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Two natural generalizations of cocycles

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TWO NATURAL GENERALIZATIONS OF COCYCLES

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ABSTRACT. In this paper, inspired by a theorem of A. Bahyrycz, Zs. Páles and M. Piszczek and a lemma of T. M. K. Davison and B. R. Ebanks, we shall introduce and investigate the following two generalizations

$$F(x, y) + F(u, y+v) + F(x+y, u+v) = F(x, u) + F(y, u+v) + F(x+u, y+v)$$

and

$$F(x, y) + F(x-u, u) + F(y-v, u) + F(y-v, v) = F(u, v) + F(u, y-v) + F(x-u, y-v) + F(x+y-u-v, u+v)$$

of the famous cocycle equation $F(x, y)+F(x+y, z) = F(x, y+z)+F(y, z)$, which seem to differ from the two most important particular cases of a Pexiderized one studied by B. R. Ebanks, C. T. Ng and T. M. K. Davison.

1. Introduction

In the proofs of Theorems 1 and 5 of [4], for a function f of one commutative group X to another Y, Bahyrycz, Páles and Piszczek have used, but not explicitly stated, the equality

$$\begin{aligned}(1)\quad f(x+y) - f(x) - f(y) &= f(x-u) + f(u) - f(x) \\ &+ f(y-v) + f(v) - f(y) + f(x+y-u-v) - f(x-u) - f(y-v) \\ &+ f(u+v) - f(u) - f(v) + f(x+y) - f(x+y-u-v) - f(u+v)\,.\end{aligned}$$

Hence, by using the Cauchy difference

$$(2)\qquad F(x, y) = f(x+y) - f(x) - f(y)\,,$$

we could note that, instead of equality (1), it is more convenient to consider the equality

$$\begin{aligned}(3)\quad F(x, y) = F(u, v) - F(x-u, u) - F(y-v, v) \\ + F(x-u, y-v) + F(x+y-u-v, u+v)\,.\end{aligned}$$

Namely thus, for instance, Theorem 1 of [4] can be easily extended to the solutions of (2). Moreover, we can prove that every symmetric cocycle F on X to Y is a solution of equation (3).

That is, if F is a function of X^2 to Y such that $F(x, y) = F(y, x)$ and

$$(4)\qquad F(x, y) + F(x+y, z) = F(x, y+z) + F(y, z)$$

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for all $x, y, z \in X$, then (3) also holds for all $x, y, u, v \in X$.

It is well-known that every Cauchy–difference is a symmetric cocycle. Moreover, in Lemma 2 of [11], Davison and Ebanks have proved that if F is a symmetric cocycle on X to Y, then

$$(5)\quad F(x+y,\, u+v) = F(x+u,\, y+v) + F(x,\, u) + F(y,\, v) - F(x,\, y) - F(u,\, v)$$

also holds for all $x, y, u, v \in X$.

At first seeing, I considered equations (3) and (5) to be very similar, but still quite independent. However, Gyula Maksa, my close colleague, has noticed that they are actually equivalent.

Namely, (5) can be immediately derived from (3) by replacing x by $x+u$ and y by $y+v$. And conversely, (3) can be immediately derived from (5) by replacing x by $x-u$ and y by $y-v$. Thus, equation (1) is a special case of (5) too.

In this paper, we shall also consider the more difficult equations

$$(6)\quad F(x,\, y) + F(u,\, y+v) + F(x+y,\, u+v) = F(x,\, u) + F(y,\, u+v) + F(x+u,\, y+v),$$

and

$$(7)\quad F(x,\, y) + F(x-u,\, u) + F(y-v,\, u) + F(y-v,\, v) = F(u,\, v) + F(u,\, y-v) + F(x-u,\, y-v) + F(x+y-u-v,\, u+v).$$

Note that if in particular F is symmetric, then equation (7) is equivalent to (3), which is in turn equivalent to (5). Moreover, it can be easily shown that if F is additive in its second variable, then equations (6) and (7) are also equivalent.

Now, we shall also prove that equations (6) and (7) are generalizations of (4) too. Therefore, it seems to be a reasonable research program to extend some of the basic theorems on equation (4) to (6) and (7). And, to establish some deeper relationships among the various generalizations of equation (4).

A most general, Pexider type generalization of (4) is the equation

$$(8)\quad F_1(x+y,\, z)+F_2(y+z,\, x)+F_3(z+x,\, y)+F_4(x,\, y)+F_5(y,\, z)+F_6(z,\, x) = 0$$

introduced and solved by Ebanks and Ng [25].

Concerning its solutions, Davison [10] has explicitly established the importance of the particular cases

$$(9)\quad F(x+y,\, z) - F(x,\, z) - F(y,\, z) = F(y+z,\, x) - F(y,\, x) - F(z,\, x)$$

and

$$(10)\quad F(x+y,\, z) - F(x,\, z) - F(y,\, z) + F(y+z,\, x) - F(y,\, x) - F(z,\, x) + F(x+z,\, y) - F(x,\, y) - F(z,\, y) = 0.$$

Note that if in particular F is symmetric, then equation (9) is equivalent to (4), and equation (10) is equivalent to

$$(11)\quad F(x+y,\, z) + F(y+z,\, x) + F(x+z,\, y) = 2\,F(x,\, y) + 2\,F(y,\, z) + 2\,F(x,\, z)$$

By Theorem 3.3 of [25], it is curious that Székelyhidi's equation [57]

$$F(x+y,\, z)+F(x-y,\, z)-2\,F(y,\, z)=F(x,\, y+z)+F(x,\, y-z)-2\,F(x,\, y) \tag{12}$$

has been considered to be still unsolved by Ebanks and Ng in [26] and [18].

