A NATURAL GALOIS CONNECTION BETWEEN GENERALIZED NORMS AND METRICS

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Abstract. Having in mind a well-known connection between norms and metrics in vector spaces, for an additively written group X, we establish a natural Galois connection between functions of X to $\mathbb{R}$ and X^2 to $\mathbb{R}$.

1. Introduction

In this paper, for an additively written group X, we shall consider the sets

$$\mathcal{N} = \mathcal{N}(X) = \mathbb{R}^X \qquad \text{and} \qquad \mathcal{M} = \mathcal{M}(X) = \mathbb{R}^{X^2}.$$

to be equipped with the usual pointwise inequality of real-valued functions.

Moreover, having in mind a well-known connection between norms and metrics in vector spaces, for any $p \in \mathcal{N}$, $d \in \mathcal{M}$ and $x, y \in X$ we define

$$p_d(x) = d(0, x) \qquad \text{and} \qquad d_p(x, y) = p(-x+y).$$

Thus, it can be easily seen that, for any $p \in \mathcal{N}$ and $d \in \mathcal{M}$,

(1) $d_p \le d \implies p \le p_d$, (2) $p \le p_d \implies d_p \le d_{p_d}$.

Moreover, if in particular

$$d(x, y) = |\varphi(x) - \varphi(y)|, \qquad \text{with} \qquad \varphi(x) = x/(1+|x|),$$

for all $x, y \in \mathbb{R}$, then d is a metric on $\mathbb{R}$ such that $d_{p_d} \not\le d$, despite that $p = p_{d_p}$ for all $p \in \mathcal{N}$.

Therefore, by defining

$$\mathcal{M}^\wedge = \mathcal{M}^\wedge(X) = \left\{ d \in \mathcal{M}(X) : \quad d_{p_d} \le d \right\},$$

we can note that the functions, defined by

$$f(p) = d_p \qquad \text{and} \qquad g(d) = p_d$$

for all $p \in \mathcal{N}$ and $d \in \mathcal{M}^\wedge$, establish an increasing Galois connection [22] between the posets $\mathcal{N}$ and $\mathcal{M}^\wedge$ in the sense that, for any $p \in \mathcal{N}$ and $d \in \mathcal{M}^\wedge$, we have

$$f(p) \le d \iff p \le g(d).$$

To feel the importance of this Galois connection, note that if in particular $p \in \mathcal{N}$ is a preseminorm [16] on X in the sense that

(1) $p(0) \le 0$, (2) $p(-x) \le p(x)$, (3) $p(x+y) \le p(x) + p(y)$

2010 *Mathematics Subject Classification.* Primary O6A15; Secondary 20A99, 54E25.

Key words and phrases. Groups, preseminorms, invariant semimetrics, Galois connections.

The work of the author has been supported by the Hungarian Scientific Research Fund (OTKA) Grant K-111651.