GENERALIZATIONS OF A RESTRICTED STABILITY THEOREM OF LOSONCZI ON CAUCHY DIFFERENCES TO GENERALIZED COCYCLES

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Abstract. As a main result of this paper, we shall show that a natural generalization of a restricted stability theorem of László Losonczi on Cauchy differences to symmetric semi-cocycles can be derived from a similar generalization of an asymptotic stability theorem of Anna Bahyrycz, Zsolt Páles and Magdalena Piszczek.

For this, by using our former results, we shall prove that if F is a symmetric semi-cocycle on an unbounded commutative preseminormed group X to an arbitrary commutative preseminormed group Y, and S is a relation on X such that the intersection of the domain and the range of S is bounded, then

$$\sup_{z \in X^2} \| F(z) \| \leq 5 \sup_{z \in S^c} \| F(z) \| .$$

1. Introduction

In [30], generalizing and sharpening a restricted stability theorem of Skof [36], László Losonczi has proved the following

Theorem 1.1. *Let X be a normed linear space, Y a Banach space and $\varepsilon \geq 0$. Let further B be a subset of X^2 such that the first (or second) coordinates of the points of B form a bounded set.*

If $g : X \to Y$ satisfies the inequality

$$\| g(x+y) - g(x) - g(y) \| \leq \varepsilon \qquad \big((x, y) \in X^2 - B \big),$$

then there exists a unique function $A : X \to Y$ additive on X^2, that is

$$A(x+y) - A(x) - A(y) = 0 \qquad \big((x, y) \in X^2 \big),$$

such that

$$\| g(x) - A(x) \| \leq 5\,\varepsilon \qquad \big(x \in X \big).$$

Remark 1.2. Here, analogously to the forthcoming theorems, it is also necessary to assume that X is unbounded. That is, $X \neq \{0\}$.

Namely, if this not the case, then by taking $B = \{(0, 0)\}$ we can see that any function g of X to Y satisfies the condition of the theorem. While, $f = B$, the only additive function of X to Y, need not have the required property.

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Now, we shall show that a natural generalization of the above theorem to symmetric semi-cocycles can be derived from a similar generalization of the following asymptotic stability theorem of Bahyrycz, Páles and Piszczek [3].

Theorem 1.3. *Let $(X, +, d)$ and $(Y, +, \rho)$ be metric abelian groups such that X is unbounded by d. Let $\varepsilon \geq 0$ and assume that $f : X \to Y$ possesses the following asymptotic stability property*

$$\limsup_{\min(\|x\|_d, \|y\|_d) \to \infty} \| f(x+y) - f(x) - f(y) \|_\rho \leq \varepsilon,$$

then

$$\| f(x+y) - f(x) - f(y) \|_\rho \leq 5\varepsilon \qquad \text{for all} \quad x, y \in X.$$

Remark 1.4. Moreover, by taking $\varepsilon > 0$ and $x_0 \in X \setminus \{0\}$, and defining $f(x_0) = 3\varepsilon$ and $f(x) = \varepsilon$ for $x \in X \setminus \{x_0\}$, they have also proved that 5 is the smallest possible constant in their theorem.

More concretely, as a main result of this paper, we shall prove the following

Theorem 1.5. *If F is a symmetric semi-cocycle on an unbounded commutative preseminormed group X to an arbitrary commutative preseminormed group Y, and S is a relation on X such that the intersection of the domain and the range of S is bounded, then*

$$\sup_{z \in X^2} \| F(z) \| \leq 5 \sup_{z \in S^c} \| F(z) \|.$$

Remark 1.6. Hence, by using a simple extension of the classical Hyers theorem [20], a straightforward generalization of Theorem 1.1 can be immediately derived.

2. A few basic facts on relations and preseminorms

A subset F of a product set $X \times Y$ is called a *relation* on X to Y. If in particular $F \subseteq X^2$, with $X^2 = X \times X$, then we simply say that F is a relation on X. In particular, $\Delta_X = \{(x, x) : x \in X\}$ is called the *identity relation* on X.

