Tech. Rep., Inst. Math., Univ. Debrecen, 2015/2, 40 pp.

THE CLOSURE-INTERIOR GALOIS CONNECTION AND ITS APPLICATIONS TO RELATIONAL EQUATIONS AND INCLUSIONS

ÁRPÁD SZÁZ

Abstract. By using the closure–interior Galois connection and the box product of relations, we show that the duals of some of our former theorems on the relational equation $R = G^{-1} \circ S \circ F$ and inclusion $R \subseteq G^{-1} \circ S \circ F$ can be easily derived from some simple basic facts on Galois connections.

To clarify the importance of this subject, we recall that a relation F on a set X is called regular (normal) if $F = F \circ S \circ F$ $\big(F = F \circ S \circ (F^c)^{-1} \big)$ for some relation S on X. Note that here S may, for instance, be the identity relation of X or the inverse relation of F. Thus, in particular, all idempotent and non-mingled-valued relations are regular.

Moreover, a pair (F, G) of relations is called properly (uniformly) mildly continuous, with respect to the relations R and S, if $R = G^{-1} \circ S \circ F$ $\big(R \subseteq G^{-1} \circ S \circ F \big)$. Thus, in particular the relation F is regular if and only if the pair $\big(F, F^{-1} \big)$ is properly mildly continuous with respect to F and some relation S. It is noteworthy that the largest such S is $\big(F^{-1} \circ F^c \circ F^{-1} \big)^c$.

1. Introduction

In [50], to have a common generalizations of some of the results of Schein [33], Hardy and Petrich [16], Xu, Liu and Luo [63, 64, 65], Jiang, Xu, Cai and Han [18, 19, 20] and Romano [28, 29, 30], we have proved the following

Theorem 1.1. *For any relations R on X to Y, F on X to Z, and G on Y to W, with the notation*

$$S = \big(G \circ R^c \circ F^{-1} \big)^c ,$$

the following assertions are equivalent:

(1) $R = G^{-1} \circ S \circ F$, *(2)* $R \subseteq G^{-1} \circ S \circ F$,

(3) $R = G^{-1} \circ \Omega \circ F$ *for some relation* Ω *on* Z *to* W,

(4) $\forall\ (x, y) \in R : \quad \exists\ (z, w) \in Z \times W : \quad (x, y) \in F^{-1}(z) \times G^{-1}(w) \subseteq R$,

---

1991 *Mathematics Subject Classification.* Primary 06A75, 06A15; Secondary 54E15, 54C60.

Key words and phrases. Binary relations, upper and lower bounds, closures and interiors, increasing functions and Galois connections, relational equalities and inclusions, regular and mildly continuous relations.

The work of the first author has been supported by the Hungarian Scientific Research Fund (OTKA) Grant K-111651.

(5) $\forall \ (x, y) \in A : \quad \exists \ (z, w) \in Z \times W :$

$$\text{(a)} \quad (x, z) \in F, \quad (y, w) \in G,$$
$$\text{(b)} \quad (s, z) \in F, \quad (t, w) \in G \implies (s, t) \in R.$$

Hint. To prove this, among others, for any $x \in X$ and $y \in Y$, we defined

$$(F \boxtimes G)(x, y) = F(x) \times G(y).$$

Moreover, we showed that, for any relations $U \subseteq X \times Y$ and $V \subseteq Y \times W$, we have

$$(F \boxtimes G)[U] = G \circ U \circ F^{-1} \qquad \text{and} \qquad (F \boxtimes G)^{-1}[V] = G^{-1} \circ V \circ F.$$

Remark 1.2. To illustrate the above theorem, we can use the diagram:

$$\begin{tikzcd} X \arrow[r, "F"] \arrow[d, "R"'] & Z \arrow[d, "S"] \\ Y \arrow[r, "G"] & W \end{tikzcd}$$

Moreover, to show the appropriateness of Theorem 1.1, we can note that assertions (1) and (2) are nothing else, but the definitions of the proper and uniform mild continuities of the pair (F, G) with respect to the relations R and S, respectively. (Concerning mild continuities, see [35, 36, 41, 58, 60].)

In this respect, it is also worth mentioning that, from Theorem 1.1, by taking $Z = Y$, $W = X$, $R = F$ and $G = F^{-1}$, we can immediately derive the following intrinsic characterizations of regular relations established first by Schein [33] and Xu and Liu [63].

Corollary 1.3. *For any relation F on X to Y, with the notation*

$$S = \left(F^{-1} \circ F^{c} \circ F^{-1} \right)^{c},$$

the following assertions are equivalent:

(1) $F = F \circ S \circ F$, *(2)* $F \subseteq F \circ S \circ F$,

(3) $F = F \circ \Omega \circ F$ *for some relation* Ω *on* Y *to* X,

(4) $\forall \ (x, y) \in F : \quad \exists \ (u, v) \in X \times Y : \quad (x, y) \in F^{-1}(v) \times F(u) \subseteq F,$

(5) $\forall \ (x, y) \in F : \quad \exists \ (u, v) \in X \times Y :$

$$\text{(a)} \quad (x, v) \in F, \quad (u, y) \in F;$$
$$\text{(b)} \quad (s, v) \in F, \quad (u, t) \in F \implies (s, t) \in F.$$

Remark 1.4. To illustrate this corollary, we can use the diagram:

$$\begin{tikzcd} X \arrow[r, "F"] \arrow[d, "F"'] & Y \arrow[d, "S"] \\ Y \arrow[r, "F^{-1}"] & X \end{tikzcd}$$

Moreover, we can note that the relation F is regular if and only if the pair $\left(F, F^{-1}\right)$ is properly or uniformly mildly continuous with respect to the relations F and S.

In the present paper, we shall show that, by using the box product $F \boxtimes G$ of relations, a dual of Theorem 1.1 can be easily derived from some simple basic facts on Galois connections between power sets.

For this, for any relation R on X to Y and sets $A \subseteq X$ and $B \subseteq Y$, we define

$$F_R(A) = \operatorname{cl}_{R^{-1}}(A) = R\,[\,A\,] \qquad \text{and} \qquad G_R(B) = \operatorname{int}_R(B) = \operatorname{cl}_R\big(B^c\big)^c .$$

Moreover, we show that the functions F_R and G_R form an increasing Galois connection between the posets $\mathcal{P}(X)$ and $\mathcal{P}(Y)$.

Thus, in particular, by some basic theorems on Galois connections [56], we can at once state that if $B \subseteq Y$, then $A = G_R(B)$ is the largest subset of X such that $F_R(A) \subseteq B$. Moreover, $B = F_R(A)$ for some $A \subseteq X$ if and only if under the notation $\Psi_R = F_R \circ G_R$ we have $B = \Psi_R(A)$ or equivalently $B \subseteq \Psi_R(A)$.

To keep the paper completely self-contained and make the reader popular with our notations and methods, in the subsequent sections, we shall carefully list several basic facts on relations, increasing functions and Galois connections.

The results presented in these preparatory sections may also be useful for those who are not very much interested in relational equations and inclusions. However, for the sake of brevity, the proofs of these results will be frequently omitted.

2. A few basic facts on relations

A subset F of a product set $X \times Y$ is called a *relation* on X to Y. If in particular $F \subseteq X^2$, with $X^2 = X \times X$, then we may simply say that F is a relation on X. In particular, $\Delta_X = \{\,(x,\, x): \ x \in X\,\}$ is called the *identity relation* on X.

If F is a relation on X to Y, then for any $x \in X$ and $A \subseteq X$ the sets $F(x) = \{\,y \in Y: \ (x,\, y) \in F\,\}$ and $F\,[\,A\,] = \bigcup_{a \in A} F(a)$ are called the *images* of x and A under F. If $(x,\, y) \in F$, then we may also write $x\,F\,y$.

Moreover, the sets $D_F = \{\,x \in X: \ F(x) \neq \emptyset\,\}$ and $R_F = F\,[\,X\,]$ are called the *domain* and *range* of F. If in particular $D_F = X$, then we say that F is a relation of X to Y, or that F is a *total relation* on X to Y.

In particular, a relation f on X to Y is called a *function* if for each $x \in D_f$ there exists $y \in Y$ such that $f(x) = \{\,y\,\}$. In this case, by identifying singletons with their elements, we may simply write $f(x) = y$ in place of $f(x) = \{\,y\,\}$.

Moreover, a function $\star$ of X to itself is called a *unary operation* on X. While, a function $*$ of X^2 to X is called a *binary operation* on X. And, for any $x,\, y \in X$, we usually write $x^\star$ and $x * y$ instead of $\star(x)$ and $*\big((x,\, y)\big)$.

If F is a relation on X to Y, then $F = \bigcup_{x \in X} \{x\} \times F(x)$. Therefore, the values $F(x)$, where $x \in X$, uniquely determine F. Thus, a relation F on X to Y can also be naturally defined by specifying $F(x)$ for all $x \in X$.

For instance, the *complement relation* F^c can be naturally defined such that $F^c(x) = F(x)^c = Y \setminus F(x)$ for all $x \in X$. Thus, we also have $F^c = X \times Y \setminus F$. Moreover, it noteworthy that $F^c\,[\,A\,]^c = \bigcap_{a \in A} F(a)$ for all $A \subseteq X$. (See [50].)

Quite similarly, the *inverse relation* F^{-1} can be naturally defined such that $F^{-1}(y) = \{x \in X: \ y \in F(x)\}$ for all $y \in Y$. Thus, we have $F^{-1}\,[\,B\,] = \{x \in X: F(x) \cap B \neq \emptyset\}$ for all $B \subseteq Y$, and hence in particular $D_F = F^{-1}\,[\,Y\,]$.

Moreover, if in addition G is a relation on Y to Z, then the *composition relation* $G \circ F$ can be naturally defined such that $(G \circ F)(x) = G[F(x)]$ for all $x \in X$. Thus, we also have $(G \circ F)[A] = G\big[F[A]\big]$ for all $A \subseteq X$.

If F is a relation on X to Y, then a function f of D_F to Y is called a *selection* of F if $f \subseteq F$, i.e., $f(x) \in F(x)$ for all $x \in D_F$. Thus, by the Axiom of Choice, every relation has a selection. Moreover, it is the union of its selections.

If F is a relation on X to Y and $U \subseteq D_F$, then the relation $F \,|\, U = F \cap (U \times Y)$ is called the *restriction* of F to U. Moreover, if F and G are relations on X to Y such that $D_F \subseteq D_G$ and $F = G \,|\, D_F$, then G is called an *extension* of F.

For any relation F on X to Y, we may naturally define two *set-valued functions*, $F^{\triangleright}$ of X to $\mathcal{P}(Y)$ and $F^{\blacktriangleright}$ of $\mathcal{P}(X)$ to $\mathcal{P}(Y)$, such that $F^{\triangleright}(x) = F(x)$ for all $x \in X$ and $F^{\blacktriangleright}(A) = F[A]$ for all $A \subseteq X$.

Functions of X to $\mathcal{P}(Y)$ can be identified with relations on X to Y. While, functions of $\mathcal{P}(X)$ to $\mathcal{P}(Y)$ are more general objects than relations on X to Y. They were briefly called *corelations* on X to Y in [52].

Now, a relation R on X may be briefly defined to be *reflexive* if $\Delta_X \subseteq R$, and *transitive* if $R \circ R \subseteq R$. Moreover, R may be briefly defined to be *symmetric* if $R^{-1} \subseteq R$, and *antisymmetric* if $R \cap R^{-1} \subseteq \Delta_X$.

Thus, a reflexive and transitive (symmetric) relation may be called a *preorder (tolerance) relation*. And, a symmetric (antisymmetric) preorder relation may be called an *equivalence (partial order) relation*.

For instance, for $A \subseteq X$, the *Pervin relation* $R_A = A^2 \cup A^c \times X$ is a preorder relation on X. (See [47].) While, for a *pseudo-metric* d on X and $r > 0$, the *surrounding* $B_r^d = \big\{ x \in X^2 : \ d(x_1, x_2) < r \big\}$ is a tolerance relation on X.

Moreover, we may recall that if $\mathcal{A}$ is a *partition* of X, i.e., a family of pairwise disjoint, nonvoid subsets of X such that $X = \bigcup \mathcal{A}$, then $S_{\mathcal{A}} = \bigcup_{A \in \mathcal{A}} A^2$ is an equivalence relation on X, which can, to some extent, be identified with $\mathcal{A}$.

According to algebra, for any relation R on X, we may naturally define $R^0 = \Delta_X$, and $R^n = R \circ R^{n-1}$ if $n \in \mathbb{N}$. Moreover, we may also naturally define $R^\infty = \bigcup_{n=0}^{\infty} R^n$. Thus, R^∞ is the smallest preorder relation containing R [14].

Now, in contrast to $(F^c)^c = F$ and $(F^{-1})^{-1} = F$, we have $(R^\infty)^\infty = R^\infty$. Moreover, analogously to $(F^c)^{-1} = (F^{-1})^c$, we also have $(R^\infty)^{-1} = (R^{-1})^\infty$. Thus, in particular R^{-1} is also a preorder on X if R is a preorder on X.

3. Structures derived from a single relation

Notation 3.1. *In this section, we shall assume that R is a relation on one set X to another Y.*

Definition 3.2. Under this assumption, for any $x \in X$ and $B \subseteq Y$, we write

(1) $x \in \operatorname{lb}_R(B)$ if $B \subseteq R(x)$, (2) $x \in \operatorname{int}_R(B)$ if $R(x) \subseteq B$.

Remark 3.3. Thus, lb_R and int_R are relations on $\mathcal{P}(Y)$ to X, which are called the *lower bound and interior relations induced by* R, respectively.

Note that, because of our former observations, lb_R and int_R may also be considered as functions of $\mathcal{P}(Y)$ to $\mathcal{P}(X)$, that is, corelations on Y to X.

The relations lb_R and int_R are not independent of each other. Namely, by using the notations $R^c = X{\times}Y \setminus R$ and $\mathcal{C}_Y(B) = B^c = Y \setminus B$ for all $B \subseteq Y$, we can easily prove the following

Theorem 3.4. *We have*

(1) $\mathrm{lb}_R = \mathrm{int}_{R^c} \circ \mathcal{C}_Y$, *(2)* $\mathrm{int}_R = \mathrm{lb}_{R^c} \circ \mathcal{C}_Y$.

Proof. For any $x \in X$ and $B \subseteq Y$, we have

$$x \in \mathrm{lb}_R(B) \iff B \subseteq R(x) \iff R(x)^c \subseteq B^c \iff R^c(x) \subseteq B^c$$
$$\iff x \in \mathrm{int}_{R^c}(B^c) \iff x \in \mathrm{int}_{R^c}\big(\mathcal{C}_Y(B)\big) \iff x \in \big(\mathrm{lb}_{R^c} \circ \mathcal{C}_Y\big)(B).$$

Therefore, $\mathrm{lb}_R(B) = \big(\mathrm{lb}_{R^c} \circ \mathcal{C}_Y\big)(B)$ for all $B \subseteq Y$, and thus (1) also holds.

