,q\big(f(0)\big)=\,q\big(2\,f(0)\big)\le q\big(f(0)\big)\,,$$

and thus $q\big(f(0)\big)\le 0$. Therefore, $q\big(f(0)\big)=0$, and thus $f(0)=0$.

Now, from (2), we can also see that

$$q\big(f(x)+f(-x)\big)\le\, q\big(f(0)\big)=q(0)=0\,,$$

and thus $f(x) + f(-x) = 0$ for all $x \in X$. Therefore, f is odd.

Moreover, from (2), by using Theorem 2.5, we can see that

$$q\big(f(x)\big)^2 + p\big(f(y)\big)^2 + 2Q_1\big(f(x),\ f(y)\big) = q\big(f(x)+f(y)\big)^2 \leq q\big(f(x+y)\big)^2,$$

and thus

$$2Q_1\big(f(x),\ f(y)\big) \leq q\big(f(x+y)\big)^2 - q\big(f(x)\big)^2 - q\big(f(y)\big)^2$$

for all $x,\ y \in X$. Therefore, (2) implies (3).

On the other hand, if (3) holds, then by using Theorem 2.10, 2.5 and 2.3 we can see that

$$\begin{aligned}
&q\big(f(x) + f(y) - f(x+y)\big)^2 - q\big(f(x)\big)^2 - q\big(f(y)\big)^2 - q\big(f(x+y)\big)^2 \\
&= 2\ Q_1\big(f(x),\ f(y)\big) + 2\ Q_1\big(f(x), -f(x+y)\big) + 2\ Q_1\big(f(y), -f(x+y)\big) \\
&= 2\ Q_1\big(f(x),\ f(y)\big) + 2\ Q_1\big(f(-x),\ f(x+y)\big) + 2\ Q_1\big(f(x+y),\ f(-y)\big) \\
&\leq q\big(f(x+y)\big)^2 - q\big(f(x)\big)^2 - q\big(f(y)\big)^2 + \\
&q\big(f(y)\big)^2 - q\big(f(-x)\big)^2 - q\big(f(x+y)\big)^2 + q\big(f(x)\big)^2 - q\big(f(x+y)\big)^2 - q\big(f(-y)\big)^2 \\
&= q\big(f(x+y)\big)^2 - q\big(f(x)\big)^2 - q\big(f(y)\big)^2 + \\
&q\big(f(y)\big)^2 - q\big(f(x)\big)^2 - q\big(f(x+y)\big)^2 + q\big(f(x)\big)^2 - q\big(f(x+y)\big)^2 - q\big(f(y)\big)^2 \\
&= -q\big(f(x)\big)^2 - q\big(f(y)\big)^2 - q\big(f(x+y)\big)^2
\end{aligned}$$

and thus

$$q\big(f(x) + f(y) - f(x+y)\big)^2 \leq 0$$

for all $x,\ y \in X$. Hence, we can already infer that

$$f(x) + f(y) - f(x+y) = 0$$

for all $x,\ y \in X$. Therefore, (1) also holds.

Remark 3.4. Note that the above proof does not requires particular tricks. Therefore, it is more simple than the one given by Maksa and Volkmann [7].

Application of Theorem 3.3 to the proof of the left-invariance of some particular generalized metrics will be given in a forthcoming technical report [15].

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