Some fundamental results of Székelyhidi [58] on equations (4) and (12) have been extended by Páles [53] to the more attractive equation

$$F(x,\, y)+\frac{1}{n}\sum_{i=1}^{n} F\big(x+\phi_i(y),\, z\big)=\frac{1}{n}\sum_{i=1}^{n} F\big(x,\ y+\phi_i(z)\big)+F(x,\, y) \tag{13}$$

with suitable functions ϕ_i. This equation could certainly be also used to extend some further results on equation (4) and establish some reasonable generalizations of equations (6) and (7) too.

Finally, we note that equations (6) and (7) may also be compared to the generalized rhombic and rectangular equations

$$\begin{aligned} &F(x+u,\, y)+F(x-u,\, y)\\ &\qquad +F(x,\, y+v)+F(x,\, y-v)=k\,F(x,\, y)+l\,F(u,\, v), \end{aligned} \tag{14}$$

and

$$\begin{aligned} &F(x+u,\, y+v)+F(x+u,\, y-v)\\ &\qquad +F(x-u,\, y+v)+F(x-u,\, y-v)=k\,F(x,\, y)+l\,F(u,\, v) \end{aligned} \tag{15}$$

considered, for some $k,\, l\in\mathbb{Z}$, by Aczél et al. [3] and Chung et al. [7].

In the sequel, we shall strive to make our presentation quite instructive and completely self-contained. In particular, we shall give more readable proofs of some elementary basic facts on cocycles.

2. Cauchy differences and symmetric cocycles

Notation 2.1. *In this section, and the subsequent Sections 3 and 5, we shall assume that X and Y are commutative groups, and F is a function of X^2 to Y.*

Remark 2.2. Now, by defining

$$(x,\, y)+(z,\, w)=(\,x+z\,,\ y+w\,)$$

for all $x,\, y,\, z,\, w\in X$, the set X^2 can also be turned into a commutative group.

Definition 2.3. The function F will be called *Cauchy* if there exists a function f of X to Y such that, for any $x,\, y\in X$, we have

$$f(x+y)=f(x)+f(y)+F(x,\, y)\,.$$

Remark 2.4. In this case, according to Davison and Ebanks [11] and Stetkaer [56, p. 280], we may also say that F is the Cauchy difference of f, or that F is a coboundary with generator function $-f$.

Moreover, analogously to the case of generalized Hyers–Ulam stability [40], we may also say that f is F–approximately additive, or more briefly F–additive. Thus, f is additive if and only if it is 0–additive.

For some functions F, solutions of the inhomogeneous Cauchy equation, considered in Definition 2.3, were given in [6, 8, 9, 29, 49, 50, 32, 24, 45, 37, 31].

Moreover, in [19] and [21], B. R. Ebanks has solved the much more general equation $f(x+y) = f(x) + f(y) + g\big(F(x, y)\big)$ for unknown functions f and g.

Note that this equation, in contrast to the one considered in Definition 2.3, allows of the investigation of the functional dependence of two Cauchy differences.

For an easy illustration of Definition 2.3, one can easily check the statements of following examples inspired by those of Prunescu [54] and Stetkaer [56, p. 288].

Example 2.5. If $X = \mathbb{Z}$ and $F(x, x) = x\,y$ for all $x, y \in X$, then F is a Cauchy function of X^2 to X with a generator function f defined by $f(x) = x\,(x+1)/2$ for all $x \in X$.

Example 2.6. If $X = \mathbb{R}^{\Gamma}$, for a nonvoid set Γ, and $F(x, y) = x\,y$ for all $x, y \in X$, then F is a Cauchy function of X^2 to X with a generator function f defined by $f(x) = x^2/2$ for all $x \in \mathbb{R}^{\Gamma}$.

Example 2.7. If X is a real inner product space and $F(x, y) = \langle\, x, y \,\rangle$ for all $x, y \in X$, then F is a Cauchy function with a generator function f defined by $f(x) = \|\, x \,\|^2/2$ for all $x \in X$.

Remark 2.8. Note the if F is a Cauchy function of X to Y and f is a generator function of F, then a function g of X to Y is another generator function of F if and only if $f - g$ is an additive function.

Thus, to get all solutions f of the inhomogeneous Cauchy equation, considered in Definition 2.3, it is enough to find only one solution of this inhomogeneous equation and all solutions of the corresponding homogeneous equation.

Definition 2.9. Under Notation 2.1, we shall also say that:

(1) F is *symmetric (skew-symmetric)* if, for any $x, y \in X$, we have

$$F(x, y) = F(y, x) \qquad \big(F(x, y) = -F(y, x)\big)$$

(2) F is *cocyclic* if, for any $x, y, z \in X$, we have

$$F(x, y) + F(x+y, z) = F(x, y+z) + F(y, z).$$

Remark 2.10. If F is skew-symmetric, then by Hosszú [42], Aczél [1], Ebanks [14] and Stetkaer [56, p. 17], we may also say that F is antisymmetric.

While, if F is a cocyclic function of X^2 to Y, then by Davison and Ebanks [11] and Stetkaer [56, p. 280], we may also say that F is a cocycle on X to Y.

The cocycle equation, consider in Definition 2.9, can also be written in the less general, but more instructive difference and Cauchy–difference forms

$$F(x+y, z) - F(y, z) = F(x, y+z) - F(x, y)$$

and

$$F(x+y, z) - F(x, z) - F(y, z) = F(x, y+z) - F(x, y) - F(x, z).$$

References, listed in [11] and [56] and the present paper, show that the cocycle equation has been utilized in several different branches of mathematics.

They seem to first appear in the context of group extensions. (See Kurosh [48, § 50], Fuchs [34, § 60], Hosszú [43] and Stetkaer [56, Lemma 16.3].)

However, in the theory of functional equations, the following theorem was certainly first established by Kurepa [47]. The simple proof is included here for the reader's convenience.

Theorem 2.11. *If F is Cauchy, then F is both symmetric and cocyclic.*

Proof. By Definition 2.3, there exists a function f of X to Y such that

$$F(s, t) = f(s+t) - f(s) - f(t)$$

for all $s, t \in X$.