Now, by writing R^c in place of R in (1), we can see that $\mathrm{lb}_{R^c} = \mathrm{int}_R \circ \mathcal{C}_Y$. Hence, by applying $\mathcal{C}_Y$ to both sides, we can already infer that $\mathrm{lb}_{R^c} \circ \mathcal{C}_Y = \mathrm{int}_R$, and thus (2) also holds.

Definition 3.5. Now, in addition to Definition 3.2, we may also naturally define

(1) $\mathrm{ub}_R = \mathrm{lb}_{R^{-1}}$, (2) $\mathrm{cl}_R = \big(\mathrm{int}_R \circ \mathcal{C}_Y\big)^c$.

Remark 3.6. Namely, thus for any $A \subseteq X$ and $y \in Y$ we have

$$y \in \mathrm{ub}_R(A) \iff y \in \mathrm{lb}_{R^{-1}}(A) \iff A \subseteq R^{-1}(y)$$
$$\iff \forall\ x \in A:\ x \in R^{-1}(y) \iff \forall\ x \in A:\ y \in R(x) \iff \forall\ x \in A:\ x\,R\,y\,.$$

Moreover, we can easily prove the following

Theorem 3.7. *For any* $B \subseteq Y$ *, we have*

$$\mathrm{cl}_R(B) = \big\{\, x \in X : \quad R(x) \cap B \neq \emptyset \,\big\} = R^{-1}[\,B\,]\,.$$

Proof. For any $x \in X$, we have

$$x \in \mathrm{cl}_R(B) \iff x \in \big(\mathrm{int}_R \circ \mathcal{C}_Y\big)^c(B) \iff x \in \big(\mathrm{int}_R \circ \mathcal{C}_Y\big)(B)^c$$
$$\iff x \notin \big(\mathrm{int}_R \circ \mathcal{C}_Y\big)(B) \iff x \notin \mathrm{int}_R\big(\mathcal{C}_Y(B)\big) \iff x \notin \mathrm{int}_R\big(B^c\big)$$
$$\iff R(x) \not\subseteq B^c \iff R(x) \cap B \neq \emptyset \iff x \in R^{-1}[\,B\,]\,.$$

Therefore, the required equalities are also true.

Later, by considering Galois connections, we shall see that several further properties of the relations lb_R , ub_R , int_R , and cl_R can be easily derived from the following two simple theorems.

Theorem 3.8. *For any* $A \subseteq X$ *and* $B \subseteq Y$, *we have*

$$A \subseteq \mathrm{lb}_R(B) \iff B \subseteq \mathrm{ub}_R(A)\,.$$

Proof. By the corresponding definitions, it is clear that

$$A \subseteq \mathrm{lb}_R(B) \iff \forall\ x \in A:\ x \in \mathrm{lb}_R(B)$$
$$\iff \forall\ x \in A:\ \forall\ y \in B:\ x\,R\,y \iff \forall\ y \in B:\ \forall\ x \in A:\ x\,R\,y$$
$$\iff \forall\ y \in B:\ y \in \mathrm{ub}_R(A) \iff B \subseteq \mathrm{ub}_R(A)\,.$$

Remark 3.9. Note that each of the above conditions is equivalent to the inclusion $A \times B \subseteq R$, which can also be naturally written in the shorter form $A \, R \, B$.

Thus, in particular, for any two subsets A and B of a poset (partially ordered set) X, we may naturally write $A \leq B$ if $x \leq y$ for all $x \in A$ and $y \in B$.

However, this extended inequality is, in general, neither reflexive nor antisymmetric. Namely, $A \not\leq A$ for all $A \subseteq X$ with $\operatorname{card}(A) > 1$, and moreover $\emptyset \leq B$ and $B \leq \emptyset$ for all $B \subseteq X$.

Theorem 3.10. *For any $A \subseteq X$ and $B \subseteq Y$, we have*

$$A \subseteq \operatorname{int}_R(B) \quad \Longleftrightarrow \quad \operatorname{cl}_{R^{-1}}(A) \subseteq B .$$

Proof. If $A \subseteq \operatorname{int}_R(B)$ holds, then for each $x \in A$ we have $x \in \operatorname{int}_R(B)$, and thus $R(x) \subseteq B$. Hence, by using that $R[A] = \bigcup_{x \in A} R(x)$, we can already infer that $R[A] \subseteq B$. Thus, by Theorem 3.7, $\operatorname{cl}_{R^{-1}}(A) \subseteq B$ also holds.

Conversely, if $\operatorname{cl}_{R^{-1}}(A) \subseteq B$ holds, then by Theorem 3.7 we have $R[A] \subseteq B$. Hence, by using that $R[A] = \bigcup_{x \in A} R(x)$, we can already infer that $R(x) \subseteq B$, and thus $x \in \operatorname{int}_R(B)$ for all $x \in A$. Therefore, $A \subseteq \operatorname{int}_R(B)$ also holds.

Remark 3.11. Note that, by Theorem 3.4, the latter two theorems are not independent of each other. However, because of their importance, it seemed now desirable to give them independent proofs.

Definition 3.12. Now, by using our former definitions, we may also naturally define

(1) $\mathcal{E}_R = \big\{ B \subseteq Y : \ \operatorname{int}_R(B) \neq \emptyset \big\}$, (2) $\mathcal{D}_R = \big\{ B \subseteq Y : \ \operatorname{cl}_R(B) = X \big\}$,

(3) $\mathcal{L}_R = \big\{ B \subseteq Y : \ \operatorname{lb}_R(B) \neq \emptyset \big\}$, (4) $\mathcal{U}_R = \big\{ A \subseteq X : \ \operatorname{ub}_R(A) \neq \emptyset \big\}$.

Remark 3.13. The families $\mathcal{E}_R$ and $\mathcal{D}_R$ are called the *R–fat and R–dense subsets* of Y, respectively.

While, the members of the families $\mathcal{L}_R$ and $\mathcal{U}_R$ are called the *lower and upper R–bounded subsets* of Y and X, respectively.

By using the corresponding definitions, we can easily prove the following

Theorem 3.14. *We have*

(1) $\mathcal{E}_R = \big\{ E \subseteq Y : \ E^c \notin \mathcal{D}_R \big\} = \big\{ E \subseteq Y : \ \forall \ D \in \mathcal{D}_R : \ E \cap D \neq \emptyset \big\}$,

(2) $\mathcal{D}_R = \big\{ D \subseteq Y : \ D^c \notin \mathcal{E}_R \big\} = \big\{ D \subseteq Y : \ \forall \ E \in \mathcal{E}_R : \ E \cap D \neq \emptyset \big\}$.

Remark 3.15. Moreover, by the corresponding definitions, it is clear that

(1) $\mathcal{L}_R = \mathcal{U}_{R^{-1}}$, (2) $\mathcal{U}_R = \mathcal{L}_{R^{-1}}$.

Concerning the families $\mathcal{L}_R$ and $\mathcal{L}_R$, for instance, we can also easily prove

Theorem 3.16. *We have*

(1) $\mathcal{L}_R = \mathcal{P}(Y) \setminus \mathcal{D}_{R^c}$, *(2)* $\mathcal{U}_R = \mathcal{P}(X) \setminus \mathcal{D}_{(R^c)^{-1}}$.

Proof. To prove (1), note that by the corresponding definitions and Theorems 3.4 and 3.14, for any $B \subseteq Y$, we have

$$B \in \mathcal{L}_R \iff \operatorname{lb}_R(B) \neq \emptyset \iff \big(\operatorname{int}_{R^c} \circ \, \mathcal{C}_Y\big)(B) \neq \emptyset$$
$$\iff \operatorname{int}_{R^c}\big(B^c\big) \neq \emptyset \iff B^c \in \mathcal{E}_{R^c} \iff B \notin \mathcal{D}_{R^c} \iff B \in \mathcal{P}(Y) \setminus \mathcal{D}_{R^c} .$$

Remark 3.17. In connection with (2), by using Definition 3.12 and Theorems 3.7 and 3.14, we can also easily prove that

(1) $\mathcal{D}_{R^{-1}} = \{ A \subseteq X : \ Y = R[A] \}$, (2) $\mathcal{E}_{R^{-1}} = \{ A \subseteq X : \ Y \neq R[A^c] \}$.

4. Some further structures derived from a single relation

Notation 4.1. *In the present and the next section, by specializing our former Notation 3.1, we shall assume that R is a relation on a set X.*

Remark 4.2. Note that if F is a relation on one set X to another Y, then F is also a relation on $X \cup Y$. However, this view of F is usually quite unnatural.

Definition 4.3. Now, by using our former definitions, for any $A \subseteq X$, we may also naturally define

(1) $\min_R(A) = A \cap \mathrm{lb}_R(A)$, (2) $\max_R(A) = A \cap \mathrm{ub}_R(A)$,

(3) $\inf_R(A) = \max_R\big(\mathrm{lb}_R(A)\big)$, (4) $\sup_R(A) = \min_R\big(\mathrm{ub}_R(A)\big)$.

Thus, we can at once state the following

Theorem 4.4. *For any $A \subseteq X$, we have*

(1) $\inf_R(A) = \mathrm{lb}_R(A) \cap \mathrm{ub}_R\big(\mathrm{lb}_R(A)\big)$,

(2) $\sup_R(A) = \mathrm{ub}_R(A) \cap \mathrm{lb}_R\big(\mathrm{ub}_R(A)\big)$.

Now, by the corresponding definitions, it is clear that we also have

Theorem 4.5. *For any $A \subseteq X$, we have*

(1) $\min_R(A) = \max_{R^{-1}}(A)$, *(2)* $\max_R(A) = \min_{R^{-1}}(A)$,

(3) $\inf_R(A) = \sup_{R^{-1}}(A)$, *(4)* $\sup_R(A) = \inf_{R^{-1}}(A)$.

Moreover, by using Theorem 3.8, we can also easily prove the following two theorems.

Theorem 4.6. *For any $A \subseteq X$, we have*

(1) $\min_R(A) = A \cap \inf_R(A)$, *(2)* $\max_R(A) = A \cap \sup_R(A)$.

Proof. To prove (1), note that, by Theorem 4.4 and Definition 4.3, we have

$$A \cap \inf_R(A) = A \cap \mathrm{lb}_R(A) \cap \mathrm{ub}_R\left(\mathrm{lb}_R(A)\right) = \min_R(A) \cap \mathrm{ub}_R\left(\mathrm{lb}_R(A)\right).$$

Moreover, by Definition 4.3, we have $\min_R(A) \subseteq A$. And, by Theorem 3.8, we have $A \subseteq \mathrm{ub}_R\left(\mathrm{lb}_R(A)\right)$. Therefore, $\min_R(A) \subseteq \mathrm{ub}_R\left(\mathrm{lb}_R(A)\right)$.

Theorem 4.7. *For any $A \subseteq X$, we have*

(1) $\inf_R(A) = \sup_R\big(\mathrm{lb}_R(A)\big)$, *(2)* $\sup_R(A) = \inf_R\big(\mathrm{ub}_R(A)\big)$.

Proof. To prove (1), note that by Definition 4.3 and Theorem 4.6, we have

$$\inf_R(A) = \max_R\big(\mathrm{lb}_R(A)\big) = \mathrm{lb}_R(A) \cap \sup_R\big(\mathrm{lb}_R(A)\big).$$

Moreover, by Theorem 3.8, we have $A \subseteq \mathrm{ub}_R\big(\mathrm{lb}_R(A)\big)$, whence it is clear that $\mathrm{lb}_R\left(\mathrm{ub}_R\big(\mathrm{lb}_R(A)\big)\right) \subseteq \mathrm{lb}_R(A)$. Therefore, by Theorem 4.4, we also have

$$\sup_R\big(\mathrm{lb}_R(A)\big) = \mathrm{ub}_R\big(\mathrm{lb}_R(A)\big) \cap \mathrm{lb}_R\left(\mathrm{ub}_R\big(\mathrm{lb}_R(A)\big)\right) \subseteq \mathrm{lb}_R(A).$$

Remark 4.8. Note that, by Theorem 3.8, $\operatorname{lb}_R(A) \subseteq \operatorname{lb}_R\big(\operatorname{ub}_R\big(\operatorname{lb}_R(A)\big)\big)$ also holds. Therefore, the corresponding equality is also true.

Now, having in mind the notion of a complete poset [3, p. 6], we may also naturally introduce the following

Definition 4.9. The set X, considered in Notation 4.1, is called $\inf_R$*–complete* ($\sup_R$*–complete*) if $\inf_R(A) \neq \emptyset$ $\big(\sup_R(A) \neq \emptyset\big)$ for all $A \subseteq X$.

Remark 4.10. Quite similarly, having in mind the notion of a well-ordered set [3, p. 180], the set X may be naturally called $\min_R$*–complete* ($\max_R$*–complete*) if $\min_R(A) \neq \emptyset$ $\big(\max_R(A) \neq \emptyset\big)$ for all $A \subseteq X$ with $A \neq \emptyset$.

Now, as an immediate consequence of Theorem 4.7, we can also state

Theorem 4.11. *The following assertions are equivalent:*

(1) X *is* $\inf_R$*–complete,* *(2)* X *is* $\sup_R$*–complete.*

Remark 4.12. Some similar equivalences of several modified inf– and sup–completeness properties of generalized ordered sets have been established in [6] and [5].

Definition 4.13. In addition Definition 3.12, we may also naturally define

(1) $\mathcal{T}_R = \big\{ A \subseteq X : \ A \subseteq \operatorname{int}_R(A) \big\}$, (2) $\mathcal{F}_R = \big\{ A \subseteq X : \ \operatorname{cl}_R(A) \subseteq X \big\}$,

(3) $\mathfrak{L}_R = \big\{ A \subseteq X : \ A \subseteq \operatorname{lb}_R(A) \big\}$, (3) $\mathfrak{U}_R = \big\{ A \subseteq X : \ A \subseteq \operatorname{ub}_R(A) \big\}$.

Remark 4.14. The members of the families $\mathcal{T}_R$ and $\mathcal{F}_R$ are called the *R–open and R–closed subsets* of X, respectively.

While, the members of the family $\mathfrak{L}_R$ and $\mathfrak{U}_R$ are called the *self lower and upper R–bounded subsets* of X, respectively.

Thus, we can at once state the following

Theorem 4.15. *We have*

(1) $\mathcal{T}_R = \big\{ A \subseteq X : \ A^c \in \mathcal{F}_R \big\}$, *(2)* $\mathcal{F}_R = \big\{ A \subseteq X : \ A^c \in \mathcal{T}_R \big\}$.