If F is a relation on X to Y, then for any $x \in X$ and $A \subseteq X$ the sets $F(x) = \{y \in Y : (x, y) \in F\}$ and $F[A] = \bigcup_{a \in A} F(a)$ are called the *images* of x and A under F, respectively.

Moreover, the sets $D_F = \{x \in X : F(x) \neq \emptyset\}$ and $R_F = F[X]$ are called the *domain* and *range* of F, respectively. If in particular $D_F = X$, then we say that F is a relation of X to Y, or that F is a *total relation* on X to Y.

In particular, a relation f on X to Y is called a *function* if for each $x \in D_f$ there exists $y \in Y$ such that $f(x) = \{y\}$. In this case, by identifying singletons with their elements, we may simply write $f(x) = y$ in place of $f(x) = \{y\}$.

Moreover, a function $\star$ of X to itself is called a *unary operation* on X. While, a function $*$ of X^2 to X is called a *binary operation* on X. And, for any $x, y \in X$, we usually write $x^\star$ and $x * y$ instead of $\star(x)$ and $*\big((x, y)\big)$.

Furthermore, a function d of X^2 to $\mathbb{R}$ is called a *distance function* on X. And, for any $r > 0$, the relation $B_r^d = \{(x, y) : d(x, y) < r\}$ is called the *r-sized surrounding* on X generated by d.

If F is a relation on X to Y, then $F = \bigcup_{x \in X} \{x\} \times F(x)$. Therefore, the values $F(x)$, where $x \in X$, uniquely determine F. Thus, a relation F on X to Y can also be naturally defined by specifying $F(x)$ for all $x \in X$.

For instance, the *complement relation* F^c can be naturally defined such that $F^c(x) = F(x)^c = Y \setminus F(x)$ for all $x \in X$. Thus, we also have $F^c = X \times Y \setminus F$. And, it is noteworthy that $F^c[A]^c = \bigcap_{a \in A} F(a)$ for all $A \subseteq X$.

Quite similarly, the *inverse relation* F^{-1} can be naturally defined such that $F^{-1}(y) = \{x \in X : \ y \in F(x)\}$ for all $y \in Y$. Thus, we also have $F^{-1} = \{(y, x) : (x, y) \in F\}$. And, it is noteworthy that $(F^c)^{-1} = (F^{-1})^c$.

Moreover, if in addition G is a relation on Y to Z, then the *composition relation* $G \circ F$ can be naturally defined such that $(G \circ F)(x) = G[F(x)]$ for all $x \in X$. Thus, we also have $(G \circ F)[A] = G\big[F[A]\big]$ for all $A \subseteq X$.

For any $A \subseteq X$, the *Pervin relation* $R_A = A^2 \cup A^c \times X$ is a preorder (reflexive and transitive) relation on X in the sense that $\Delta_X \subseteq R_A$ and $R_A \circ R_A \subseteq R_A$ [41]. Thus, R_A is idempotent in the sense that $R_A = R_A^2$ with $R_A^2 = R_A \circ R_A$.

A distance function d on X will be called a *semimetric* if it is symmetric and triangular in the sense that $d(x, y) = d(y, x)$ and $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$. (The assumption $d(x, x) = 0$ will not be needed.)

In particular, a function p on a group X to $\mathbb{R}$ will be called a *preseminorm* on X if it is even and subadditive in the sense that $p(-x) = p(x)$ and $p(x+y) \leq p(x) + p(y)$ for all $x, y \in X$. (The assumption $p(0) = 0$ will not be needed.)

If a semimetric d on X is *left-invariant* in the sense that $d(z+x, z+y) = d(x\ y)$ for all $x, y, z \in X$, and $p_d(x) = d(0, x)$ for all $x \in X$, then it can be easily seen that p_d is a preseminorm on X.

Conversely, if p is a preseminorm on X and $d_p(x, y) = p(-x+y)$ for all $x\ y \in X$, then it can be easily seen that d_p is a left-invariant semimetric on X such that $|\,p(x) - p(y)\,| \leq d_p(x, y)$ for all $x, y \in X$.