Hence, it is clear that F is symmetric. Moreover, we can easily check that

$$\begin{aligned} &F(x, y) + F(x+y, z) - F(x, y+z) - F(y, z) \\ &\qquad = f(x+y) - f(x) - f(y) + f(x+y+z) - f(x+y) - f(z) \\ &\qquad\qquad - f(x+y+z) + f(x) + f(y+z) - f(y+z) + f(y) + f(z) = 0 \end{aligned}$$

for all $x, y, z \in X$. Therefore, F is also cocyclic.

In this respect, it is also worth mentioning the following theorem which was already used by Erdős [30]. Later, Aczél [1, p. 66] and Stetkaer [56, p. 283] stated it explicitly by leaving the proof to the reader.

Theorem 2.12. *If F is biadditive, then F is cocyclic.*

Proof. Now, we also have

$$\begin{aligned} &F(x, y) + F(x+y, z) - F(x, y+z) - F(y, z) \\ &\qquad = F(x, y) + F(x, z) + F(y, z) - F(x, y) - F(x, z) - F(y, z) = 0 \end{aligned}$$

for all $x, y, z \in X$. Therefore, F is cocyclic.

In this respect, it is also worth proving the following theorem which seems not to be stated explicitly in the existing literature.

Theorem 2.13. *If F is cocyclic and additive in one of its variables, then F is biadditive.*

Proof. Since F is cocyclic, for any $x, y, z \in X$ we have

$$F(x, y) + F(x+y, z) = F(x, y+z) + F(y, z).$$

Hence, if for instance F is additive in its second variable, we can infer that

$$F(x, y) + F(x+y, z) = F(x, y) + F(x, z) + F(y, z),$$

and thus $F(x+y, z) = F(x, z) + F(y, z)$. Therefore, F is additive in its first variable too, and thus it is also biadditive.

Now, as an immediate consequence of the latter two theorems, we can also state

Corollary 2.14. *The following assertions are equivalent:*

(1) F is biadditive,

(2) F is cocyclic and additive in one of its variables.

In addition to Theorem 2.13, it is also worth mentioning the following widely used theorem.

Theorem 2.15. *If F is additive in one of its variables and F is either symmetric or skew-symmetric, then F is biadditive.*

Proof. If for instance F is skew-symmetric and additive in its first variable, then for any $x, y, z \in X$, we have

$$\begin{aligned} F(x, y+z) = -F(y+z, x) = -\big(F(y, x) + F(z, x)\big) \\ = -F(y, x) - F(z, x) = F(x, y) + F(x, z). \end{aligned}$$

Therefore, F is additive in its second variable too, and thus it is also biadditive.

From this theorem, by using Theorem 2.12, we can immediately derive

Corollary 2.16. *If F is as in Theorem 2.15, then F is cocyclic.*

Now, by using Theorem 2.12 and the following example, we can also see that even a cocycle on $\mathbb{R}$ to itself need not be symmetric, and thus a Cauchy function. This was a conjecture of Kurepa [47] justified by Erdős [30].

Example 2.17. By using Hamel bases, it can be shown that there exist an additive function φ of $\mathbb{R}$ to itself which is not homogeneous [41, 5]. Thus, there exists $\lambda \in \mathbb{R}$ such that $\varphi(\lambda) \neq \lambda\,\varphi(1)$.

Now, by defining $\Phi(x, y) = \varphi(x)\,y$ for all $x, y \in \mathbb{R}$, we can easily see that Φ is a biadditive function of $\mathbb{R}^2$ to $\mathbb{R}$ such that $\Phi(1, \lambda) = \varphi(1)\,\lambda \neq \varphi(\lambda) = \Phi(\lambda, 1)$, and thus Φ is not symmetric.

Remark 2.18. By weakening the differentiability assumption of Kurepa [47], János Aczél in 1958 already proved that a continuous cocycle on $\mathbb{R}$ to itself is necessarily symmetric.

This result was announced by Erdős [30] who, by using a Hamel basis, directly defined a non-symmetric biadditive function to show the necessity of some regularity conditions such as continuity, monotonicity, boundedness or measurability in one of the variables.

Because of Theorem 2.11 and the existence of nonsymmetric cocycles, we may naturally introduce the following

Definition 2.19. The function F, considered in Notation 2.1, will be called *quasi–Cauchy* if it is both symmetric and cocyclic.

Remark 2.20. Thus, the main problem on cocycles can be briefly reformulated by asking that when a quasi-Cauchy function is a Cauchy function.

For instance, by a theorem of J. Erdős [30], we can state that if Y is $\mathbb{N}$-divisible and F is quasi-Cauchy, then F is already Cauchy.

Remark 2.21. Here, the group Y is called $\mathbb{N}$–divisible if it is n–divisible for all $n \in \mathbb{N}$ in the sense that $Y = nY$, or equivalently $Y \subseteq nY$.

In a detailed form, this means only that for any $y \in Y$ there exists $z \in Y$ such that $y = nz$. That is, the set $n^{-1}y = \{ z \in Y : \ y = nz \}$ is not empty.

Quite similarly the group Y is called n–cancelable if $ny = nz$ implies $y = z$ for all $y, z \in Y$. Thus, Y is uniquely n–divisible if and only if its both n–divisible and n–cancelable. (For some more general observations see [36].)

The following example, inspired by those of Ebanks [15] and Statkaer [56, p. 286], shows that a quasi-Cauchy function F of X^2 to Y need not be a Cauchy function, even when $\operatorname{card}(X) = 2$ and $Y = \mathbb{Z}$.