Moreover, we can also easily prove the following two theorems.

Theorem 4.16. *For any* $A \subseteq X$*, the following assertions are equivalent:*

(1) $A \in \mathfrak{L}_R$, *(2)* $A \subseteq \min_R(A)$, *(3)* $A \subseteq \inf_R(A)$,

(4) $A \in \mathfrak{U}_R$, *(5)* $A \subseteq \max_R(A)$, *(6)* $A \subseteq \sup_R(A)$.

Proof. By the corresponding definitions, it is clear that

$$A \in \mathfrak{L}_R \iff A \subseteq \operatorname{lb}_R(A) \iff A \subseteq A \cap \operatorname{lb}_R(A) \iff A \subseteq \min_R(A).$$

Moreover, by using Theorems 3.8 and 4.4, we can also easily see that

$$A \in \mathfrak{L}_R \iff A \subseteq \operatorname{lb}_R(A) \iff A \subseteq \operatorname{lb}_R(A) \cap \operatorname{ub}_R\big(\operatorname{lb}_R(A)\big) \iff A \subseteq \inf_R(A).$$

Therefore, assertions (1), (2), and (3) are equivalent.

The remaining equivalences is now quite obvious by Theorems 3.8 and 4.5. Namely, by these theorems, we have $\mathfrak{L}_R = \mathfrak{U}_R$ and $\mathfrak{U}_R = \mathfrak{L}_{R^{-1}}$.

Theorem 4.17. *We have*

(1) $\mathfrak{L}_R = \big\{ \min_R(A) : \ \ A \subseteq X \big\}$, *(2)* $\mathfrak{L}_R = \big\{ \max_R(A) : \ \ A \subseteq X \big\}$.

Proof. To prove (1), note that if $V \in \mathfrak{L}_R$, then by Theorem 4.16, we have $V \subseteq \min_R(V)$, and thus also $V = \min_R(V)$. Therefore, V is in the family $\mathcal{A} = \{ \min_R(A) : \ A \subseteq X \}$.

Conversely, if $V \in \mathcal{A}$, then there exists $A \in \mathcal{A}$ such that $V = \min_R(A)$. Hence, by definition, it follows that $V \subseteq A$ and $V \subseteq \mathrm{lb}_R(A)$. Now, we can also see that $\mathrm{lb}_R(A) \subseteq \mathrm{lb}_R(V)$. Therefore, $V \subseteq \mathrm{lb}_R(V)$, and thus $V \in \mathfrak{L}_R$ also holds.

5. The importance of reflexivity, transitivity, and antisymmetry

Most of the subsequent theorems have already been proved in [57]. Some of the proofs will only be included here for the reader's convenience.

Theorem 5.1. *The following assertions are equivalent:*

(1) R *is reflexive on* X,

(2) $x \in \mathrm{lb}_R(x)$ *for all* $x \in X$,

(3) $x \in \mathrm{ub}_R(x)$ *for all* $x \in X$,

(4) $A \subseteq \mathrm{cl}_R(A)$ *for all* $A \subseteq X$,

(5) $\mathrm{int}_R(A) \subseteq A$ *for all* $A \subseteq X$,

(6) $\mathrm{int}_R\big(\mathrm{ub}_R(x)\big) \subseteq \mathrm{ub}_R(x)$ *for all* $x \in X$.

Proof. Note that $\mathrm{ub}_R(x) = \mathrm{ub}_R\big(\{x\}\big) = R(x)$ for all $x \in X$. Therefore, (1) and (3) are equivalent. Moreover, if $A \subseteq X$ and $x \in \mathrm{int}_R(A)$, then $R(x) \subseteq A$, and thus $\mathrm{ub}_R(x) \subseteq A$. Therefore, if (3) holds, then $x \in A$, and thus (5) also holds.

Furthermore, we can note that (5) trivially implies (6). Moreover, for any $x \in X$, we have $R(x) = \mathrm{ub}_R(x) \subseteq \mathrm{ub}_R(x)$, and hence $x \in \mathrm{int}_R\big(\mathrm{ub}_R(x)\big)$. Therefore, (6) also implies (3).

Thus, we have proved that (1), (3), (5), and (6) are equivalent. Moreover, by using Definition 3.5, we can also easily see that (2) is equivalent to (3), and (4) is equivalent to (5). However, the equivalence of (1) to (4) is even more obvious from Theorem 3.7.

Now, because of this theorem and Definition 4.13, we can also state

Corollary 5.2. *If* R *is reflexive on* X, *then*

(1) $\mathcal{T}_R = \big\{ A \subseteq X : \ A = \mathrm{int}_R(A) \big\}$, *(2)* $\mathcal{F}_R = \big\{ A \subseteq X : \ A = \mathrm{cl}_R(A) \big\}$.

Theorem 5.3. *The following assertions are equivalent:*

(1) R *is transitive*,

(2) $\mathrm{ub}_R(x) \in \mathcal{T}_R$ *for all* $x \in X$,

(3) $\mathrm{cl}_R(A) \in \mathcal{F}_R$ *for all* $A \subseteq X$,

(4) $\mathrm{int}_R(A) \in \mathcal{T}_R$ *for all* $A \subseteq X$,

(5) $\mathrm{int}_R\big(\mathrm{ub}_R(x)\big) \in \mathcal{T}_R$ *for all* $x \in X$,

(6) $x \in \mathrm{int}_R\Big(\mathrm{int}_R\big(\mathrm{ub}_R(x)\big)\Big)$ *for all* $x \in X$.

Proof. If $x \in X$ and (1) holds, then for any $y \in \operatorname{ub}_R(x)$ we have

$$R(y) \subseteq R\,[\,\operatorname{ub}_R(x)\,] = R\,[\,R(x)\,] = (\,R \circ R\,)(x) \subseteq R(x) = \operatorname{ub}_R(x)\,,$$

and thus $y \in \operatorname{int}_R\big(\operatorname{ub}_R(x)\big)$. Therefore, $\operatorname{ub}_R(x) \subseteq \operatorname{int}_R\big(\operatorname{ub}_R(x)\big)$, and thus (2) also holds.

While, if $A \subseteq X$, then for any $x \in \operatorname{int}_R(A)$ we have $R(x) \subseteq A$, and thus $\operatorname{ub}_R(x) \subseteq A$. Hence, we can infer that $\operatorname{int}_R\big(\operatorname{ub}_R(x)\big) \subseteq \operatorname{int}_R(A)$. Moreover, if (2) holds, then $\operatorname{ub}_R(x) \subseteq \operatorname{int}_R\big(\operatorname{ub}_R(x)\big)$. Therefore, $R(x) = \operatorname{ub}_R(x) \subseteq \operatorname{int}_R(A)$, and thus $x \in \operatorname{int}_R\big(\operatorname{int}_R(A)\big)$. This shows that $\operatorname{int}_R(A) \subseteq \operatorname{int}_R\big(\operatorname{int}_R(A)\big)$, and thus (4) also holds.

Clearly, (4) implies (5). Moreover, if $x \in X$ and (5) holds, then by the corresponding definitions it is clear that

$$x \in \operatorname{int}_R\big(\,R(x)\big) = \operatorname{int}_R\big(\operatorname{ub}_R(x)\big) \subseteq \operatorname{int}_R\,\big(\operatorname{int}_R\big(\operatorname{ub}_R(x)\big)\big)\,,$$

and thus (6) also holds.

While, if (6) holds, then for any $x \in X$ we have

$$R(x) \subseteq \operatorname{int}_R\big(\operatorname{ub}_R(x)\big) = \operatorname{int}_R\big(\,R(x)\big)\,.$$

Therefore, for any $y \in R(x)$, we have $y \in \operatorname{int}_R\big(\,R(x)\big)$, and thus $R(y) \subseteq R(x)$. Hence, we can see that $(\,R \circ R\,)(x) = R\,[\,R(x)\,] \subseteq R(x)$. Therefore, $R \circ R \subseteq R$, and thus (1) also holds.

Now, to complete the proof, it remains to note only that by Definition 3.5 and Theorem 3.14 assertions (2) and (3) are also equivalent.

Now, because of this theorem and Corollary 5.2, we can also state

Corollary 5.4. *If R is a preorder on X, then*

(1) $\mathcal{T}_R = \big\{\operatorname{int}_R(A) : \quad A \subseteq X\,\big\}$, *(2)* $\mathcal{F}_R = \big\{\operatorname{cl}_R(A) : \quad A \subseteq X\,\big\}$.

Moreover, by using Theorems 5.1 and 5.3, and some set theoretic properties of the families of $\mathcal{T}_R$ $\mathcal{F}_R$, we can also easily prove the following two theorems.

Theorem 5.5. *The following assertions are equivalent:*

(1) R *is a preorder on* X,

(2) $\operatorname{int}_R(A) = \bigcup\ \mathcal{T}_R \cap \mathcal{P}(A)$ *for all* $A \subseteq X$,

(3) $\operatorname{cl}_R(A) = \bigcap\ \mathcal{F}_R \cap \mathcal{P}^{-1}(A)$ *for all* $A \subseteq X$.

Theorem 5.6. *If R is a preorder on X, then for any $A \subseteq X$, we have*

(1) $A \in \mathcal{E}_R$ *if and only if* $B \subseteq A$ *for some* $B \in \mathcal{T}_R \setminus \{\emptyset\}$,

(2) $A \in \mathcal{D}_R$ *if and only if* $A \setminus B \neq \emptyset$ *for all* $B \in \mathcal{F}_R \setminus \{\,X\,\}$.

However, it is now more important to note that, by using the corresponding definitions and Theorem 4.17, we can also easily prove the following

Theorem 5.7. *If in particular R is reflexive on X, then the following assertions are equivalent:*

(1) R *is antisymmetric*, *(2)* $\operatorname{card}(A) \leq 1$ *for all* $A \in \mathfrak{L}_R$,

(3) $\operatorname{card}(\Phi(A)) \leq 1$ *for all* $A \subseteq X$ *and* $\Phi \in \{\min_R, \max_R, \inf_R, \sup_R\}$.

Proof. To prove the implication (2) $\Longrightarrow$ (1), note that

$$\mathfrak{L}_R = \left\{ A \subseteq X : \quad \forall\ x, y \in A : \quad x\,R\,y \right\}.$$

Thus, if $x, y \in X$ such that $x \leq y$ and $y \leq x$, then because of the assumptions $x\,R\,x$ and $y\,R\,y$, we also have $\{x, y\} \in \mathfrak{L}_R$. Therefore, if (2) holds, then we necessarily have $x = y$, and thus (1) also holds.

Remark 5.8. Note that the really important implications (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) do not require R to be reflexive.

6. A few basic facts on increasing functions

Definition 6.1. According to [43], an ordered pair $X(\leq) = (X, \leq)$, consisting of a set X and a relation $\leq$ on X, is called a *goset (generalized ordered set)*.

Remark 6.2. Now, the goset $X(\leq)$ may, for instance, be naturally called *reflexive* if the relation $\leq$ is reflexive on X.

Moreover, it may be naturally called a *proset (preordered set)* if $\leq$ is a preorder on X. Thus, a poset is an antisymmetric proset.

Examples 6.3. Note that every set X is a poset with the identity relation Δ_X. And, X is a proset with the universal relation X^2. Moreover, the power set $\mathcal{P}(X)$ is a poset with set inclusion $\subseteq$.

In this respect, it is also worth mentioning that if in particular $X(\leq)$ is a goset, and for any $A, B \subseteq X$ we define $A \leq B$ if $x \leq y$ for all $x \in A$ and $y \in B$, then $\mathcal{P}(X)$ is also a goset with this extended inequality.

Moreover, if $X(\leq)$ is a goset and $Y \subseteq X$, then by taking $\leq_Y = \leq \cap Y^2$ we can also get a goset $Y(\leq_Y)$. This subgoset inherits several properties of the original goset. Thus, for instance, every family of sets is a poset with set inclusion.

However, it is now more important to note that if $X(\leq)$ is a goset, and for instance $X' = X$ and $\leq' = \leq^{-1}$, then $X'(\leq')$ is also a goset. This dual goset also inherits several properties of the original goset. Thus, for instance, every family of sets is also a poset with the reverse set inclusion $\supseteq$.

Remark 6.4. If $X(\leq)$ is a goset, then in the sequel we shall simply write X instead of $X(\leq)$.

Moreover, for instance, we shall also simply write ub and int instead of $\text{lb}_{\leq}$ and $\text{int}_{\leq}$, respectively. However, sometimes the ground set X has to be indicated.

Definition 6.5. A function f of one goset X to another Y is called *increasing* if $u \leq_X v$ implies $f(u) \leq_Y f(v)$ for all $u, v \in X$.

Remark 6.6. Now, the function f may be briefly defined to be *decreasing* if it is increasing as a function of X to Y'. Thus, the study of decreasing functions can be traced back to that of the increasing ones.

Concerning increasing functions, in [57] we have prove the following theorem which shows that increasingness is also a continuity property [58].

Theorem 6.7. *For a function f of one goset X to another Y, the following assertions are equivalent:*

(1) f is increasing,

(2) $f[\mathrm{cl}_X(A)] \subseteq \mathrm{cl}_Y\big(f[A]\big)$ *for all* $A \subseteq X$,

(3) $f[\mathrm{lb}_X(A)] \subseteq \mathrm{lb}_Y\big(f[A]\big)$ *for all* $A \subseteq X$,

(4) $f[\mathrm{ub}_X(A)] \subseteq \mathrm{ub}_Y\big(f[A]\big)$ *for all* $A \subseteq X$,

(5) $\mathrm{cl}_X\big(f^{-1}[B]\big) \subseteq f^{-1}[\mathrm{cl}_Y(B)]$ *for all* $B \subseteq Y$,

(6) $f^{-1}[\mathrm{int}_Y(B)] \subseteq \mathrm{int}_X\big(f^{-1}[B]\big)$ *for all* $B \subseteq Y$.

Proof. To prove the implication (1) $\Longrightarrow$ (2), note that if $A \subseteq X$ and $y \in f[\mathrm{cl}_X(A)]$, then there exists $x \in \mathrm{cl}_X(A)$ such that $y = f(x)$. Thus, by Theorem 3.7, we have $\leq_X(x) \cap A \neq \emptyset$. Therefore, there exists $a \in A$ such that $a \in \leq_X(x)$, and thus $x \leq_X a$. Hence, if (1) holds, we can infer that $f(x) \leq_Y f(a)$. Therefore, $y \leq_Y f(a)$, and thus $f(a) \in \leq_Y(y)$. Hence, since $f(a) \in f[A]$, we can already infer that $\leq_Y(y) \cap f[A] \neq \emptyset$. Therefore, by Theorem 3.7, we have $y \in \mathrm{cl}_Y\big(f[A]\big)$, and thus (2) also holds.