Under the pointwise inequality, for any preseminorm p and left-invariant semimetric d on X, we have $d_p \leq d$ if and only if $p \leq p_d$. Therefore, the mappings $p \mapsto d_p$ and $d \mapsto p_d$ form a *Galois connection* [43].

Thus, in particular we have $p = p_{d_p}$ and $d = d_{p_d}$ for any preseminorm p and left-invariant semimetric d on X. Therefore, in the group X, preseminorms and left-invariant semimetrics are equivalent tools.

However, the former ones, being functions of only one variable, are more convenient tools than the latter ones. Therefore, in contrast to several former authors, we shall use preseminorms instead of left-invariant semimetrics.

If p is a preseminorm on X, then by using induction and the corresponding definitions we can easily see that $p(nx) \leq n\,p(x)$, and thus $p((-n)x) = p(n(-x)) \leq n\,p(-x) = n\,p(x)$ for all $n \in \mathbb{N}$ and $x \in X$.

Therefore, the preseminorm $\|\ \|$ may be naturally called a *seminorm* if $n\|x\| \leq \|nx\|$ for all $x \in X$. Namely, thus we also have $\|kx\| = |k|\,\|x\|$ for all $x \in X$ and $k \in \mathbb{Z} \setminus \{0\}$. (Clearly, if $\|0\| = 0$, then this also holds for $k = 0$.)

To see one advantage of preseminorms over seminorms, note that a nonzero seminorm $\|\ \|$ on X cannot be bounded. While, for instance, the function $\|\ \|^\star$ defined by $\|x\|^\star = \min\{1, \|x\|\}$ for all $x \in X$ is a bounded preseminorm on X.

Now, a seminorm (preseminorm) p on X may be naturally called a *norm (prenorm)* if $p(x) = 0$ implies $x = 0$ for all $x \in X$. Note that if $X = \mathbb{Z}x$ for all $x \in X \setminus \{0\}$, then each nonzero preseminorm on X is a prenorm.

In the sequel, for instance, an ordered pair $X(p) = (X, p)$ consisting a group X and a preseminorm p, will be called a preseminormed group. Moreover, we shall write $\| x \|$ instead of $p(x)$ for any point x and preseminorm p.

3. A few basic facts on cocycles

Notation 3.1. *In this and the subsequent section, we shall assume that F is a function of one commutative group X to another Y.*

Remark 3.2. Note that now, by defining $(x, y) + (z, w) = (x + z, y + w)$ for all $x, y, z, w \in X$, the set X^2 can also be turned into a commutative group.

Definition 3.3. Under Notation 3.1, we shall say that:

(1) F is *symmetric (skew-symmetric)* if

$$F(x, y) = F(y, x) \qquad \big(F(x, y) = -F(y, x) \big)$$

for all $x, y \in X$,

(2) F is *cocyclic* if

$$F(x, y) + F(x+y, z) = F(x, y+z) + F(y, z)$$

for all $x, y, z \in X$.

Remark 3.4. If F is a cocyclic function of X^2 to Y, then according to Davison and Ebanks [10] and Stetkaer [39, p. 280], we may also say that F is a cocycle on X to Y.

The following two simple theorems give some useful examples for cocycles. The first one was already used by Erdős [12, p. 5] and Aczél [1, p. 66].

Theorem 3.5. *If F is biadditive in the sense that it is additive in each of its variable, then F is cocyclic.*

Remark 3.6. Conversely, in [44], it has been proved that if F is cocyclic and additive in one of its variables, then F is biadditive.

Moreover, it can be shown that if Y is 2–cancelable in the sense that $2y = 2z$ implies $y = z$ for all $y, z \in Y$, then every skew-symmetric cocycle on X to Y is biadditive. (For the origins of this result, see [12, 17, 1, 10].)

The following theorem has been first proved in [44] by the present author.