Example 2.22. Define $X = \{u, v\}$ and

$$u + u = u, \qquad u + v = v, \qquad v + u = v, \qquad v + v = u.$$

Then, it is clear that this addition is commutative. Moreover, by considering several possible cases for the variables $x, y, z \in X$, it can be easily seen that it is also associative. Hence, it is clear that X is a commutative group with zero element u, and inverse elements $-u = u$ and $-v = v$.

Moreover, define a function F of X^2 such that

$$F(u, u) = 0, \qquad F(u, v) = 0, \qquad F(v, u) = 0, \qquad F(v, v) = 1.$$

Then, it is clear that F is a symmetric function of X to $\mathbb{Z}$. Moreover, analogously to the proof of the associativity of the addition, it can be easily seen that F is cocyclic, and thus it is also quasi-Cauchy.

However, F cannot be a Cauchy function of X^2 to Z. Namely, if it is Cauchy, then for any generator function f of F we have

$$f(u) = f(u + u) = f(u) + f(u) + F(u, u) = 2 f(u),$$

and thus $f(u) = 0$. Moreover, we also have

$$0 = f(u) = f(v + v) = f(v) + f(v) + F(v, v) = 2 f(v) + 1,$$

and thus $f(v) = -1/2$. And, this contradicts the assumption that $f(v) \in \mathbb{Z}$.

Remark 2.23. Note that now we have

$$F(v + v, v) = F(u, v) = 0 \qquad \text{and} \qquad F(v, v) + F(v, v) = 2 F(v, v) = 2.$$

Therefore, F is not additive in its first variable, and thus also in its second variable.

Moreover, by using the forthcoming Theorem 3.1, every quasi-Cauchy function F of X^2 to Y can be easily determined with $a = F(u, u)$ and $b = F(v, v)$ in place of 0 and 1, respectively.

3. Some further useful results on cocycles

Among the many known results on cocycles, we shall only mention here a few elementary ones which are closely connected with additivity and biadditivity.

The following preliminary observation has already been stated in [11, Lemma 1] by Davison and Ebanks and [56, Lemma 16.2] by Stetkaer. The simple proof is included here for the reader's convenience.

Theorem 3.1. *If F is cocyclic, then for any $x, y \in X$ we have*

$$F(x, 0) = F(0, 0) \qquad \textit{and} \qquad F(0, y) = F(0, 0).$$

Proof. From (2) in Definition 2.9, by taking $y = 0$, we can infer that

$$F(x, 0) + F(x, z) = F(x, z) + F(0, z),$$

and thus $F(x, 0) = F(0, z)$ for all $x, z \in X$. Hence, by putting first $z = 0$ and then $x = 0$, we can already see that the required assertions are also true.

Hence, by Theorem 2.11, it is clear that in particular we also have

Corollary 3.2. *If F is a Cauchy function with generator function f, then for any $x, y \in X$ we have $F(x, 0) = -f(0)$ and $F(0, y) = -f(0)$.*

In addition to Theorems 2.11 and 2.12, we can now also easily prove the following theorem which seems not to be stated in the existing literature.

Theorem 3.3. *If F is additive and $F(x, 0) = F(0, y)$ for all $x, y \in X$, then F is cocyclic.*

Proof. Now, we also have

$$\begin{aligned}
&F(x, y) + F(x+y, z) - F(x, y+z) - F(y, z) \\
&\qquad = F(x, y) + F\big((x, 0) + (y, z)\big) - F\big((x, y) + (0, z)\big) - F(y, z) \\
&\qquad = F(x, y) + F(x, 0) + F(y, z) - F(x, y) - F(0, z) - F(y, z) \\
&\qquad\qquad = F(x, 0) - F(0, z) = 0
\end{aligned}$$

for all $x, y, z \in X$. Therefore, F is also cocyclic.

Now, as an immediate consequence of the latter two theorems, we can also state

Corollary 3.4. *If F is additive, then the following assertions are equivalent:*

(1) F is cocyclic, *(2) $F(x, 0) = F(0, y)$ for all $x, y \in X$.*

In this respect, it is also worth mentioning [56, Theorem 16.6] of Stetkaer, whose origin goes back to the calculations of Aczél presented in [30], and also those of Hosszú [42] and Davison and Ebanks [11, p. 146]. Our subsequent proof is much longer, but more straightforward than those of the above mentioned authors.

Theorem 3.5. *If F is both cocyclic and skew-symmetric, then the function $G = 2F$ is biadditive.*

Proof. If $x, y \in X$, then by (1) in Definition 2.9, we have

$$G(x, y) = 2F(x, y) = F(x, y) + F(x, y) = F(x, y) - F(y, x).$$

Hence, by putting $x + y$ in place of x, and z in place of y, we can see that, for any x $y, z \in X$, we have

$$G(x+y, z) = \underline{F(x+y, z)} - \underline{F(z, x+y)}.$$

Moreover, by (2) in Definition 2.9, we have

$$F(x, y) + \underline{F(x+y, z)} = F(x, y+z) + F(y, z).$$

Hence, by putting z in place of x, x in place of y, and y in place of z, we can see that

$$F(z, x) + F(z+x, y) = \underline{F(z, x+y)} + F(x, y).$$

Therefore, after some simplifications and rearrangements, we have

$$\begin{aligned}
&G(x+y, z) = F(x+y, z) - F(z, x+y) \\
&\quad = F(x, y+z) + F(y, z) - F(x, y) - F(z, x) - F(z+x, y) + F(x, y) \\
&\qquad\qquad = F(x, z) + F(y, z) - \big(\underline{F(x+z, y)} - \underline{F(x, z+y)}\big).
\end{aligned}$$

Moreover, again from (2) in Definition 2.9, by putting z in place of y and y in place of z, we can see that

$$F(x, z) + \underline{F(x+z, y)} = \underline{F(x, z+y)} + F(z, y).$$

Therefore, again by (1) in Definition 2.9 and the definition of G, we have

$$\begin{aligned} G(x+y,\, z) &= F(x,\, z) + F(y,\, z) - \big(F(x+z,\, y) - F(x,\, z+y)\big) \\ &= F(x,\, z) + F(y,\, z) - F(z,\, y) + F(x,\, z) \\ &= 2\,F(x,\, z) + 2\,F(y,\, z) = G(x,\, z) + G(y,\, z)\,. \end{aligned}$$

This shows that G is additive in its first variable.