From this theorem, by using Definitions 3.12 and 4.13, we can easily derive the following two corollaries.

Corollary 6.8. *If* f *is an increasing function of one goset* X *to another* Y, *then*

(1) $A \in \mathcal{L}_X$ *implies* $f[A] \in \mathcal{L}_Y$, *(2)* $A \in \mathcal{U}_X$ *implies* $f[A] \in \mathcal{U}_Y$,

(3) $A \in \mathfrak{L}_X$ *implies* $f[A] \in \mathfrak{L}_Y$, *(4)* $A \in \mathfrak{U}_X$ *implies* $f[A] \in \mathfrak{U}_Y$,

(5) $B \in \mathcal{T}_Y$ *implies* $f^{-1}[B] \in \mathcal{T}_X$, *(6)* $A \in \mathcal{F}_Y$ *implies* $f^{-1}[B] \in \mathcal{F}_X$.

Corollary 6.9. *If* f *is an increasing function of one goset* X *onto another* Y, *then*

(1) $A \in \mathcal{D}_X$ *implies* $f[A] \in \mathcal{D}_Y$, *(2)* $B \in \mathcal{E}_Y$ *implies* $f^{-1}[B] \in \mathcal{E}_X$.

Now, by using our former results, we can also easily prove the following

Theorem 6.10. *For a function* f *of a goset* X *to a proset* Y, *the following assertions are equivalent:*

(1) f *is increasing*,

(2) $B \in \mathcal{T}_Y$ *implies* $f^{-1}[B] \in \mathcal{T}_X$, *(3)* $B \in \mathcal{F}_Y$ *implies* $f^{-1}[B] \in \mathcal{F}_X$.

However, it is now more important to note that, in addition to Theorem 6.7, we can also prove the following

Theorem 6.11. *For a function* f *of a reflexive goset* X *to an arbitrary one* Y, *the following assertions are equivalent:*

(1) f *is increasing*,

(2) $f[\max_X(A)] \subseteq \mathrm{ub}_Y\big(f[A]\big)$ *for all* $A \subseteq X$,

(3) $f[\max_X(A)] \subseteq \max_Y\big(f[A]\big)$ *for all* $A \subseteq X$.

Proof. If (1) holds, then by Theorem 6.7, for any $A \subseteq X$, we have

$$f[\max_X(A)] = f[A \cap \mathrm{ub}_X(A)] \subseteq f[A] \cap f[\mathrm{ub}_X(A)] \subseteq f[A] \cap \mathrm{ub}_Y\big(f[A]\big) = \max_Y\big(f[A]\big).$$

Therefore, (3) also holds even if X is not assumed to be reflexive.

Thus, since the implication (3) $\Longrightarrow$ (2) trivially hold, we need only show that (2) also implies (1). For this, note that if $u, v \in X$ such that $u \leq v$, then by taking $A = \{u, v\}$ and using the reflexivity of X we can see that $v \in \mathrm{ub}_X(A)$, and thus $v \in A \cap \mathrm{ub}_X(A) = \max_X(A)$. Hence, if (2) holds, we can infer that

$$f(v) \in f[\max_X(A)] \subseteq \mathrm{ub}_Y\big(f[A]\big) = \mathrm{ub}_Y\big(\{f(u),\, f(v)\}\big).$$

Thus, in particular $f(u) \leq f(v)$, and thus (1) also holds.

From this theorem, by using Theorems 4.5 and 4.4, we can immediately derive

Corollary 6.12. *If f is a function on a reflexive goset X to an arbitrary one Y such that*

$$f[\sup_X(A)] \subseteq \sup_Y\big(f[A]\big)$$

for all $A \subseteq X$, then f is already increasing.

Proof. Namely, by the above assumption and Theorems 4.7 and 4.4, we have

$$f[\max_X(A)] \subseteq f[\sup_X(A)] \subseteq \sup_Y\big(f[A]\big) \subseteq \mathrm{ub}_Y\big(f[A]\big).$$

for all $A \subseteq X$. Thus, Theorem 6.11 can be applied.

Remark 6.13. Note that in Theorem 6.11, and thus also in Corollary 6.12, we may restrict ourselves to the particular case $\mathrm{card}(A) \leq 2$.

Finally, we note that, from Theorem 6.7, we can also immediately derive

Theorem 6.14. *If f is an increasing function of one goset X to another Y, then for any $A \subseteq X$ we have*

$$\mathrm{lb}_Y\left(\mathrm{ub}_Y\big(f[A]\big)\right) \subseteq \mathrm{lb}_Y\left(f[\mathrm{ub}_X(A)]\right).$$

Moreover, by using Theorem 6.7, we can also easily prove the following

Theorem 6.15. *If f is an increasing function of one sup-complete, antisymmetric goset X to another Y, then for any $A \subseteq X$ we have*

$$\sup_Y\big(f[A]\big) \leq_Y f\big(\sup_X(A)\big).$$

Proof. If $\alpha = \sup_X(A)$, then by Theorems 5.7 and 4.4, and the usual identification of singletons with their elements, we also have $\alpha \in \mathrm{ub}_X(A)$, and thus $f(\alpha) \in f[\mathrm{ub}_X(A)]$. Hence, by using Theorem 6.7, we can already infer that $f(\alpha) \in \mathrm{ub}_Y\big(f[A]\big)$.

While, if $\beta = \sup_Y\big(f[A]\big)$, then by Theorems 5.7 and 4.4, and the usual identification of singletons with their elements, we also have $\beta \in \mathrm{lb}_Y\left(\mathrm{ub}_Y\big(f[A]\big)\right)$. Hence, by using that $f(\alpha) \in \mathrm{ub}_Y\big(f[A]\big)$, we can already infer that $\beta \leq_Y f(\alpha)$, and thus the required inequality is also true.

7. A few basic facts on closure operations

The importance of increasing functions in algebra lies mainly in the extensive theory of closure operations and Galois connections. (See [3] and [9].)

Definition 7.1. For an unary operation φ on a goset X, we say that:

(1) φ is *extensive* if $\Delta_X \leq \varphi$,

(2) φ is *upper idempotent* if $\varphi \leq \varphi^2$,

(3) φ is *upper involutive* if $\Delta_X \leq \varphi^2$.

(Note that, by definition, the latter inequality in a detailed form means only that $\Delta_X(x) \leq \varphi^2(x)$, i.e., $x \leq \varphi\big(\varphi(x)\big)$ for all $x \in X$.)

Remark 7.2. Now, the operation φ may be briefly called *intensive, lower idempotent*, and *lower involutive* if it is extensive, upper idempotent, and upper involutive as an operation on X', respectively. Thus, the study of former ones can be traced back to the latter ones.

In this respect, we can also easily establish the following theorem which shows that extensivity is a more fundamental notion than idempotency and involutiveness.

Theorem 7.3. *For an unary operation φ on a goset X, the following assertions hold:*

(1) φ is upper involutive if and only if φ^2 is extensive,

(2) φ is upper idempotent if and only if $\varphi \mid \varphi[X]$ is extensive.

Corollary 7.4. *An extensive operation φ on a goset X is upper idempotent.*

The importance of extensivity is also apparent from the next theorem of [57].

Theorem 7.5. *If φ is a strictly increasing operation on a* min*–complete, antisymmetric goset X, then φ is extensive.*

Proof. For this, note that if φ is not extensive, then $A = \{ x \in X : \ x \not\leq \varphi(x) \} \neq \emptyset$, and thus $\min(A) \neq \emptyset$. Therefore, by Theorem 5.7, we have $a = \min(A)$ for some $a \in X$. Moreover, to get a contradiction, note that now, for any $u, v \in X$, we have $u \not\leq v$ if and only if $v < u$ in the sense that $v \leq u$ and $v \neq u$.

Remark 7.6. Furthermore, note also that a function f of one goset X to another Y is called *strictly increasing* if $u < v$ implies $f(u) < f(v)$ for all $u, v \in X$.

Thus, an injective, increasing function f of one goset X to another Y is strictly increasing. Therefore, as a useful consequence of Theorem 7.5, we can also state

Corollary 7.7. *If φ is an injective, increasing operation on a min-complete, antisymmetric goset X, then φ is extensive.*

Remark 7.8. The importance of the latter results lies mainly in the fact that if φ is an extensive operation an antisymmetric goset X, then each maximal element of X is a fixed point of φ in the sense that $\varphi(x) = x$.

This fact, in the context of posets, was already strongly stressed by Brøndsted [7]. Note that now an element u of a goset X is rather called maximal if $u \leq v$ implies $v \leq u$. However, this is equivalent to $v = u$ if X is antisymmetric.

Definition 7.9. For an unary operation φ on a goset X, we say that:

(1) φ is an *involution* if its increasing and both upper and lower involutive,

(2) φ is a *projection* if it is increasing and both upper and lower idempotent,

(3) φ is a *closure (interior) operation* if it is an extensive (intensive) projection.

Remark 7.10. Thus, φ is an interior operation on X if and only if φ is a closure on X'. Therefore, the study of interior operations can be traced back to that of the closure ones.

Concerning closure operations, we can easily prove the following

Theorem 7.11. *If φ is a closure operation on an inf-complete, antisymmetric goset X, then for any $A \subseteq X$ we have*

$$\inf\big(\varphi\,[A\,]\big) = \varphi\,\big(\inf\big(\varphi\,[\,A\,]\big)\big)\,.$$

Proof. Now, by the dual of Theorem 6.15, we have $\varphi\big(\inf(A)\big) \leq \inf\big(\varphi\,[A\,]\big)$. Hence, by writing $\varphi\,[\,A\,]$ in place of A, we can see that

$$\varphi\,\big(\inf\big(\varphi\,[\,A\,]\big)\big) \leq \inf\,(\,\varphi\,[\,\varphi\,[\,A\,]\,]\,)\,.$$

Moreover, by using the antisymmetry of X, we can see that φ is now idempotent. Therefore, $\varphi\,[\,\varphi\,[\,A\,]\,] = (\varphi \circ \varphi)\,[\,A\,] = \varphi^2\,[\,A\,] = \varphi\,[\,A\,]$. Thus, we actually have

$$\varphi\,\big(\inf\big(\varphi\,[\,A\,]\big)\big) \leq \inf\,(\varphi\,[\,A\,]\,)\,.$$

Moreover, by extensivity of φ, the converse inequality is also true. Thus, by the antisymmetry of X, the required equality is also true.

Corollary 7.12. *If φ is as in Theorem 7.11, then for any $A \subseteq \varphi\,[X]$ we have*

$$\inf\big(A\big) = \varphi\big(\inf(A)\big)\,.$$

Proof. Namely, if $a \in A$, then by using $A \subseteq \varphi\,[X]$ and $\varphi^2 = \varphi$, we can easily see that $\varphi(a) = a$. Thus, $\varphi\,[\,A\,] = A$ also holds.

Corollary 7.13. *If φ is as in Theorem 7.11 and $Y = \varphi\,[\,X\,]$, then for any $A \subseteq Y$ we have*

$$\inf{}_Y(\,A\,) = \inf{}_X(\,A\,)\,.$$

Proof. Namely, if $\alpha = \inf_X(\,A\,)$, then by Corollary 7.12 we have $\alpha = \varphi(\alpha)$. Therefore, $\alpha \in \varphi\,[\,X\,]$, and thus $\alpha \in Y$ also holds.

Hence, by using theorem 4.11, we can immediately derive

Corollary 7.14. *If φ is as in Theorem 7.11, then the subgoset $Y = \varphi\,[\,A\,]$ is also complete.*

Remark 7.15. Note that now, for any $x \in X$ we have $x \in Y$ if and only if $\varphi(x) \leq x$. Therefore, Y is just the family of all "φ–closed elements" of X.

In [57], in addition to Theorem 7.11 and Corollary 7.12, we have also proved the following two theorems.

Theorem 7.16. *If φ is a closure operation on a sup-complete, transitive, and antisymmetric goset X, then for any $A \subseteq X$ we have*

$$\varphi\big(\sup(A)\big) = \varphi\,\big(\sup\big(\varphi\,[A\,]\big)\big)\,.$$

Theorem 7.17. *If φ is as in Theorem 7.16 and $Y = \varphi\,[\,X\,]$, then for any $A \subseteq Y$ we have*

$$\sup{}_Y(\,A\,) = \varphi\big(\sup{}_X(\,A\,)\big)\,.$$

Remark 7.18. Unfortunately, the latter theorem cannot be derived from the former one.

From Theorem 7.16, we can only infer that, for any $A \subseteq X$, the equalities

(1) $\varphi\big(\sup(A)\big) = \sup\big(\varphi[A]\big)$, (2) $\sup\big(\varphi[A]\big) = \varphi\big(\sup\big(\varphi[A]\big)\big)$

are equivalent.

8. The order and interior relations derived from a function

Notation 8.1. *In this section, we shall assume that f is a function on a set X to a goset Y.*

Definition 8.2. Under this assumption, for any $u, v \in X$ and $y \in Y$, we write:

(1) $v \in \operatorname{Ord}_f(u)$ if $f(u) \leq f(v)$, (2) $u \in \operatorname{Int}_f(y)$ if $f(u) \leq y$.

Remark 8.3. Thus, Ord_f is a relation on X and Int_f is a relation on Y to X. These relations will be called the *natural order and proximal interior induced by f*, respectively.

Note that if φ is a function of Y to itself, then for any $y \in Y$ we have $\varphi(y) \leq y$ if and only if $y \in \operatorname{Int}_\varphi(y)$. Therefore, if φ is a closure-like operation on Y, then y is "φ–closed" if and only if it is "proximally φ–open".

The relations Ord_f and Int_f are again not independent of each other. Namely, by using the corresponding definitions, we can easily prove the following

Theorem 8.4. *We have*

$$\operatorname{Ord}_f = \big(\operatorname{Int}_f \circ f\big)^{-1} = f^{-1} \circ \operatorname{Int}_f^{-1}.$$

Proof. By the corresponding definitions, it is clear that

$$v \in \operatorname{Ord}_f(u) \iff f(u) \leq f(v) \iff u \in \operatorname{Int}_f\big(f(v)\big)$$
$$\iff u \in \big(\operatorname{Int}_f \circ f\big)(v) \iff u \in \big(\operatorname{Int}_f \circ f\big)^{-1}(v)$$

for all $u, v \in X$. Therefore, $\operatorname{Ord}_f(v) = \big(\operatorname{Int}_f \circ f\big)^{-1}(v)$ for all $v \in X$, and thus the equality $\operatorname{Ord}_f = \big(\operatorname{Int}_f \circ f\big)^{-1}$ is also true.