Theorem 3.7. *If F is additive as a function of X^2 to Y and $F(x, 0) = F(0, y)$ for all $x, y \in X$, then F is cocyclic.*

Remark 3.8. Conversely, it can be shown that if F is cocyclic, then

$$F(x, 0) = F(0, 0) \qquad \text{and} \qquad F(0, y) = F(0, 0)$$

for all $x, y \in X$. (This was already stated in [23, p. 258] and [10, Lemma 1].)

Definition 3.9. The function F will be called *Cauchy* if there exists a function f of X to Y such that

$$F(x, y) = f(x+y) - f(x) - f(y)$$

for all $x, y \in X$.

Remark 3.10. In this case, by Davison and Ebanks [10] and Stetkaer [39, pp. 16, 280], we should say that F is the Cauchy difference (or kernel) of f, or that F is a coboundary with generator function $-f$.

In [44], in addition to Definition 3.9, we have also introduced the following

Definition 3.11. The function F will be called *quasi-Cauchy* if it is both symmetric and cocyclic.

The appropriateness of this definition is apparent from the following simple, but important theorem.

Theorem 3.12. *If F is Cauchy, then it is quasi-Cauchy.*

Remark 3.13. The $X = Y = \mathbb{R}$ particular case of this theorem was already established by Kurepa [29] who, having in mind the case of additive functions, conjectured that the converse statement need not be true.

Jenő Erdős [12], answering the question of Kurepa [29], proved that an arbitrary cocycle on $\mathbb{R}$ need not be symmetric. Moreover, by using a theorem of O. Schreier on group extensions and a theorem R. Baer on direct sums, he proved that every quasi-Cauchy function of $\mathbb{R}^2$ to $\mathbb{R}$ is Cauchy.

By the proof of Erdős [12] and a theorem of Jessen, Karp and Thorup [23], more generally we can also state that if Y is n–divisible for all $n \in \mathbb{N}$ in the sense that $Y = nY$ (i.e., for each $y \in Y$ there exists $z \in Y$ such that $y = nz$), then every quasi-Cauchy function of X^2 to Y is already Cauchy.

4. A few basic facts on generalized cocycles

In [44], motivated by some observations of Davison and Ebanks [10] and Bahyrycz and Páles and Piszczek [3], we have also introduced the following

Definition 4.1. In addition to (2) in Definition 3.3, we shall say that:

(1) F is *semi-cocyclic* if

$$F(x, y) + F(u, y+v) + F(x+y, u+v) = F(x, u) + F(y, u+v) + F(x+u, y+v)$$

for all $x, y, u, v \in X$,

(2) F is *pseudo-cocyclic* if

$$F(x, y) + F(x-u, u) + F(y-v, u) + F(y-v, v) = F(u, v) + F(u, y-v) + F(x-u, y-v) + F(x+y-u-v, u+v)$$

for all $x, y, u, v \in X$.

Remark 4.2. Now, analogously to Definition 3.11, the function F may also be naturally called *semi-Cauchy (pseudo-Cauchy)* if it is both symmetric and semi-cocyclic (pseudo-cocyclic).

Moreover, because of the terms $F(y-v, u)$ and $F(u, y-v)$ in (2), we may also naturally introduce the following

Definition 4.3. The function F will be called *Cauchy-like* if

$$F(x, y) + F(x-u, u) + F(y-v, v)$$
$$= F(u, v) + F(x-u, y-v) + F(x+y-u-v, u+v).$$

for all $x, y, u, v \in X$.

Remark 4.4. Namely, thus we can at once state that a pseudo-Cauchy function is Cauchy-like, and a symmetric Cauchy-like function is pseudo-Cauchy.

Moreover, by using the substitutions $x = s+u$ and $y = t+v$, suggested by Gyula Maksa, we can easily prove the following two theorems of [44].

Theorem 4.5. *If F is additive in its second variable, then the following assertions are equivalent:*

(1) F is semi-cocyclic, *(2) F is pseudo-cocyclic.*

Theorem 4.6. *The following assertions are equivalent:*

(1) F is Cauchy-like,

(2) for any $x, y, u, v \in X$, we have

$$F(x, y) + F(u, v) + F(x+y, u+v)$$
$$= F(x, u) + F(y, v) + F(x+u, y+v).$$

Remark 4.7. Note that the latter equation, which is closely related to (1) in Definition 4.1, is only a rearrangement of an equation of Davison and Ebanks given in [10, Lemma 2].