Moreover, we can easily see that now G is also skew-symmetric. Therefore, by Theorem 2.15, we can also state that G is biadditive.

From this theorem, we can immediately derive

Corollary 3.6. *If Y is 2–cancelable and F is cocyclic and skew-symmetric, then F is biadditive.*

Proof. Now, by Theorem 3.5, for any $x,\, y,\, z \in X$ we have

$$2\,F(x+y,\, z) = 2\,F(x,\, z) + 2\,F(y,\, z) = 2\big(F(x,\, z) + F(y,\, z)\big)\,.$$

Hence, by using that Y is 2–cancelable, we can already infer that

$$F(x+y,\, z) = F(x,\, z) + F(y,\, z)\,.$$

This shows that F is additive in its first variable. Hence, by Theorem 2.15, we can see that F is biadditive.

Now, as an immediate of this corollary and Theorem 2.12, we can also state

Theorem 3.7. *If Y is 2–cancelable and F is skew-symmetric, then the following assertions are equivalent:*

(1) F is cocyclic, *(2) F is biadditive.*

However, the importance of Corollary 3.6 lies mainly in the following observation of Hosszú [42] inspired by a calculation of Aczél presented in [30].

Theorem 3.8. *If Y is uniquely 2–divisible and F is cocyclic, and moreover*

$$A(x,\, y) = \frac{F(x,\, y) + F(y,\, x)}{2} \qquad \text{and} \qquad B(x,\, y) = \frac{F(x,\, y) - F(y,\, x)}{2}$$

for all $x,\, y \in X$, then A is a quasi-Cauchy and B is a skew-symmetric biadditive function of X^2 to Y such that $F = A + B$.

Proof. By the above definitions, we evidently have $F(x,\, y) = A(x,\, y) + B(x,\, y)$ for all $x,\, y \in X$. Moreover, it is clear that A is symmetric and B is skew-symmetric. Furthermore, it can be easily seen that both A and B are also cocyclic. Thus, A is quasi-Cauchy. Moreover, by Corollary 3.6, B is biadditive.

Now, as a close analogue of Theorem 3.5, we can also prove the following striking theorem which shows that a non-zero additive function of X^2 to Y cannot, in general, be biadditive.

Theorem 3.9. *If F is both additive and biadditive, then $2\,F(x,\, y) = 0$ for all $x,\, y \in X$.*

Proof. By the corresponding definitions, we have

$$F(x, y) + F(u, v) = F\big((x, y) + (u, v)\big) = F(x + u, y + v)$$
$$= F(x, y + v) + F(u, y + v) = F(x, y) + F(x, v) + F(u, y) + F(u, v),$$

and thus $F(x, v) + F(u, y) = 0$ for all $x, y, u, v \in X$.

Hence, by taking $u = x$ and $v = y$, we can already infer that $2F(x, y) = F(x, y) + F(x, y) = 0$ for all $x, y \in X$.

From this theorem, it is clear that in particular we also have

Corollary 3.10. *If Y is 2–cancelable and F is both additive and biadditive, then $F(x, y) = 0$ for all $x, y \in X$.*

Now, by using Theorems 3.3, 3.5 and 3.9, we can also easily prove

Theorem 3.11. *If F is both additive and skew-symmetric, and $F(x, 0) = F(0, y)$ for all $x, y \in X$, then $4F(x, y) = 0$ for all $x, y \in X$.*

Proof. From Theorem 3.3, we can see that F is cocyclic. Thus, by Theorem 3.5, the function $G = 2F$ is biadditive. Moreover, from the additivity of F, it is clear that G is also additive. Therefore, by Theorem 3.9, we have $2G(x, y) = 0$ for all $x, y \in X$. Hence, we can already infer that $4F(x, y) = 2\big(2F(x, y)\big) = 2G(x, y) = 0$ for all $x, y \in X$.

From this theorem, it is clear that in particular we also have

Corollary 3.12. *If in addition to the conditions of Theorem 3.11, Y is 4–cancelable, then $F(x, y) = 0$ for all $x, y \in X$.*

Remark 3.13. Note that if Y is 2–cancellable, then Y is also 4–cancellable. However, the converse statement is certainly not true.

4. Two particular methods of constructing Cauchy and quasi-Cauchy functions

The following method for constructing Cauchy functions has been suggested by Shapiro [55].

Theorem 4.1. *If φ is a locally integrable, additive function of $\mathbb{R}$ to itself and*

$$f(x) = \int_0^x \varphi(s)\, ds \qquad \textit{and} \qquad F(x, y) = \varphi(x)\, y$$

for all $x, y \in \mathbb{R}$, then F is a Cauchy-function with generator function f.

Proof. By some basic properties of the integral, for any $x, y \in \mathbb{R}$, we have

$$f(x + y) = \int_0^{x+y} \varphi(s)\, ds = \int_0^{x} \varphi(s)\, ds + \int_x^{x+y} \varphi(s)\, ds = f(x) + \int_0^{y} \varphi(x + t)\, dt$$
$$= f(x) + \int_0^{y} \big(\varphi(x) + \varphi(t)\big)\, dt = f(x) + \int_0^{y} \varphi(x)\, dt + \int_0^{y} \varphi(t)\, dt$$
$$= f(x) + \varphi(x)\, y + f(y) = f(x) + F(x, y) + f(y),$$

and thus $F(x, y) = f(x + y) - f(x) - f(y)$.