Moreover, we can also easily prove the following

Theorem 8.5. *We have*

(1) $\operatorname{Int}_f(y) = f^{-1}[\operatorname{lb}(y)]$ *for all* $y \in Y$,

(2) $\operatorname{Int}_f^{-1}(x) = \operatorname{ub}\big(f(x)\big)$ *for all* $x \in X$.

Proof. If $y \in Y$, then by the corresponding definitions and the usual identifications of singletons with their elements, for any $x \in X$, we have

$$x \in \operatorname{Int}_f(y) \iff f(x) \leq y \iff f(x) \in \operatorname{lb}(y) \iff x \in f^{-1}\big[\operatorname{lb}(y)\big].$$

Therefore, (1) is true. The proof of (2) is similarly simple.

Now, as an immediate consequence of the above theorems, we can also state

Corollary 8.6. *For any $x \in X$, we have*

$$\operatorname{Ord}_f^{-1}(x) = \operatorname{Int}_f\big(f(x)\big) = f^{-1}\big[\operatorname{lb}\big(f(x)\big)\big].$$

Remark 8.7. Moreover, it is noteworthy that, for any $y \in Y$, we also have

$$y \in \mathrm{ub}\left(f\left[\mathrm{Int}_f(y)\right]\right).$$

Namely, by the definition of Int_f, for every $x \in \mathrm{Int}_f(y)$, we have $f(x) \leq y$.

Concerning the relations Ord_f and Int_f, we can also easily prove the following two theorems.

Theorem 8.8. Ord_f *is the largest relation on* X *making* f *to be increasing.*

Proof. Namely, under the notation $\leq_f = \mathrm{Ord}_f$, for any $u, v \in X$, we have $u \leq_f v$ if and only if $f(u) \leq f(v)$.

Therefore, if $\leq$ is a relation on X making f to be increasing, then $u \leq v$ implies $f(u) \leq f(v)$ implies $u \leq_f v$ for all $u, v \in X$. Therefore, $\leq \subseteq \leq_f$, and thus $\leq \subseteq \mathrm{Ord}_f$ is also true.

Remark 8.9. Note that if in particular Y is a proset, then Ord_f is a preorder on X. While, Ord_f is partial order on X if and only if f is injective and Y is a poset.

Theorem 8.10. *If* R *is a relation on* X *to* Y *and*

$$F(A) = R[A]$$

for all $A \subseteq X$, *then for any* $B \subseteq Y$ *we have*

$$\mathrm{int}_R(B) = \left\{x \in X : \quad \{x\} \in \mathrm{Int}_F(B)\right\}.$$

Proof. By the corresponding definitions, for any $x \in X$, we have

$$\begin{aligned} \{x\} \in \mathrm{Int}_F(B) \iff F\left(\{x\}\right) \subseteq B \iff R\left[\{x\}\right] \subseteq B \\ \iff R(x) \subseteq B \iff x \in \mathrm{int}_R(B). \end{aligned}$$

Remark 8.11. Therefore, if F is a function of $\mathcal{P}(X)$ to Y, then for any $x \in X$ and $y \in Y$ we may naturally write $x \in \mathrm{int}_F(y)$ if $\{x\} \in \mathrm{Int}_F(y)$.

Thus, the assertion of the above theorem can be briefly expressed by writing that $\mathrm{int}_R = \mathrm{int}_{R^\blacktriangleright}$. (Note that now, in addition to $F(A) = R^\blacktriangleright(A)$, we also have $F(A) = \mathrm{cl}_{R^{-1}}(A)$ for all $A \subseteq X$.)

However, it is now more important to note that, concerning the relation Int_f, we can also prove the following

Theorem 8.12. *If in particular* X *is also a goset and*

$$f\left[\sup(A)\right] \subseteq \mathrm{lb}\left(\mathrm{ub}\left(f[A]\right)\right)$$

for all $A \subseteq X$, *then for any* $y \in Y$ *we have*

$$\max\left(\mathrm{Int}_f(y)\right) = \sup\left(\mathrm{Int}_f(y)\right).$$

Proof. By Theorem 4.6, it is enough to show only that, for any $y \in Y$,

$$\sup\left(\mathrm{Int}_f(y)\right) \subseteq \mathrm{Int}_f(y).$$

For this, note that if $x \in \sup\left(\mathrm{Int}_f(y)\right)$, then by the assumed property of f

$$f(x) \in f\left[\sup\left(\mathrm{Int}_f(y)\right)\right] \subseteq \mathrm{lb}\left(\mathrm{ub}\left(f\left[\mathrm{Int}_f(y)\right]\right)\right).$$

Moreover, by Remark 8.7, we also have $y \in \mathrm{ub}\left(f\left[\mathrm{Int}_f(y)\right]\right)$. Therefore, we necessarily have $f(x) \leq y$, and thus also $x \in \mathrm{Int}_f(y)$.

9. Increasingly regular and normal functions

Notation 9.1. *In this and the next four sections, we shall assume that:*

(a) X *and* Y *are posets,*

(b) φ *is a function of* X *to itself,*

(c) f *is a function of* X *to* Y *and* g *is a function of* Y *to* X.

Remark 9.2. Because of Theorems 3.8 and 3.10, this generality is actually more than that is sufficient for all practical purposes.

However, by our recent papers [53, 54], X and Y could as well be naturally assumed to be arbitrary gosets, or even relator spaces.

Definition 9.3. Now, according to our former paper [48], we say that:

(1) f is *increasingly* g*–normal* if for any $x \in X$ and $y \in Y$ we have

$$f(x) \leq y \quad \Longleftrightarrow \quad x \leq g(y)\,,$$

(2) f is *increasingly* φ*–regular* if for any $u, v \in X$ we have

$$f(u) \leq f(v) \quad \Longleftrightarrow \quad u \leq \varphi(v)\,.$$

Remark 9.4. If (2) holds, then having in mind the classical terminology [3, p. 124], we may also say that f and g form an *increasing Galois connection* between X and Y.

While, if (1) holds, then because of the latter terminology and the fundamental work of Pataki [24] we may also say that f and φ form an *increasing Pataki connection* between X and Y.

Now, by using the arguments given in [48], we can easily prove the following two theorems.

Theorem 9.5. *If* f *is increasingly* g*–normal and* $\varphi = g \circ f$, *then* f *is increasingly* φ*–regular.*

Theorem 9.6. *If* f *is increasingly* φ*–regular,* $\varphi = g \circ f$, *and* $Y = f\,[\,X\,]$, *then* f *is increasingly* g*–normal.*

Proof. Note that if $x \in X$, $y \in Y$, and $v \in X$ such that $y = f(v)$, then

$$\begin{aligned} f(x) \leq y \iff f(x) \leq f(v) \iff x \leq \varphi(v) \\ \iff x \leq \big(g \circ f\,\big)(v)\big) \iff x \leq g\big(f(v)\big) \iff x \leq g(y)\,. \end{aligned}$$

Remark 9.7. Note that if f is an increasingly g–normal function of X to Y, $Z = f\,[\,X\,]$, and $h = g\,|\,Z$, then f is an increasingly h–normal function of X onto Z.

However, the converse statement need not be true. Therefore, the increasingly normal functions are somewhat more general objects than the increasingly regular ones.

In this respect, it is also worth mentioning that concerning the increasingly regular functions we cannot prove a certain counterpart of the following dualization principle.

Theorem 9.8. *The following assertions are equivalent:*

(1) f *is an increasingly* g*–normal function of* X *to* Y*,*

(2) g *is an increasingly* f*–normal function of* Y' *to* X'*.*

Proof. Recall that $X' = X'(\leq') = X(\leq^{-1})$. Moreover, note that if (1) holds, then for any $y \in Y$ and $x \in X$, we have

$$g(y) \leq' x \iff x \leq g(y) \iff f(x) \leq y \iff y \leq' f(x).$$

Therefore, (2) also holds. The converse implication can be proved quite similarly.

From this theorem, by using Theorem 9.5, we can immediately derive

Corollary 9.9. *If* f *is an increasingly* g*–normal function of* X *to* Y*, and* $\psi = f \circ g$*, then* g *is an increasingly* ψ*–regular function of* Y' *to* X'*.*

By using the following definition, the latter two results can be reformulated in somewhat simpler forms.

Definition 9.10. Under Notation 9.1, we may also naturally say that:

(1) f is *decreasingly* g*–normal* if f is increasingly g–normal as a function of function of X to Y',

(2) f is *decreasingly* φ*–regular* if f is increasingly φ–regular as a function of function of X to Y'.

Remark 9.11. Now, if (2) holds, then we may also say that f and g form an *decreasing Galois connection* between X and Y.

While, if (1) holds, then we may also say that f and φ form a *decreasing Pataki connection* between X and Y.

Moreover, as an immediate consequence of Theorem 3.8, we can also state the following theorem which already shows the importance of decreasingly normal functions.

Theorem 9.12. *If* R *is a relation on* X *to* Y*, and*

$$f_R(A) = \mathrm{ub}_R(A) \qquad \text{and} \qquad g_R(B) = \mathrm{lb}_R(B)$$

for all $A \subseteq X$ *and* $B \subseteq Y$*, then* f_R *is a decreasingly* g_R*–normal function of* $\mathcal{P}(X)$ *to* $\mathcal{P}(Y)$*.*

Proof. By Theorem 3.8, for any $A \subseteq X$ and $B \subseteq Y$, we have

$$f_R(A) \supseteq B \iff B \subseteq \mathrm{ub}_R(A) \iff A \subseteq \mathrm{lb}_R(B) \iff A \subseteq g_R(B).$$

Therefore, by Definition 9.10, the required assertion is also true.

Remark 9.13. The above "upper and lower bound Galois connection", in different formulation, was already studied by Garrett Birkhoff in 1940 under the name polarity [3, p. 122].

For some further developments of the subject, see the papers [23, 11], and the books [4, 13, 12]. However, some more recent results on Galois connections and their applications are in [9] and [10].

10. Some further theorems on increasingly regular and normal functions

Now, by using the arguments given in [48], we can also easily prove the following two theorems and their corollaries.

Theorem 10.1. *If f is increasingly φ–regular, then*

(1) f *is increasing,* *(2)* $f = f \circ \varphi$.

Proof. For any $u \in X$, we have $f(u) \leq f(u)$ and $\varphi(u) \leq \varphi(u)$. Hence, by using the assumed regularity of f, we can infer that $u \leq \varphi(u)$ and $f\big(\varphi(u)\big) \leq f(u)$.

Moreover, if $v \in X$ such that $u \leq v$, then because of $v \leq \varphi(v)$, we also have $u \leq \varphi(v)$. Hence, by using the assumed regularity of f, we can infer that $f(u) \leq f(v)$. Therefore, (1) is true. Now, from $u \leq \varphi(u)$, we can also see that $f(u) \leq f\big(\varphi(u)\big)$, and thus (2) also holds.

Corollary 10.2. *If f is increasingly φ–regular, then the following assertions are equivalent:*

(1) f *is injective,* *(2)* $\varphi = \Delta_X$.

Theorem 10.3. *The following assertions are equivalent:*

(1) φ *is a closure,*

(2) φ *is increasingly* φ*–regular;*

(3) there exists an increasingly φ*–regular function* h *of* X *to a poset* Z .

Proof. To prove the implication (3) $\Longrightarrow$ (1), note that if (3) holds, then because of $h(u) \leq h(u)$ we have $u \leq \varphi(u)$ for all $u \in X$. Therefore, φ is extensive. Moreover, $\varphi(u) \leq \varphi\big(\varphi(u)\big)$ also holds for all $u \in X$.

On the other hand, from Theorem 10.1 we can see that $h\big(\varphi(u)\big) = h(u)$ for all $u \in X$. Thus, $h\left(\varphi\big(\varphi(u)\big)\right) = h\big(\varphi(u)\big) = h(u) \leq h(u)$ also holds for all $u \in X$. Hence, by using (3), we can infer that $\varphi\big(\varphi(u)\big) \leq \varphi(u)$ for all $u \in X$. Therefore, φ is idempotent.

Moreover, if $u, v \in X$ such that $u \leq v$, then by Theorem 10.1 we have $h\big(\varphi(u)\big) = h(u) \leq h(v)$. Hence, by using (3), we can infer that $\varphi(u) \leq \varphi(v)$. Therefore, φ is increasing, and thus (1) also holds.

Remark 10.4. The origin of (2) goes back to R. Dedekind by Erné [10, p. 50]. While, (3) is mainly due to Pataki [24, Theorem 1.9].

Corollary 10.5. *The following assertions are equivalent:*

(1) f *is increasingly* φ*–regular,*

(2) φ *is a closure and* $\operatorname{Ord}_f = \operatorname{Ord}_\varphi$.

Remark 10.6. Note that the equality $\operatorname{Ord}_f = \operatorname{Ord}_\varphi$ in a detailed form means only that for any $u, v \in X$ we have $f(u) \leq f(v)$ if and only if $\varphi(u) \leq \varphi(v)$.

From Theorems 10.1 and 10.3, by Theorem 9.5, it is clear that we also have

Theorem 10.7. *If f is increasingly g–normal and $\varphi = g \circ f$, then*

(1) f *is increasing,* *(2)* φ *is a closure,* *(3)* $f = f \circ \varphi$.

Hence, by using Theorem 9.8, we can immediately derive the following

Corollary 10.8. *If f is increasingly g–normal and $\psi = f \circ g$, then*

(1) g *is increasing,* *(2)* ψ *is an interior,* *(3)* $g = g \circ \psi$.

Now, in addition to Theorem 10.3, we can also easily prove the following

Theorem 10.9. *The following assertions are equivalent:*

(1) φ *is an involution;* *(2)* φ *is increasingly φ–normal.*

Proof. To prove the implication (2) $\Longrightarrow$ (1), note that if (2) holds, then by Theorem 10.7 φ is increasing and φ^2 is extensive. Moreover, by Corollary 10.8, φ^2 is intensive. Therefore, $\varphi^2 = \Delta_X$, and thus (1) also holds.

Moreover, in addition to Corollary 10.2, we can easily prove the following

Theorem 10.10. *If f is increasingly g–normal and $\psi = f \circ g$, then the following assertions are equivalent:*

(1) $Y = f[X]$, *(2)* g *is injective,* *(3)* $\psi = \Delta_Y$.

In this respect, it is also mentioning that we also have the following

Theorem 10.11. *If f is increasingly g–normal and $\psi = f \circ g$, then for any $y \in Y$ the following assertions are equivalent:*

(1) $y \in f[X]$, *(2)* $y = \psi(y)$, *(3)* $y \leq \psi(y)$.