In [44], by using some lengthy computations, we have also proved the following theorem which shows the appropriateness of our former definitions.

Theorem 4.8. *If F is cocyclic, then it is both semi-cocyclic and pseudo-cocyclic.*

Hence, by using Remark 4.2 and Theorem 4.6, we can already derive

Corollary 4.9. *If F is quasi-Cauchy, then its is also semi-Cauchy, pseudo-Cauchy and Cauchy-like.*

Remark 4.10. Note that the latter statement is only a reformulation of [10, Lemma 2] of Davison and Ebanks.

5. Some natural analogues and generalizations of Theorem 1.3

Notation 5.1. *In this and the subsequent section, we shall assume that F is a function of an unbounded commutative preseminormed group X to an arbitrary commutative preseminormed group Y.*

Remark 5.2. Note that now, by defining $\|(x, y)\| = \|x\| + \|y\|$ or

$$\|(x, y)\| = \|x\| \vee \|y\| = \max\{\|x\|, \|y\|\}$$

for all $x, y \in X$, the group X^2 can also be turned into an unbounded preseminormed group.

In [45], by using some more simple arguments than that used by Bahyrycz, Páles and Piszczek in [3], we have proved the following natural analogue of Theorem 1.3.

Theorem 5.3. *If F is semi-cocyclic (pseudo-cocyclic) and*

$$\varepsilon = \overline{\lim}_{\| z \| \to +\infty} \| F(z) \| ,$$

then

$$\| F(z) \| \leq 5\varepsilon \qquad \Big(\| F(z) \| \leq 7\varepsilon \Big)$$

for all $z \in X^2$.

Thus, in particular, we can also state

Corollary 5.4. *If Y is prenormed, F is either semi-cocyclic or pseudo-cocyclic, and*

$$\overline{\lim}_{\|z\| \to +\infty} \| F(z) \| = 0,$$

then $F(z) = 0$ for all $z \in X^2$.

Remark 5.5. In [45] , in addition to the natural preseminorms given in Remark 5.2, we have also defined

$$\nparallel (x, y) \nparallel = \| x \| \wedge \| y \| = \min\big\{ \| x \| , \| y \| \big\} ,$$

for all $x, y \in X$.

Thus, the function $\nparallel \ \nparallel$ is not a preseminorm on X^2. However, it can be used to prove the following natural generalization of Theorem 1.3. (See [45] .)

Theorem 5.6. *If F is pseudo-Cauchy (pseudo-cocyclic) and*

$$\varepsilon = \overline{\lim}_{\nparallel z \nparallel \to +\infty} \| F(z) \| ,$$

then

$$\| F(z) \| \leq 5\varepsilon \qquad \big(\| F(z) \| \leq 7\varepsilon \big)$$

for all $z \in X^2$.

Thus, in particular, we can also state

Corollary 5.7. *If Y is prenormed, F is pseudo-cocyclic and*

$$\overline{\lim}_{\nparallel z \nparallel \to +\infty} \| F(z) \| = 0,$$

then $F(z) = 0$ for all $z \in X^2$.

Remark 5.8. Note that $\| z \| \geq \nparallel z \nparallel$ for all $z \in X^2$. Therefore,

$$\big\{ F(z) : \ \nparallel z \nparallel > r \big\} \subseteq \big\{ F(z) : \ \| z \| > r \big\} ,$$

and thus

$$\sup_{\nparallel z \nparallel > r} \| F(z) \| \leq \sup_{\| z \| > r} \| F(z) \|$$

for all $r > 0$. Consequently,

$$\overline{\lim}_{\nparallel z \nparallel \to +\infty} \| F(z) \| = \inf_{r>0} \sup_{\nparallel z \nparallel > r} \| F(z) \| \leq \inf_{r>0} \sup_{\| z \| > r} \| F(z) \| = \overline{\lim}_{\| z \| \to +\infty} \| F(z) \| .$$

Therefore, for instance, the second part of Theorem 5.6 is stronger than that of Theorem 5.3.