Remark 4.2. Therefore, $F(x,\, y) = F(y,\, x)$, and thus $\varphi(x)\, y = \varphi(y)\, x$ for all $x,\, y \in \mathbb{R}$. Hence, by taking $y = 1$, we can get $\varphi(x) = \varphi(1)\, x$ for all $x \in \mathbb{R}$.

The above particular theorem can certainly be generalized by using the Kurzweil integral, or an invariant mean, instead of the Riemann or Lebesgue one.

The subsequent method for constructing quasi-Cauchy functions is taken from the book of Aczél and Daróczy [2, p. 94]. It already indicates that cocycles on commutative semigroups have to be also considered. (For such investigations, see [44, 15, 11, 12, 22].)

Definition 4.3. A function φ of $[\,0,\, 1\,]$ to $\mathbb{R}$ is called an *information function* if

(1) $\varphi(0) = \varphi(1)$,

(2) for any $u,\, v \in [\,0,\, 1\,[$ with $u + v \leq 1$, we have

$$\varphi(u) + (1-u)\, \varphi\Big(\frac{v}{1-u}\Big) = \varphi(v) + (1-v)\, \varphi\Big(\frac{u}{1-v}\Big).$$

Remark 4.4. Note that the above restrictions on u and v are necessary and sufficient in order that the values $v/(1-u)$ and $u/(1-v)$ could have meanings and belong to the domain of φ

For the origins of the above fundamental equation (2) of information, see [2, pp. 71–74] where in addition to (1) and (2) it is assumed that $\varphi(1/2) = 1$.

Moreover, the reader can get a rapid overview on the subject by consulting a recent book [38] of E. Gselmann which contains several further references.

The following theorem is only a simplification of [2, Proposition (3.1.24)]. The proof is included here for the reader's convenience.

Theorem 4.5. *If φ is an information function, then*

(1) $\varphi(0) = 0$ *and* $\varphi(1) = 0$,

(2) $\varphi(u) = \varphi(1-u)$ *for all* $u \in [\,0,\, 1\,]$.

Proof. From (2) in Definition 4.3, by taking $v = 0$, we can see that

$$\varphi(u) + (1-u)\, \varphi(0) = \varphi(0) + \varphi(u),$$

and thus $u\, \varphi(0) = 0$ for all $u \in [\,0,\, 1\,[$. Hence, it is clear that $\varphi(0) = 0$, and thus by (1) in Definition 4.3 the equality $\varphi(1) = 0$ also holds.

Moreover, if $u \in \,]\,0,\, 1\,[$, then from (2) in Definition 4.3, by taking $v = 1-u$ we can also see that

$$\varphi(u) + (1-u)\, \varphi(1) = \varphi(1-u) + u\, \varphi(1).$$

Hence, by using that $\varphi(1) = 0$, we can already infer that $\varphi(u) = \varphi(1-u)$.

Now, to complete the proof it remains only to note that, by (1) in Definition 4.3, the latter equality is also true for $u = 0$ and $u = 1$.

From this theorem, we can immediately derive the following

Corollary 4.6. *If φ is an information function, then for any $x > 0$, $y \geq 0$ and $z \geq 0$ we have*

(1) $\varphi\Big(\dfrac{x}{x+y}\Big) = \varphi\Big(\dfrac{y}{x+y}\Big)$, *(2)* $\varphi\Big(\dfrac{x}{x+y+z}\Big) = \varphi\Big(\dfrac{y+z}{x+y+z}\Big)$.

The following theorem is only a simplified reformulation of [2, Lemma (3.5.2)]. However, the proof given here is much more readable than the one given by the above mentioned authors.

Theorem 4.7. *If* φ *is an information function and*

$$F(x,\,y) = (x+y)\,\varphi\Big(\frac{x}{x+y}\Big)$$

for all $x > 0$ *and* $y > 0$, *then* F *is a positively homogeneous, quasi-Cauchy function of* $\mathbb{R}^2_+$ *to* $\mathbb{R}$.

Proof. From the definition of F, by Corollary 4.6, it is clear that F is symmetric. Moreover, we can also at once see that

$$F(\lambda x,\,\lambda y) = \lambda F(x,\,y)$$

for all $\lambda > 0$, $x > 0$ and $y > 0$, and thus F is positively homogeneous.

Therefore, we need actually prove that F is also cocyclic. For this, note that if $x > 0$, $y > 0$ and $z > 0$, then by the symmetry and the definition of F we have

$$\begin{aligned}
F(x,\,y) + F(x+y,\,z) &= F(x,\,y) + F(z,\,x+y)\\
&= (x+y)\,\varphi\Big(\frac{x}{x+y}\Big) + (x+y+z)\,\varphi\Big(\frac{z}{x+y+z}\Big)\\
&= (x+y+z)\left(\varphi\Big(\frac{z}{x+y+z}\Big) + \frac{x+y}{x+y+z}\,\varphi\Big(\frac{x}{x+y}\Big)\right).
\end{aligned}$$

Hence, by using the notations

$$u = \frac{z}{x+y+z} \qquad \text{and} \qquad v = \frac{x}{x+y}\,(1-u) = \frac{x}{x+y+z},$$

and Definition 4.3, Corollary 4.6 and the symmetry of F, we can already see that

$$\begin{aligned}
&F(x,\,y) + F(x+y,\,z) =\\
&(x+y+z)\left(\varphi(u) + (1-u)\,\varphi\Big(\frac{v}{1-u}\Big)\right) = (x+y+z)\left(\varphi(v) + (1-v)\,\varphi\Big(\frac{u}{1-v}\Big)\right)\\
&= (x+y+z)\left(\varphi\Big(\frac{x}{x+y+z}\Big) + \frac{y+z}{x+y+z}\,\varphi\Big(\frac{z}{y+z}\Big)\right)\\
&= (x+y+z)\,\varphi\Big(\frac{x}{x+y+z}\Big) + (y+z)\,\varphi\Big(\frac{z}{y+z}\Big)\\
&= (x+y+z)\,\varphi\Big(\frac{y+z}{x+y+z}\Big) + (y+z)\,\varphi\Big(\frac{y}{y+z}\Big)\\
&= F(y+z,\,x) + F(y,\,z) = F(x,\,y+z) + F(y,\,z).
\end{aligned}$$