Proof. From Corollary 10.8, we know that $\psi(y) \leq y$. Therefore, (2) and (3) are equivalent. Moreover, if (2) holds, then $y = \psi(y) = (f \circ g)(y) = f\big(g(y)\big)$. Therefore, (1) also holds.

On the other hand, if (1) holds, then there exists $x \in X$ such that $f(x) = y$, and thus $f(x) \leq y$. Hence, by using the increasing g–normality of f, we can infer that $x \leq g(y)$. Now, by Theorem 10.7, we can also state that $f(x) \leq f\big(g(y)\big)$. Hence, by using that $y = f(x)$ and $f\big(g(y)\big) = (f \circ g)(y) = \psi(y)$, we can see that (3) also holds.

Now, in addition to Corollary 10.5, we can also easily prove the following

Theorem 10.12. *Under the notations $\varphi = g \circ f$ and $\psi = f \circ g$, the following assertions are equivalent:*

(1) f *is increasingly g–normal;*

(2) f *and* g *are increasing,* φ *is extensive, and* ψ *is intensive.*

Remark 10.13. This theorem shows that the modern definition of Galois connections [9, p. 155], suggested by Schmidt [34, p. 209], is equivalent to the classical one given by Ore [23].

11. Maximum properties of increasingly normal functions

Now, by using the arguments given in [48], we can also easily prove the following

Theorem 11.1. *The following assertions are equivalent:*

(1) f *is increasingly g–normal,*

(2) $\operatorname{Int}_f(y) = \operatorname{lb}\big(g(y)\big)$ *for all* $y \in Y$,

(3) f *is increasing and* $g(y) = \max\big(\operatorname{Int}_f(y)\big)$ *for all* $y \in Y$.

Proof. By the corresponding definitions, for any $x \in X$ and $y \in Y$, we have

$$f(x) \le y \iff x \in \operatorname{Int}_f(y) \qquad \text{and} \qquad x \le g(y) \iff x \in \operatorname{lb}\big(g(y)\big).$$

Therefore, (1) and (2) are equivalent.

Moreover, if (1) holds, then by Theorem 10.7 the function f is increasing. And, by Corollary 10.8, for any $y \in Y$, we have $f\big(g(y)\big) \le y$, and thus $g(y) \in \operatorname{Int}_f(y)$.

On the other hand, if (1) holds, then by our former observation (2) also holds. Thus, in particular $\operatorname{Int}_f(y) \subseteq \operatorname{lb}\big(g(y)\big)$. Hence, by using Theorem 3.8, we can infer that $g(y) \in \operatorname{ub}\big(\operatorname{Int}_f(y)\big)$. Therefore,

$$g(y) \in \operatorname{Int}_f(y) \cap \operatorname{ub}\big(\operatorname{Int}_f(y)\big) = \max\big(\operatorname{Int}_f(y)\big).$$

Hence, by using Theorem 5.7, we can infer that $g(y) = \max\big(\operatorname{Int}_f(y)\big)$, and thus (3) also holds.

While, if (3) holds, then by Definition 4.3 and Theorem 5.7, for any $y \in Y$, we have $g(y) \in \operatorname{Int}_f(y)$ and $g(y) \in \operatorname{ub}\big(\operatorname{Int}_f(y)\big)$. Hence, by using Theorem 3.8, we can infer that $\operatorname{Int}_f(y) \subseteq \operatorname{lb}\big(g(y)\big)$. Therefore, $f(x) \le y$ implies $x \le g(y)$. Moreover, if $x \le g(y)$, then by the increasingness of f, we also have $f(x) \le f\big(g(y)\big)$. Moreover, because of $g(y) \in \operatorname{Int}_f(y)$, we now also have $f\big(g(y)\big) \le y$. Therefore, $f(x) \le y$, and thus (1) also holds.

From this theorem, it is clear that in particular we also have

Corollary 11.2. *If f is increasingly g–normal, then g is uniquely determined by f.*

Moreover, from Theorem 11.1, by using Theorems 8.5, we can immediately derive

Theorem 11.3. *The following assertions are equivalent:*

(1) f *is increasingly* g*–normal*,

(2) $\operatorname{lb}\big(g(y)\big) = f^{-1}[\operatorname{lb}(y)]$ *for all* $y \in Y$.

Hence, by Theorem 10.9, it is clear that in particular we also have

Corollary 11.4. *The following assertions are equivalent:*

(1) φ *is an involution*,

(2) $\operatorname{lb}\big(\varphi(x)\big) = \varphi^{-1}[\operatorname{lb}(x)]$ *for all* $x \in X$.

However, it is now more important to note that, from Theorem 11.1, we can also easily derive

Theorem 11.5. *The following assertions are equivalent:*

(1) f *is increasingly normal*,

(2) f *is increasing and* $\max\big(\operatorname{Int}_f(y)\big) \ne \emptyset$ *for all* $y \in Y$.

Proof. To prove the implication $(2) \Longrightarrow (1)$, note that if (2) holds, then by Theorem 5.7 we can see that

$$h(y) = \max\big(\operatorname{Int}_f(y)\big)$$

is a singleton for all $y \in Y$. Therefore, h is a function of Y to X. Moreover, if $y_1, y_2 \in Y$ such that $y_1 \leq y_2$, then by using the increasingness of f, we can see that $\operatorname{Int}_f(y_1) \subseteq \operatorname{Int}_f(y_2)$. Therefore, $h(y_1) \leq h(y_2)$, and thus h is also increasing. Hence, by Theorem 11.1, we can see that f is increasingly h–normal, and thus (1) also holds.

From this theorem, it is clear that in particular we also have

Corollary 11.6. *If X is* max*–complete, then following assertions are equivalent:*

(1) f is increasingly normal,

(2) f is increasing and $f[X]$ is descending in Y.

Proof. To prove the implication $(2) \Longrightarrow (1)$, note that if $f[X]$ is descending in Y, then for every $y \in Y$ there exists $x \in X$ such that $f(x) \leq y$, and thus $x \in \operatorname{Int}_f(y)$. Hence, by using the max–completeness of X, we can infer that $\max\big(\operatorname{Int}_f(y)\big) \neq \emptyset$. Therefore, if in addition f is increasing, then by Theorem 11.5 assertion (1) also holds.

Now, more specially we can also state

Corollary 11.7. *If X is* max*–complete and $Y = f[X]$, then following assertions are also equivalent:*

(1) f is increasingly normal, (2) f is increasing.

Remark 11.8. Note that the implications $(1) \Longrightarrow (2)$ in the above corollaries do not require X to be max–complete.

12. Supremum properties of increasingly normal functions

Now, in addition to Theorem 6.14, we can also prove the following

Theorem 12.1. *If f is increasingly normal, then for any $A \subseteq X$ we have*

$$f\left[\operatorname{lb}\big(\operatorname{ub}(A)\big)\right] \subseteq \operatorname{lb}\big(\operatorname{ub}\big(f[A]\big)\big).$$

Proof. If $y \in f\left[\operatorname{lb}\big(\operatorname{ub}(A)\big)\right]$, then there exists $x \in \operatorname{lb}\big(\operatorname{ub}(A)\big)$ such that $y = f(x)$. Moreover, if $v \in \operatorname{ub}\big(f[A]\big)$, then for any $a \in A$ we have $f(a) \leq v$. Hence, by using that f is increasingly h–normal for some function h of Y to X, we can infer that $a \leq h(v)$. Therefore, $h(v) \in \operatorname{ub}(A)$, and thus because of $x \in \operatorname{lb}\big(\operatorname{ub}(A)\big)$ we have $x \leq h(v)$. Hence, by using the increasing h–normality of f, we can infer that $f(x) \leq v$, and thus $y \leq v$. Therefore, $y \in \operatorname{lb}\big(\operatorname{ub}\big(f[A]\big)\big)$ also holds.

Remark 12.2. Note that this proof does not also need any particular properties of the inequality relations in X and Y.

From Theorem 12.1, by using Theorems 10.7 and 6.14, we can immediately derive

Corollary 12.3. *If f is increasingly normal, then for any $A \subseteq X$ we have*

$$f\left[\operatorname{lb}\big(\operatorname{ub}(A)\big)\right] \subseteq \operatorname{lb}\big(f[\operatorname{ub}(A)]\big).$$

However, it is now more important to note that, by using Theorem 12.1 and some former results, we can also prove the following

Theorem 12.4. *If X is* sup*–complete, then the following assertions are equivalent:*

(1) f *is increasingly normal,*

(2) $f\big(\sup(A)\big) = \sup\big(f[A]\big)$ *for all* $A \subseteq X$.

(3) f *is increasing and* $\sup\big(\operatorname{Int}_f(y)\big) \in \operatorname{Int}_f(y)$ *for all* $y \in Y$,

(4) f *is increasing and* $\max\big(\operatorname{Int}_f(y)\big) = \sup\big(\operatorname{Int}_f(y)\big)$ *for all* $y \in Y$,

(5) f *is increasing and* $f\,\big(\sup(A)\big) \in \operatorname{lb}\big(\operatorname{ub}\big(f[A]\big)\big)$ *for all* $A \subseteq X$,

(6) f *is increasing and* $f\big[\operatorname{lb}\big(\operatorname{ub}(A)\big)\big] \subseteq \operatorname{lb}\big(\operatorname{ub}\big(f[A]\big)\big)$ *for all* $A \subseteq X$.

Proof. If (1) holds, then from Theorems 10.7 and 12.1 we can see that (6) also holds. While, if (6) holds, then by using Theorem 4.4 we can see that

$$f[\sup(A)] \subseteq f\big[\operatorname{lb}\big(\operatorname{ub}(A)\big)\big] \subseteq \operatorname{lb}\big(\operatorname{ub}\big(f[A]\big)\big)$$

for all $A \subseteq X$. Hence, by Theorem 5.7 and the sup–completeness of X, it is clear that (5) also holds.

While, if (5) holds, then from Theorem 8.11 we can see that (4) also holds. Moreover, from Theorem 4.6 we can see that, for any $y \in Y$, we have

$$\max\big(\operatorname{Int}_f(y)\big) = \sup\big(\operatorname{Int}_f(y)\big) \quad \Longleftrightarrow \quad \sup\big(\operatorname{Int}_f(y)\big) \subseteq \operatorname{Int}_f(y).$$

Hence, by Theorem 5.7 and the sup–completeness of X, it is clear that (4) and (3) are equivalent. Furthermore, if (4) holds, then from Theorem 11.5 and the sup-completeness of we can see that (1) also holds.

Thus, we have proved that assertions (1), (3), (4), (5) and (6) are equivalent. Therefore, to complete the proof, it remains only to show that (1) and (2) are also equivalent.

For this note that, if (1) holds, then by Theorems 4.4, 6.7 and 12.1 we have

$$\begin{aligned} f\big[\sup(A)\big] = f\big[\operatorname{ub}(A) \cap \operatorname{lb}\big(\operatorname{ub}(A)\big)\big] &\subseteq f\big[\operatorname{ub}(A)\big] \cap f\big[\operatorname{lb}\big(\operatorname{ub}(A)\big)\big] \\ &\subseteq \operatorname{ub}\big(f[A]\big) \cap \operatorname{lb}\big(\operatorname{ub}\big(f[A]\big)\big) = \sup\big(f[A]\big). \end{aligned}$$

for all $A \subseteq X$. Hence, by Theorem 5.7 and the sup–completeness of X, it is clear that (2) also holds.

While, if (2) holds, then from Corollary 6.12, we know that f is increasing. Moreover, by using Theorem 4.4, we can see that

$$f\big(\sup(A)\big) = \sup\big(f[A]\big) \subseteq \operatorname{lb}\big(\operatorname{ub}\big(f[A]\big)\big)$$

for all $A \subseteq X$. Therefore, (5) and thus (1) also holds.

Remark 12.5. Some further supremum properties of increasingly normal functions can be found in [46].

13. Maximum and supremum properties of increasingly regular functions

Now, analogously to Theorem 11.1, we can also easily prove the following

Theorem 13.1. *The following assertions are equivalent:*

(1) f *is increasingly* φ*–regular,*

(2) $\operatorname{Ord}_f^{-1}(x) = \operatorname{lb}\big(\varphi(x)\big)$ *for all* $x \in X$*,*

(3) f *is increasing and* $\varphi(x) = \max\big(\operatorname{Ord}_f^{-1}(x)\big)$ *for all* $x \in X$*.*

Proof. By the corresponding definitions, for any x, $u \in X$, we have

$$f(u) \leq f(x) \iff x \in \operatorname{Ord}_f(u) \iff u \in \operatorname{Ord}_f^{-1}(x)$$

and $u \leq \varphi(x) \iff u \in \operatorname{lb}\big(\varphi(x)\big)$. Therefore, (1) and (2) are equivalent.

The proof of the equivalence $(1) \iff (2)$ is similar to that in Theorem 11.1.

From this theorem, it is clear that in particular we also have

Corollary 13.2. *If* f *is increasingly* φ*–regular, then* φ *is uniquely determined by* f*.*

Moreover, from Theorem 13.1, by using Corollary 8.6, immediately derive

Theorem 13.3. *The following assertions are equivalent:*

(1) f *is increasingly* φ*–regular,*

(2) $\operatorname{lb}\big(\varphi(x)\big) = f^{-1}\big[\operatorname{lb}\big(f(x)\big)\big]$ *for all* $x \in X$*.*

Hence, by Theorem 10.3, it is clear that in particular we also have

Corollary 13.4. *The following assertions are equivalent:*

(1) φ *is a closure,*

(2) $\operatorname{lb}\big(\varphi(x)\big) = \varphi^{-1}\big[\operatorname{lb}\big(\varphi(x)\big)\big]$ *for all* $x \in X$*.*

However, it is now more important to note that, from Theorem 13.1, we can also easily derive

Theorem 13.5. *The following assertions are equivalent:*

(1) f *is increasingly regular,*

(2) f *is increasing and* $\max\big(\operatorname{Ord}_f^{-1}(x)\big) \neq \emptyset$ *for all* $x \in X$*.*

From this theorem, it is clear that in particular we also have

Corollary 13.6. *If* X *is* max*–complete, then following assertions are equivalent:*

(1) f *is increasingly regular,* *(2)* f *is increasing.*

Proof. Namely now, by the corresponding definitions, for any $x \in X$ we have $x \in \operatorname{Ord}_f^{-1}(x)$, and thus $\operatorname{Ord}_f^{-1}(x) \neq \emptyset$. Therefore, by the max–completeness of X, we also have $\max\big(\operatorname{Ord}_f^{-1}(x)\big) \neq \emptyset$.