6. A natural generalization of Theorem 1.1

Now, we shall show that Theorem 1.1 can already be derived from a particular case Theorem 5.3.

For this, it is convenient to prove first the following intermediate theorem which is of some interest for itself.

Theorem 6.1. *If F is semi-Cauchy and S is a relation on X such that $D_S \cap R_S$ is a bounded, then*

$$\sup_{z \in X^2} \| F(z) \| \leq 5 \sup_{z \in S^c} \| F(z) \| .$$

Proof. Define

$$\varepsilon = \sup_{z \in S^c} \| F(z) \| .$$

Then, in particular, we have $\| F(x, y) \| \leq \varepsilon$ for all $(x, y) \in S^c$. (Note that, because of the boundedness conditions on X and $D_S \cap R_S$, we have $S \neq X^2$, and thus $S^c \neq \emptyset$. Therefore, $\varepsilon \neq -\infty$, and thus $0 \leq \varepsilon \leq +\infty$ by the nonnegativity of the preseminorm in Y.)

Hence, by using that

$$(x, y) \in \big(S^{-1} \big)^c \implies (x, y) \notin S^{-1} \implies (y, x) \notin S$$
$$\implies (y, x) \in S^c \implies \| F(y, x) \| \leq \varepsilon \implies \| F(x, y) \| \leq \varepsilon$$

for all $x, y \in X$, we can see that $\| F(x, y) \| \leq \varepsilon$ also holds for all $(x, y) \in S^c \cup \big(S^{-1} \big)^c$, and thus also for all $(x, y) \in \big(S \cap S^{-1} \big)^c$.

Moreover, we can note that

$$D_{S \cap S^{-1}} \subseteq D_S \cap D_{S^{-1}} = D_S \cap R_S \quad \text{and} \quad R_{S \cap S^{-1}} = D_{(S \cap S^{-1})^{-1}} = D_{S \cap S^{-1}} \subseteq D_S \cap R_S .$$

Therefore,

$$S \cap S^{-1} \subseteq D_{S \cap S^{-1}} \times R_{S \cap S^{-1}} \subseteq \big(D_S \cap R_S \big)^2 .$$

Now, since $D_S \cap R_S$ is bounded, we can see that there exists $r > 0$ such that $D_S \cap R_S \subseteq B_r(0)$, and thus

$$S \cap S^{-1} \subseteq \big(D_S \cap R_S \big)^2 \subseteq B_r(0)^2 , \qquad \text{whence} \qquad \big(B_r(0)^2 \big)^c \subseteq \big(S \cap S^{-1} \big)^c .$$

Moreover, we can note that

$$\big(B_r(0)^2 \big)^c = \big(B_r(0) \times B_r(0) \big)^c = B_r(0)^c \times X \,\cup\, X \times B_r(0)^c .$$

Therefore,

$$B_r(0)^c \times X \,\cup\, X \times B_r(0)^c = \big(B_r(0)^2 \big)^c \subseteq \big(S \cap S^{-1} \big)^c .$$

Now, if $x, y \in X$ such that $\| (x, y) \| > r$ holds with

$$\| (x, y) \| = \| x \| \vee \| y \| = \max\big\{ \| x \| , \| y \| \big\} ,$$

then we can note that either $\| x \| > r$ or $\| y \| > r$, and thus $x \in B_r(0)^c$ or $y \in B_r(0)^c$. Therefore,

$$(x, y) \in \big(B_r(0)^c \times X \big) \cup \big(X \times B_r(0)^c \big) \qquad \text{and thus} \qquad (x, y) \in \big(S \cap S^{-1} \big)^c .$$

Hence, by our former observation, it follows that $\| F(x, y) \| \leq \varepsilon$.