Remark 4.8. By using the proof of [2, Lemma (3.5.7)], it can be shown that if F is a positively homogeneous, quasi-Cauchy function of $\mathbb{R}^2_+$ to $\mathbb{R}$, and

$$\varphi(0) = \varphi(1) = 0 \qquad \text{and} \qquad \varphi(x) = F(x,\,1-x) \quad \text{for} \quad x \in \,]0,\,1[\,,$$

then φ is already an information function generating F in the former sense that $F(x,\,y) = (x+y)\,\varphi\big(x\,/\,(x+y)\big)$ for all $x > 0$ and $y > 0$.

Remark 4.9. Moreover, it is also noteworthy that if F is only a positively homogeneous cocyclic function of $\mathbb{R}^2_+$ to $\mathbb{R}$, then by a results of Ebanks [14], the function F is already a Cauchy-function with a generator function f having the useful derivation property $f(x\,y) = f(x)\,y + x\,f(y)$ for all $x > 0$ and $y > 0$.

5. Two natural generalizations of cocycles

The results collected in above three sections already give ample reasons for teaching, investigating and generalizing cocyclic functions.

The Pexider type generalization (8) of equation (4), and its important particular cases (9) and (10), have been studied by Ebanks and Ng [25] and Davison [10].

However, our subsequent generalizations of equation (4) are more closely related to equation (5) of Davison and Ebanks stated in [11, Lemma 2].

Definition 5.1. In addition to (2) in Definition 2.9, we shall also say that:

(1) F is *semi-cocyclic* if, for any $x, y, u, v \in X$, we have

$$F(x, y) + F(u, y+v) + F(x+y, u+v) = F(x, u) + F(y, u+v) + F(x+u, y+v),$$

(2) F is *pseudo-cocyclic* if, for any $x, y, u, v \in X$, we have

$$F(x, y) + F(x-u, u) + F(y-v, u) + F(y-v, v) = F(u, v) + F(u, y-v) + F(x-u, y-v) + F(x+y-u-v, u+v).$$

Remark 5.2. Now, analogously to Definition 2.19, the function F may also be naturally called *semi-Cauchy (pseudo-Cauchy)* if it is both symmetric and semi-cocyclic (pseudo-cocyclic).

Moreover, because of the terms $F(y-v, u)$ and $F(u, y-v)$ in (2), we may also naturally introduce the following

Definition 5.3. The function F will be called *Cauchy-like* if

$$F(x, y) + F(x-u, u) + F(y-v, v) = F(u, v) + F(x-u, y-v) + F(x+y-u-v, u+v).$$

for all $x, y, u, v \in X$.

Remark 5.4. Namely, thus we can at once state that a pseudo-Cauchy function is Cauchy-like, and a symmetric Cauchy-like function is pseudo-Cauchy.

Moreover, by using the substitutions of Gyula Maksa mentioned in the Introduction, we can easily prove the following two theorems.

Theorem 5.5. *The following assertions are equivalent:*

(1) F *is Cauchy-like,*

(2) for any $x, y, u, v \in X$, *we have*

$$F(x, y) + F(u, v) + F(x+y, u+v) = F(x, u) + F(y, v) + F(x+u, y+v).$$

Remark 5.6. Note that the latter equation, which is closely related to (1) in Definition 5.1, is only a convenient rearrangement of equation (5) of Davison and Ebanks.

Theorem 5.7. *If F is additive in its second variable, then the following assertions are equivalent:*

(1) F is semi-cocyclic, *(2) F is pseudo-cocyclic.*

Proof. If (2) holds, then from (2) in Definition 5.1 by putting $x+u$ and $y+v$ in place of x and y, respectively, we can infer that

$$F(x+u, y+v)+F(x, u)+F(y, u)+F(y, v) = F(u, v)+F(u, y)+F(x, y)+F(x+y, u+v)$$

for all $x, y, u, v \in X$. Hence, by using that

$$F(y, u)+F(y, v)=F(y, u+v) \quad \text{and} \quad F(u, v)+F(u, y)=F(u, v+y)$$

for all $y, u, v \in X$, we can infer that

$$F(x+u, y+v)+F(x, u)+F(y, u+v) = F(u, v+y)+F(x, y)+F(x+y, u+v)$$

for all $x, y, u, v \in X$. Therefore, (1) in Definition 5.1 is also satisfied, and thus (1) also holds.

While, if (1) holds, then from (1) in Definition 5.1 by putting $x-u$ and $y-v$ in place of x and y, respectively, we can infer that

$$F(x-u, y-v)+F(u, y)+F(x+y-u-v, u+v) = F(x-u, u)+F(y-v, u+v)+F(x, y),$$

for all $x, y, u, v \in X$. Hence, by using that

$$F(u, y)=F(u, v)+F(u, y-v) \quad \text{and} \quad F(y-v, u+v)=F(y-v, u)+F(y-v, v)$$

for all $y, u, v \in X$, we can infer that

$$F(x-u, y-v)+F(u, v)+F(u, y-v)+F(x+y-u-v, u+v) = F(x-u, u)+F(y-v, u)+F(y-v, v)+F(x, y),$$

for all $x, y, u, v \in X$. Therefore, (2) in Definition 5.1 is also satisfied, and thus (1) also holds.

The appropriateness of terminology (1) in Definition 5.1 is also apparent from the following theorem inspired by a less convenient calculation of Davison and Ebanks [11, p. 139].