Now, by using Theorem 13.5, we can also prove the following

Theorem 13.7. *If* $Y = f[X]$*, then the following assertions are equivalent:*

(1) f *is increasingly regular,* *(2)* f *is increasingly normal.*

Proof. If (1) holds, then by Theorem 13.5 the function f is increasing, and moreover $\max\big(\operatorname{Ord}_f^{-1}(x)\big) \neq \emptyset$ for all $x \in X$. Moreover, from Theorem 8.4, we can see that $\max\big(\operatorname{Int}_f\big(f(x)\big)\big) = \max\big(\operatorname{Ord}_f^{-1}(x)\big) \neq \emptyset$ for all $x \in X$. Hence, by using that

$Y = f[X]$, we can already infer that $\max\big(\operatorname{Int}_f(y)\big) \neq \emptyset$ for all $y \in Y$. Thus, by Theorem 11.5, assertion (2) also holds.

The converse implication $(2) \Longrightarrow (1)$ is immediate from Theorem 9.5.

Remark 13.8. By using this theorem, from the results of Section 12 we can immediately derive some useful supremum properties of increasingly regular functions.

For instance, from Theorem 12.4, by using Theorem 13.7 and Theorem 8.4, we can immediately derive the following partial generalization of [46, Corollary 8.2].

Theorem 13.9. *If X is* sup*–complete, then under the notation $Z = f[X]$ the following assertions are equivalent:*

(1) f is increasingly regular,

(2) $f[\sup(A)] = \sup_Z\big(f[A]\big)$ for all $A \subseteq X$,

(3) f is increasing and $\sup\big(\operatorname{Ord}_f^{-1}(x)\big) \in \operatorname{Ord}_f^{-1}(x)$ for all $x \in X$,

(4) f is increasing and $\max\big(\operatorname{Ord}_f^{-1}(x)\big) = \sup\big(\operatorname{Ord}_f^{-1}(x)\big)$ for all $x \in X$.

Hence, by using Theorem 10.3, we can immediately derive the following partial generalization of Theorem 7.17.

Corollary 13.10. *If X is* sup*-complete, then under the notation $Z = \varphi[X]$, the following assertions are equivalent:*

(1) φ is a closure,

(2) $\varphi[\sup(A)] = \sup_Z\big(\varphi[A]\big)$ for all $A \subseteq X$,

(3) φ is increasing and $\sup\big(\operatorname{Ord}_\varphi^{-1}(x)\big) \in \operatorname{Ord}_\varphi^{-1}(x)$ for all $x \in X$,

(4) φ is increasing and $\max\big(\operatorname{Ord}_\varphi^{-1}(x)\big) = \sup\big(\operatorname{Ord}_\varphi^{-1}(x)\big)$ for all $x \in X$.

14. The closure–interior Galois connection

Notation 14.1. *In this and the next three sections, we shall assume that R is a relation on one set X to another Y.*

Definition 14.2. Under this assumption, we define

$$F_R(A) = \operatorname{cl}_{R^{-1}}(A) \qquad \text{and} \qquad G_R(B) = \operatorname{int}_R(B)$$

for all $A \subseteq X$ and $B \subseteq Y$.

Remark 14.3. Thus, by Definition 3.2 and Theorem 3.7, we have

(1) $G_R(B) = \big\{ x \in X : \quad R(x) \subseteq B \big\}$ for all $B \subseteq Y$,

(2) $F_R(A) = \big\{ y \in Y : \quad R^{-1}(y) \cap A \neq \emptyset \big\}$ for all $A \subseteq X$.

Moreover, by using Theorem 3.7 and Definition 3.5, we can easily establish

Theorem 14.4. *We have*

(1) $F_R(A) = R[A]$ for all $A \subseteq X$,

(2) $G_R(B) = R^{-1}[B^c]^c$ for all $B \subseteq Y$.

Proof. To check (2), note that by Definition 3.5 we have $\operatorname{int}_R = \operatorname{cl}_R^c \circ \mathcal{C}_Y$. Thus, by the corresponding definitions and Theorem 3.7, we also have

$$G_R(B) = \operatorname{int}_R(B) = \big(\operatorname{cl}_R^c \circ \mathcal{C}_Y\big)(B) = \operatorname{cl}_R^c(B^c) = \operatorname{cl}_R(B^c)^c = R^{-1}[B^c]^c$$

for all $B \subseteq Y$.

Now, by this theorem and Remark 14.3, we can also state

Corollary 14.5. *Under the notation* $F_R(x) = F_R\big(\{x\}\big)$, *we have*

(1) $R(x) = F_R(x)$ *for all* $x \in X$,

(2) $G_R(B) = \big\{ x \in X : \quad F_R(x) \subseteq B \big\}$ *for all* $B \subseteq Y$.

However, it is now more important to note that, by using Theorem 3.10, we can also easily prove the following

Theorem 14.6. F_R *is an increasingly* G_R*–normal function of* $\mathcal{P}(X)$ *to* $\mathcal{P}(Y)$.

Proof. By Definition 14.2 and Theorem 3.10, for any $A \subseteq X$ and $B \subseteq Y$, we have

$$F_R(A) \subseteq B \iff \operatorname{cl}_{R^{-1}}(A) \subseteq B \iff A \subseteq \operatorname{int}_R(B) \iff A \subseteq G_R(B).$$

Therefore, by Definition 9.3, the required assertion is also true.

Remark 14.7. By using our former terminology, this theorem can also be expressed by saying that the functions F_R and G_R form an increasing Galois connection between the posets $\mathcal{P}(X)$ and $\mathcal{P}(Y)$.

Because of Theorem 3.4, this "closure–interior Galois connection" is not independent from the "upper and lower bound Galois connection" considered in Theorem 9.12. It has been first introduced in [56] with reference to [9, Exercise 7.18].

From Theorem 14.6, by using our former theorems on increasingly normal functions, we can immediately derive several theorems about the functions F_R and G_R. However, it is sometimes more convenient to give some direct proofs.

For instance, from Theorem 14.6, by using Theorems 11.1 and 12.4, we can immediately derive the following two theorems, which can however be more easily proved directly.

Theorem 14.8. *If* G *is a function of* $\mathcal{P}(Y)$ *to* $\mathcal{P}(X)$ *such that* F_R *is* G*–normal, then* $G = G_R$.

Proof. By Definition 9.3 and Corollary 14.5, for any $x \in X$ and $B \subseteq Y$, we have

$$x \in G(B) \iff \{x\} \subseteq G(B) \iff F_R\big(\{x\}\big) \subseteq B \iff x \in G_R(B).$$

Theorem 14.9. F_R *is union-preserving and* G_R *is intersection-preserving.*

Proof. If $A_i \subseteq X$ for all $i \in I$, then by Theorem 14.4 and a basic property of relations, it is clear that

$$F_R\Big(\bigcup_{i\in I} A_i\Big) = R\Big[\bigcup_{i\in I} A_i\Big] = \bigcup_{i\in I} R[A_i] = \bigcup_{i\in I} F_R(A_i).$$

While, if $B_j \subseteq Y$ for all $j \in J$, then by using Theorem 14.4 and the DeMorgan laws, we can easily see that

$$G_R\Big(\bigcap_{j\in J} B_j\Big) = R^{-1}\Big[\Big(\bigcap_{j\in J} B_j\Big)^c\Big]^c = R^{-1}\Big[\bigcup_{j\in J} B_j^c\Big]^c$$
$$= \Big(\bigcup_{j\in J} R^{-1}[B_j^c]\Big)^c = \bigcap_{j\in J} R^{-1}[B_j^c]^c = \bigcap_{j\in J} G_R(B_j).$$

Hence, it is clear that in particular we can also have

Corollary 14.10. *F_R and G_R are increasing with respect to set inclusion.*

Proof. Namely, if for instance $B_1, B_2 \subseteq Y$ such that $B_1 \subseteq B_2$, then by Theorem 14.9 we have $G_R(B_1) = G_R(B_1 \cap B_2) = G_R(B_1) \cap G_R(B_2)$, and thus also $G_R(B_1) \subseteq G_R(B_2)$.

Remark 14.11. Note that the increasingness of G_R is also quite obvious from Definitions 13.2 and 3.2.

15. The induced closure and interior operations

Definition 15.1. Now, by Theorem 10.7 and Corollary 10.8, we may also naturally define

$$\Phi_R = G_R \circ F_R \qquad \text{and} \qquad \Psi_R = F_R \circ G_R.$$

Remark 15.2. Thus, we have

(1) $\Phi_R(A) = G_R\big(F_R(A)\big)$ for all $A \subseteq X$,

(2) $\Psi_R(B) = F_R\big(G_R(B)\big)$ for all $B \subseteq Y$.

Hence, by using Definition 14.2 and Theorem 14.4, we can immediately derive the following two theorems.

Theorem 15.3. *We have*

(1) $\Phi_R(A) = \operatorname{int}_R\big(\operatorname{cl}_{R^{-1}}(A)\big)$ *for all* $A \subseteq X$,

(2) $\Psi_R(B) = \operatorname{cl}_{R^{-1}}\big(\operatorname{int}_R(B)\big)$ *for all* $B \subseteq Y$.

Theorem 15.4. *We have*

(1) $\Phi_R(A) = R^{-1}\big[R[A]^c\big]^c$ *for all* $A \subseteq X$,

(2) $\Psi_R(B) = R\big[R^{-1}[B^c]^c\big]$ *for all* $B \subseteq Y$.

Now, as an immediate consequence of Theorems 14.6 and 9.5 and Corollary 9.9, we also can state the following

Theorem 15.5. *The following assertions hold:*

(1) *F_R is an increasingly Φ_R–regular function of $\mathcal{P}(X)$ to $\mathcal{P}(Y)$,*

(2) *G_F is an increasingly Ψ_R–regular function of $\mathcal{P}(Y)'$ to $\mathcal{P}(X)'$.*

Remark 15.6. The above assertions in detailed forms mean only that:

(1) $F_R(U) \subseteq F_R(V) \iff U \subseteq \Phi_R(V)$ for all $U, V \subseteq X$,

(2) $G_R(\Omega) \subseteq G_R(W) \iff \Psi_R(\Omega) \subseteq W$ for all $\Omega, W \subseteq Y$.

Hence, by using Definition 14.2 and Theorems 14.4, 15.3 and 15.4, we can immediately derive the following two corollaries.

Corollary 15.7. *We have*

(1) $\operatorname{int}_R(\Omega) \subseteq \operatorname{int}_R(W) \iff \operatorname{cl}_{R^{-1}}\big(\operatorname{int}_R(\Omega)\big) \subseteq W$ *for all* $\Omega, W \subseteq Y$,

(2) $\operatorname{cl}_{R^{-1}}(U) \subseteq \operatorname{cl}_{R^{-1}}(V) \iff U \subseteq \operatorname{int}_R\big(\operatorname{cl}_{R^{-1}}(V)\big)$ *for all* $U, V \subseteq X$.

Corollary 15.8. *We have*

(1) $R[U] \subseteq R[V] \iff R^{-1}\big[R[V]^c\big] \subseteq U^c$ *for all* $U, V \subseteq X$,

(2) $R^{-1}[\Omega^c] \subseteq R^{-1}[W^c] \iff R\big[R^{-1}[W^c]^c\big] \subseteq \Omega$ *for all* $\Omega, W \subseteq Y$.

Proof. To check (2), note that, by Theorem 14.4 Remark 15.6 and Theorem 15.4, we have

$$R^{-1}[\Omega^c] \subseteq R^{-1}[W^c] \iff R^{-1}[W^c]^c \subseteq R^{-1}[\Omega^c]^c$$
$$\iff G_R(W) \subseteq G_R(\Omega) \iff \Psi_R(W) \subseteq \Omega \iff R\big[R^{-1}[W^c]^c\big] \subseteq \Omega$$

for all $\Omega, W \subseteq Y$.

Now, as an immediate consequence of Theorems 14.6 and and 10.7 and Corollary 10.8, we can also state

Theorem 15.9. *The following assertions hold:*

(1) Φ_R *is a closure on* $\mathcal{P}(X)$ *such that* $F_R = F_R \circ \Phi_R$,

(2) Ψ_R *is an interior on* $\mathcal{P}(Y)$ *such that* $G_R = G_R \circ \Psi_R$.

Remark 15.10. Thus, in particular we have

(1) $A \subseteq \Phi_R(A)$ for all $A \subseteq X$, (2) $\Psi_R(B) \subseteq B$ for all $B \subseteq Y$.

Hence, by using Theorems 15.3 and 15.4 we can immediately derive the following two corollaries.

Corollary 15.11. *We have*

(1) $A \subseteq \operatorname{int}_R\big(\operatorname{cl}_{R^{-1}}(A)\big)$ *for all* $A \subseteq X$,

(2) $\operatorname{cl}_{R^{-1}}\big(\operatorname{int}_R(B)\big) \subseteq B$ *for all* $B \subseteq Y$.

Corollary 15.12. *We have*

(1) $A \subseteq R^{-1}\big[R[A]^c\big]^c$ *for all* $A \subseteq X$,

(2) $R\big[R^{-1}[B^c]^c\big] \subseteq B$ *for all* $B \subseteq Y$.

Remark 15.13. By Theorem 15.9, we also have

(1) $F_R(A) = F_R\big(\Phi_R(A)\big)$ for all $A \subseteq X$,

(2) $G_R(B) = G_R\big(\Psi_R(B)\big)$ for all $B \subseteq Y$.

Hence, by using Definition 14.2 and Theorems 14.4, 15.3 and 15.4, we can immediately derive the following two corollaries.

Corollary 15.14. *We have*

(1) $\operatorname{int}_R(B) = \operatorname{int}_R\big(\operatorname{cl}_{R^{-1}}\big(\operatorname{int}_R(B)\big)\big)$ *for all* $B \subseteq Y$,

(2) $\operatorname{cl}_{R^{-1}}(A) = \operatorname{cl}_{R^{-1}}\big(\operatorname{int}_R\big(\operatorname{cl}_{R^{-1}}(A)\big)\big)$ *for all* $A \subseteq X$.

Corollary 15.15. *We have*

(1) $R[A] = R\left[R^{-1}\left[R[A]^c\right]^c\right]$ *for all* $A \subseteq X$*,*

(2) $R^{-1}[B^c] = R^{-1}\left[R\left[R^{-1}[B^c]^c\right]^c\right]$ *for all* $B \subseteq Y$*.*

Proof. To check (2), note that by Theorem 14.4, Remark 15.12, and Theorem 15.4 we have

$$R^{-1}[B^c] = \left(R^{-1}[B^c]^c\right)^c = G_R(B)^c = G_R\big(\Psi_R(B)\big)^c$$
$$= \left(R^{-1}\left[\Psi_R(B)^c\right]^c\right)^c = R^{-1}\left[\Psi_R(B)^c\right] = R^{-1}\left[R\left[R^{-1}[B^c]^c\right]^c\right]$$

for all $B \subseteq Y$.