This shows that there exists $r > 0$ such that $\| F(z) \| \leq \varepsilon$ for all $z \in X^2$ with $\| z \| > r$. Hence, it is clear that

$$\overline{\lim_{\| z \| \to +\infty}} \| F(z) \| = \inf_{r>0} \sup_{\|z\|>r} \| F(z) \| \leq \varepsilon .$$

Therefore, by Theorem 5.3, we have $\| F(z) \| \leq 5\varepsilon$ for all $z \in X^2$. Thus, the required inequality is also true.

From the above theorem, by using a simple extension of the classical Hyers theorem [20], we can immediately derive the following straightforward generalization of Theorem 1.1.

Corollary 6.2. *If f is a function of X to a Banach space Z and $\varepsilon \geq 0$ such that there exists a relation S on X such that $D_S \cap R_S$ is bounded and*

$$\| f(x+y) - f(x) - f(y) \| \leq \varepsilon$$

for all $(x, y) \in S^c$, then there exists a unique additive function g of X to Z such that

$$\| f(x) - g(x) \| \leq 5\,\varepsilon$$

for all $x, y \in X$.

Proof. Define

$$F(x, y) = f(x+y) - f(x) - f(y)$$

for all $x, y \in X$. Then, we can note that F is quasi-Cauchy function such that $\| F(z) \| \leq \varepsilon$ for all for all $z \in S^c$.

Therefore, by the corresponding particular case of Theorem 6.1, we also have $\| F(z) \| \leq 5\varepsilon$ for all $z \in X^2$, and thus

$$\| f(x+y) - f(x) - f(y) \| \leq 5\,\varepsilon$$

for all $x, y \in X$. Hence, by an immediate generalization of the classical Hyers theorem [20], it is clear that the required assertion is also true.

Remark 6.3. An interesting partial generalization of Theorem 1.1 of Losonczi has also been been proved by Jung [25]. (See also the $a = b = 1$ particular case of [26, Theorem 3.1].)

7. Some reasonable modifications of Theorem 6.1

By using a more simple argument than that in the proof of Theorem 6.1, we can also prove the following

Theorem 7.1. *If F is semi-cocyclic (pseudo-cocyclic) and S is a relation on X such that both D_S and R_S are bounded, then*

$$\sup_{z\in X^2} \| F(z) \| \leq 5 \sup_{z\in S^c} \| F(z) \| \qquad \Big(\sup_{z\in X^2} \| F(z) \| \leq 7 \sup_{z\in S^c} \| F(z) \| \Big).$$

Hence, analogously to Corollary 6.2, we can also derive

Corollary 7.2. *If f is a function of X to a Banach space Z and $\varepsilon \geq 0$ such that there exists a relation S on X such that both D_S and R_S are bounded and*

$$\| f(x+y) - f(x) - f(y) \| \leq \varepsilon$$

for all $(x, y) \in S^c$, then there exists a unique additive function g of X to Z such that

$$\| f(x) - g(x) \| \leq 5\,\varepsilon \qquad \Big(\| f(x) - g(x) \| \leq 7\,\varepsilon \Big).$$

for all $x, y \in X$.

Moreover, in addition to Theorems 6.1 and 7.1, we can also prove the following

Theorem 7.3. *If F is pseudo-Cauchy (pseudo-cocyclic) and S is a relation on X such that either D_S or R_S is bounded, then*

$$\sup_{z \in X^2} \| F(z) \| \leq 5 \sup_{z \in S^c} \| F(z) \| \qquad \Big(\sup_{z \in X^2} \| F(z) \| \leq 7 \sup_{z \in S^c} \| F(z) \| \Big).$$

Proof. As in the proof of Theorem 6.1, define

$$\varepsilon = \sup_{z \in S^c} \| F(z) \|.$$

Then, in particular, we have $\| F(x, y) \| \leq \varepsilon$ for all $(x, y) \in S^c$. (Note that, because of the boundedness conditions on X, D_S and R_S, now we again have $S \neq X^2$. Therefore, we can also state that $0 \leq \varepsilon \leq +\infty$.)