Theorem 5.8. *If F is cocyclic, then F is also semi-cocyclic.*

Proof. From (2) in Definition 2.5, by taking first $z=u+v$, and then $y=u$ and $z=y+v$, we can see that

$$F(x, y)+F(x+y, u+v)=\underline{F(x, y+u+v)}+F(y, u+v),$$
$$F(x, u)+F(x+u, y+v)=\underline{F(x, u+y+v)}+F(u, y+v),$$

and thus

$$F(x, y)+F(u, y+v)+F(x+y, u+v)=F(x, u)+F(y, u+v)+F(x+u, y+v)$$

for all $x, y, u, v \in X$. Therefore, by (1) in Definition 5.1, F is semi-cocyclic.

By this theorem, in particular, we can also state

Corollary 5.9. *If F is quasi-Cauchy, then it is also semi-Cauchy.*

Moreover, from Theorem 5.8 we can more easily derive the following reformulation of [11, Lemma 2] of Davison and Ebanks.

Theorem 5.10. *If F is quasi-Cauchy, then F is also Cauchy-like.*

Proof. Because of Theorem 5.8 and the symmetry of F, we now have

$$\begin{aligned} F(x, y) + \underline{F(y+v, u)} + F(x+y, u+v) \\ = F(x, u) + \underline{F(u+v, y)} + F(x+u, y+v) \end{aligned}$$

for all $x, y, u, v \in X$.

Moreover, from (2) in Definition 2.9, by taking $x = y$, $y = v$ and $z = u$, and using the symmetry of F, we can see that

$$F(y, v) + \underline{F(y+v, u)} = F(y, v+u) + F(v, u) = \underline{F(u+v, y)} + F(u, v)$$

for all $x, y, u, v \in X$. Therefore, (2) in Theorem 5.5 also holds.

Now, by using an even more instructive proof, we can also prove a natural generalization of an observation of Bahyrycz, Páles and Piszczek on Cauchy differences [4], which shows the appropriateness of terminology (2) in Definition 5.1.

Theorem 5.11. *If F is cocyclic, then F is also pseudo-cocyclic.*

Proof. By (2) in Definition 2.9, we have

$$F(s, t) = F(t, z) + F(s, t+z) - F(s+t, z)$$

for all $s, t, z \in X$.

Now, by using this formula and the notation

$$\begin{aligned} \Delta(x, y) = F(x, y) + F(x-u, u) + F(y-v, v) \\ - F(u, v) - F(x-u, y-v) - F(x+y-u-v, u+v) \end{aligned}$$

with $x, y, u, v \in X$, we can see that

$$\begin{aligned} \Delta(x, y) = {} & F(y, z_1) + F(x, y+z_1) - F(x+y, z_1) \\ & + F(u, z_2) + F(x-u, u+z_2) - F(x, z_2) \\ & + \underline{F(v, z_3)} + F(y-v, v+z_3) - F(y, z_3) \\ & - \underline{F(v, z_4)} - F(u, v+z_4) + F(u+v, z_4) \\ & - F(y-v, z_5) - F(x-u, y-v+z_5) + F(x+y-u-v, z_5) \\ & - F(u+v, z_6) - F(x+y-u-v, u+v+z_6) + F(x+y, z_6) \end{aligned}$$

for all $x, y, u, v \in X$ and $z_i \in X$ with $i = 1, \ldots, 6$.

Hence, by taking $z_3 = z_1$, $z_4 = z_3$ and $z_6 = z_4$, we can see that

$$\begin{aligned}\Delta(x,\,y) = {} & F(x,\,y+z_1) \\ & + F(u,\,z_2) + F(x-u,\,u+z_2) - F(x,\,z_2) \\ & + F(y-v,\,v+z_1) \\ & - F(u,\,v+z_1) \\ & - F(y-v,\,z_5) - F(x-u,\,y-v+z_5) + \underline{F(x+y-u-v,\,z_5)} \\ & - \underline{F(x+y-u-v,\,u+v+z_1)}\end{aligned}$$

for all $x,\,y,\,u,\,v \in X$ and $z_i \in X$ with $i = 1,\,2,\,5$.

Hence, by taking $z_2 = y + z_1$ and $z_5 = u + v + z_1$, we can see that

$$\begin{aligned}\Delta(x,\,y) = F(u,\,y+z_1) + F(y-v,\,\underline{v+z_1}) \\ - F(u,\,\underline{v+z_1}) - F(y-v,\,u+\underline{v+z_1})\end{aligned}$$

for all $x,\,y,\,u,\,v \in X$ and $z_1 \in X$.

Hence, by taking $z_1 = -v$ and using Theorem 3.1, we can already infer that

$$\begin{aligned}& \Delta(x,\,y) = F(u,\,y-v) + F(y-v,\,0) - F(u,\,0) - F(y-v,\,u) \\ & = F(u,\,y-v) + F(0,\,0) - F(0,\,0) - F(y-v,\,u) = F(u,\,y-v) - F(y-v,\,u)\end{aligned}$$

for all $x,\,y,\,u,\,v \in X$. Therefore, (2) in Definition 4.1 also holds.

By this theorem, in particular, we can also state

Corollary 5.12. *If F is quasi-Cauchy, then it is also pseudo-Cauchy.*

Remark 5.13. Because of Theorems 5.8 and 5.11, it seems to be a reasonable research program to extend some of the basic theorems on cocycles to semi-cocycles and pseudo-cocycles, and to establish some deeper relationships among the various generalizations of cocycles.

However, a more important research program is suggested by the unifying theorems of Páles [53]. Namely, some further results on the cocycle equation (4) should be extended to the equation (13) of Páles. Moreover, equation (6) and (7) should also be generalized with the help of equation (13).

Unfortunately, even the rather general [53, Theorem 2] does not include characterizations of some further important differences such as, for instance, the Leibniz one $F(x,\,y) = f(x\,y) - x\,f(y) - y\,f(x)$ treated in [39, Corollary 2] of [14, Theorem 3].

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