Moreover, since either D_S or R_S is bounded, we can see that there exists $r > 0$ such that either

$$D_S \subseteq B_r(0) \qquad \text{or} \qquad R_S \subseteq B_r(0).$$

(Namely, otherwise for any $r > 0$ we would have both $D_S \not\subseteq B_r(0)$ and $R_S \not\subseteq B_r(0)$. Thus, both D_S and R_S would be unbounded.)

Therefore, we have either

$$S \subseteq D_S \times R_S \subseteq B_r(0) \times X \qquad \text{or} \qquad S \subseteq D_S \times R_S \subseteq X \times B_r(0).$$

Hence, we can infer that

$$S \subseteq B_r(0) \times X \cup X \times B_r(0), \quad \text{and thus} \quad \big(B_r(0) \times X \cup X \times B_r(0) \big)^c \subseteq S^c.$$

Moreover, by using some useful laws on complements, we can see that

$$\begin{aligned} \big(B_r(0) \times X \cup X \times B_r(0) \big)^c &= \big(B_r(0) \times X \big)^c \cap \big(X \times B_r(0) \big)^c \big)^c \\ &= B_r(0)^c \times X \cap X \times B_r(0)^c = B_r(0)^c \times B_r(0)^c = \big(B_r(0)^c \big)^2. \end{aligned}$$

Therefore,

$$\big(B_r(0)^c \big)^2 = \big(B_r(0) \times X \cup X \times B_r(0) \big)^c \subseteq S^c.$$

Now, if $x, y \in X$ such that $\nparallel (x, y) \nparallel > r$ holds with

$$\nparallel (x, y) \nparallel = \| x \| \wedge \| y \| = \min\{ \| x \|, \| y \| \},$$

then we can note that $\| x \| > r$ and $\| y \| > r$, and thus $x \in B_r(0)^c$ and $y \in B_r(0)^c$. Therefore,

$$(x, y) \in \big(B_r(0)^c \big)^2 \subseteq S^c, \qquad \text{and thus} \qquad \| F(x, y) \| \leq \varepsilon.$$

This shows that there exists $r > 0$ such that $\| F(z) \| \leq \varepsilon$ for all $z \in X^2$ with $\nparallel z \nparallel > r$. Hence, it is clear that

$$\overline{\lim_{\nparallel z \nparallel \to +\infty}} \| F(z) \| = \inf_{r>0} \sup_{\nparallel z \nparallel > r} \| F(z) \| \leq \varepsilon .$$

Therefore, by Theorem 5.6, we have

$$\| F(z) \| \leq 5\varepsilon \qquad \Big(\| F(z) \| \leq 7\varepsilon \Big)$$

for all $z \in X^2$. Thus, the required equalities are also true.

From this theorem, analogously to Corollary 6.2, we can also derive

Corollary 7.4. *If f is a function of X to a Banach space Z and $\varepsilon \geq 0$ such that there exists a relation S on X such that either D_S or R_S is bounded and*

$$\| f(x+y) - f(x) - f(y) \| \leq \varepsilon$$

for all $(x, y) \in S^c$, then there exists a unique additive function g of X to Z such that

$$\| f(x) - g(x) \| \leq 5\,\varepsilon \qquad \Big(\| f(x) - g(x) \| \leq 7\,\varepsilon \Big).$$

for all $x, y \in X$.

Remark 7.5. Note that if $\eta \geq 0$ and f is an arbitrary and g is an additive function of X to Y such that

$$\| f(x) - g(x) \| \leq \eta$$

for all $x \in X$, then we can only state that

$$\begin{aligned} \| f(x+y) - f(x) - f(y) \| &= \| f(x, y) - g(x+y) + g(x) - f(x) + g(y) - f(y) \| \\ &\leq \| f(x+y) - g(x+y) \| + \| g(x) - f(x) \| + \| g(y) - f(y) \| \leq 3\,\eta \end{aligned}$$

for all $x, y \in X$. Therefore, the corresponding particular case of Theorem 7.3 is sharper than Corollary 7.4.

This clearly reveal that the corresponding theorems on restricted stability have to split into two parts. The same idea is also apparent from the proofs of those theorems.

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