16. Coincidence of the values of F_R and G_R with the fixed points of Ψ_R and Φ_R

Now, as an immediate consequence of Theorems 14.6 and 10.11, we can also state the following

Theorem 16.1. *For any* $B \subseteq Y$*, the following assertions are equivalent:*

(1) $B = F_R(A)$ *for some* $A \subseteq X$*,*

(2) $B = \Psi_R(B)$*,* *(3)* $B \subseteq \Psi_R(B)$*.*

Hence, by using Definition 14.2 and Theorems 14.4, 15.3 and 15.4, we can immediately derive the following two corollaries.

Corollary 16.2. *For any* $B \subseteq Y$*, the following assertions are equivalent:*

(1) $B = \operatorname{cl}_{R^{-1}}(A)$ *for some* $A \subseteq X$*,*

(2) $B = \operatorname{cl}_{R^{-1}}\big(\operatorname{int}_R(B)\big)$*,* *(3)* $B \subseteq \operatorname{cl}_{R^{-1}}\big(\operatorname{int}_R(B)\big)$*.*

Corollary 16.3. *For any* $B \subseteq Y$*, the following assertions are equivalent:*

(1) $B = R[A]$ *for some* $A \subseteq X$*,*

(2) $B = R\left[R^{-1}[B^c]^c\right]$*,* *(3)* $B \subseteq R\left[R^{-1}[B^c]^c\right]$*.*

Now, in addition to the above corollaries, we can also easily prove

Theorem 16.4. *For any* $B \subseteq Y$*, the following assertions are equivalent:*

(1) $B = R[A]$ *for some* $A \subseteq X$*,*

(2) $B = \operatorname{cl}_{R^{-1}}(A)$ *for some* $A \subseteq X$*,*

(3) $\forall\ y \in B:\ \exists\ x \in X:\ \ y \in R(x) \subseteq B$*.*

Proof. To prove the implication (2) $\Longrightarrow$ (3), note that if (2) holds, then by Corollary 16.2 we have $B \subseteq \operatorname{cl}_{R^{-1}}\big(\operatorname{int}_R(B)\big)$. Therefore, for every $y \in B$, we have $y \in \operatorname{cl}_{R^{-1}}\big(\operatorname{int}_R(B)\big)$. Hence, by using Theorem 3.7, we can already infer that $R^{-1}(y) \cap \operatorname{int}_R(B) \neq \emptyset$. Therefore, there exists $x \in X$ such that $x \in R^{-1}(y)$ and $x \in \operatorname{int}_R(B)$. Hence, by using the corresponding definitions, we can infer that $y \in R(x)$ and $R(x) \subseteq B$, and thus (3) also holds.

Remark 16.5. From Theorem 10.11, by using Theorem 9.8, we can easily see that if f is an increasingly g–normal function f of one poset X to another Y, then under the notation $\varphi = g \circ f$, for any $x \in X$, the following assertions are equivalent:

(1) $x \in g[Y]$, (2) $x = \varphi(x)$, (3) $\varphi(x) \leq x$.

Hence, by Theorem 14.6, it is clear that in particular we also have the following dual of Theorem 16.1.

Theorem 16.6. *For any $A \subseteq X$, the following assertions are equivalent:*

(1) $A = G_R(B)$ *for some* $B \subseteq Y$*,*

(2) $A = \Phi_R(A)$*,* *(3)* $\Phi_R(A) \subseteq A$*.*

Now, by using Definition 14.2 and Theorems 14.4, 15.3 and 15.4, we can also easily establish following two corollaries.

Corollary 16.7. *For any $A \subseteq X$, the following assertions are equivalent:*

(1) $A = \operatorname{int}_R(B)$ *for some* $B \subseteq Y$*,*

(2) $A = \operatorname{int}_R\big(\operatorname{cl}_{R^{-1}}(A)\big)$*,* *(3)* $\operatorname{int}_R\big(\operatorname{cl}_{R^{-1}}(A)\big) \subseteq A$*.*

Corollary 16.8. *For any $A \subseteq X$, the following assertions are equivalent:*

(1) $A = R^{-1}[B^c]^c$ *for some* $B \subseteq Y$*,*

(2) $A = R^{-1}\big[R[A]^c\big]^c$*,* *(3)* $R^{-1}\big[R[A]^c\big]^c \subseteq A$*.*

Moreover, in addition to the latter corollaries, we can also easily prove

Theorem 16.9. *For any $A \subseteq X$, the following assertions are equivalent:*

(1) $A = \operatorname{int}_R(B)$ *for some* $B \subseteq Y$*,*

(2) $A = R^{-1}[B^c]^c$ *for some* $B \subseteq Y$*,*

(3) $\forall\ x \in A^c: \quad \exists\ y \in R(x): \quad y \notin R[A]$*.*

Proof. To prove the implication (1) $\Longrightarrow$ (3), note that if (1) holds, then by Corollary 16.7 we have $\operatorname{int}_R\big(\operatorname{cl}_{R^{-1}}(A)\big) \subseteq A$, and thus $A^c \cap \operatorname{int}_R\big(\operatorname{cl}_{R^{-1}}(A)\big) = \emptyset$. Therefore, for every $x \in A^c$, we have $x \notin \operatorname{int}_R\big(\operatorname{cl}_{R^{-1}}(A)\big)$, and thus also $R(x) \not\subseteq \operatorname{cl}_{R^{-1}}(A)$. Therefore, there exists $y \in R(x)$ such that $y \notin \operatorname{cl}_{R^{-1}}(A)$. Hence, by using Theorem 3.7, we can infer that $R^{-1}(y) \cap A = \emptyset$. Therefore, $y \notin R[A]$, and thus (3) also holds.

Remark 16.10. Note that requirement $y \notin R[A]$ in a detailed form means only that $y \notin R(a)$ for all $a \in A$.

17. Maximum properties of the functions F_R and Φ_R

Now, by Theorems 14.6 and 11.1, we can also at once state the following

Theorem 17.1. *For any $B \subseteq Y$, we have*

(1) $\operatorname{Int}_{F_R}(B) = \operatorname{lb}\big(G_R(B)\big)$*,* *(2)* $G_R(B) = \max\big(\operatorname{Int}_{F_R}(B)\big)$*.*

Remark 17.2. Moreover, by the corresponding definitions, we also have

(1) $\operatorname{lb}\big(G_R(B)\big) = \mathcal{P}\big(G_R(B)\big)$, (2) $\operatorname{Int}_{F_R}(B) = \big\{ A \subseteq X : \ F_R(A) \subseteq B \big\}$.

Thus, in particular, by Theorem 17.1, we can also state the following

Corollary 17.3. *If* $B \subseteq Y$, *then* $A = G_F(B)$ *is the largest subset of* X *such that* $F_R(A) \subseteq B$.

Hence, by using Definition 14.2 and Theorem 14.4, we can immediately derive the following two corollaries.

Corollary 17.4. *If* $B \subseteq Y$, *then* $A = \operatorname{int}_R(B)$ *is the largest subset of* X *such that* $\operatorname{cl}_{R^{-1}}[A] \subseteq B$.

Corollary 17.5. *If* $B \subseteq Y$, *then* $A = R^{-1}[B^c]^c$ *is the largest subset of* X *such that* $R[A] \subseteq B$.

Remark 17.6. To check this statement, note that $x \in A \implies x \in R^{-1}[B^c]^c \implies x \notin R^{-1}[B^c] \implies R(x) \cap B^c = \emptyset \implies R(x) \subseteq B$. Therefore, $R[A] \subseteq B$.

Moreover, if $U \subseteq X$ such that $R[U] \subseteq B$, then $x \in U \implies R(x) \subseteq B \implies R(x) \cap B^c = \emptyset \implies x \notin R^{-1}[B^c] \implies x \in R^{-1}[B^c]^c \implies x \in A$. Therefore, $U \subseteq A$.

Now, by Theorems 15.5 and 13.1, we can also at once state the following

Theorem 17.7. *For every* $A \subseteq X$, *we have*

(1) $\operatorname{Ord}_{F_R}^{-1}(A) = \operatorname{lb}\big(\Phi_R(A)\big)$, *(2)* $\Phi_R(A) = \max\big(\operatorname{Ord}_{F_R}^{-1}(A)\big)$.

Remark 17.8. Moreover, by Theorem 15.9 and the corresponding definitions, we also have

(1) $F_R(A) = F_R\big(\Phi_R(A)\big)$,

(2) $\operatorname{Ord}_{F_R}^{-1}(A) = \big\{ U \subseteq X : \ F_R(U) \subseteq F_R(A) \big\}$.

Thus, in particular, by Theorem 17.7, we can also state the following

Corollary 17.9. *If* $A \subseteq X$, *then* $U = \Phi_R(A)$ *is the largest subset of* X *such that* $F_R(U) \subseteq F_R(A)$ $\big(F_R(U) = F_R(A)\big)$.

Hence, by using Theorem 15.3 and 15.4, we can immediately derive the following two corollaries.

Corollary 17.10. *If* $A \subseteq X$, *then* $U = \operatorname{int}_R\big(\operatorname{cl}_{R^{-1}}(A)\big)$ *is the largest subset of* X *such that* $\operatorname{cl}_{R^{-1}}(U) \subseteq \operatorname{cl}_{R^{-1}}(A)$ $\big(\operatorname{cl}_{R^{-1}}(U) = \operatorname{cl}_{R^{-1}}(A)\big)$.

Corollary 17.11. *If* $A \subseteq X$, *then* $U = R^{-1}\big[R[A]^c\big]^c$ *is the largest subset of* X *such that* $R[U] \subseteq R[A]$ $\big(R[U] = R[A]\big)$.

Remark 17.12. Note that the first statement of the latter corollary is also an immediate consequence of Corollary 17.5.

Therefore, to give a direct proof of Corollary 17.11, we need only show that $R[A] \subseteq R\big[R^{-1}\big[R[A]^c\big]^c\big]$ also holds.

For this, note that, by Remark 17.6, for any $x \in X$ we have $x \in R^{-1}\big[R[A]^c\big]^c \iff R(x) \subseteq R[A]$. Thus, in particular $A \subseteq R^{-1}\big[R[A]^c\big]^c$.

18. Some basic facts on the box product of relations

To apply our former results on the closure–interior Galois connection to relational inclusions and equalities, in addition to composition, we shall also need the pointwise Cartesian product of relations. For this, we shall fix the following

Notation 18.1. *In this section, we shall assume that* F *is a relation on* X *to* Z *and* G *is a relation on* Y *to* W.

Definition 18.2. Under this assumption, we define

$$(F \boxtimes G)(x, y) = F(x) \times G(y)$$

for all $x \in X$ and $y \in Y$.

Remark 18.3. Thus, $F \boxtimes G$ is a relation on $X \times Y$ to $Z \times W$, which has been called the *box product* of F and G in [50]. This was already considered in a thesis of J. Riquet in 1951 who called it tensor product.

The importance of the box product is already apparent from the following

Theorem 18.4. *For any* $R \subseteq X \times Y$, *we have*

$$(F \boxtimes G)[R] = G \circ R \circ F^{-1}.$$

Proof. If $(z, w) \in (F \boxtimes G)[R]$, then there exists $(x, y) \in R$ such that

$$(z, w) \in (F \boxtimes G)(x, y) = F(x) \times G(y),$$

and thus $z \in F(x)$ and $w \in G(y)$. Hence, by noticing that $x \in F^{-1}(z)$, we can already see that

$$y \in R(x) \subset R\left[F^{-1}(z)\right] = \left(R \circ F^{-1}\right)(y),$$

and thus

$$w \in G(y) \subseteq G\left[\left(R \circ F^{-1}\right)(z)\right] = \left(G \circ \left(R \circ F^{-1}\right)\right)(z).$$

Therefore, $(z, w) \in G \circ \left(R \circ F^{-1}\right) = G \circ R \circ F^{-1}$ also holds.

Thus, we have proved that $(F \boxtimes G)[R] \subseteq G \circ R \circ F^{-1}$. The converse inclusion can be proved quite similarly.

From Theorem 18.4, by taking $R = \{(x, y)\}$, we can immediately derive

Corollary 18.5. *For any* $x \in X$ *and* $y \in Y$, *we have*

$$(F \boxtimes G)(x, y) = G \circ \{(x, y)\} \circ F^{-1}.$$

Moreover, by using Theorem 18.4, we can also easily prove the following

Corollary 18.6. *In the* $Y = Z$ *particular case, we have*

$$G \circ F = \left(F^{-1} \boxtimes G\right)[\Delta_Y].$$

Proof. By the corresponding definitions and Theorem 18.4, it is clear that

$$G \circ F = G \circ \Delta_Y \circ \left(F^{-1}\right)^{-1} = \left(F^{-1} \boxtimes G\right)[\Delta_Y].$$

Remark 18.7. The above corollaries show that the box and composition products of relations are actually equivalent tools.

However, in contrast to the composition product, the box product of relations can be immediately defined for an arbitrary family of relations.

Moreover, concerning the box product, we can prove a simpler inversion formula.

Theorem 18.8. *We have*

$$(F \boxtimes G)^{-1} = F^{-1} \boxtimes G^{-1}.$$

Proof. For any $(x, y) \in X \times Y$ and $(z, w) \in Z \times W$, we have

$$(x, y) \in (F \boxtimes G)^{-1}(z, w) \iff (z, w) \in (F \boxtimes G)(x, y) \iff$$
$$(z, w) \in F(x) \times G(y) \iff z \in F(x), \ w \in G(y) \iff x \in F^{-1}(z), \ y \in G^{-1}(w)$$
$$\iff (x, y) \in F^{-1}(z) \times G^{-1}(w) \iff (x, y) \in \big(F^{-1} \boxtimes G^{-1}\big)(z, w).$$

Therefore, $(F \boxtimes G)^{-1}(z, w) = \big(F^{-1} \boxtimes G^{-1}\big)(z, w)$ for all $(z, w) \in Z \times W$, and thus the required equality is also true.

Now, by using Theorems 18.4 and 18.8, we can also easily prove the following

Theorem 18.9. *For any $S \subseteq Z \times W$, we have*

$$(F \boxtimes G)^{-1}[S] = G^{-1} \circ S \circ F.$$

Proof. By Theorems 18.8 and 18.4, it is clear that

$$(F \boxtimes G)^{-1}[S] = \big(F^{-1} \boxtimes G^{-1}\big)[S] = G^{-1} \circ S \circ \big(F^{-1}\big)^{-1} = G^{-1} \circ S \circ F.$$

From this theorem, by using Theorem 3.7 we can immediately derive

Corollary 18.10. *For any $S \subseteq Z \times W$ we have*

$$\mathrm{cl}_{F \boxtimes G}(S) = G^{-1} \circ S \circ F.$$

19. Applications of the box product and Corollary 17.3 to relational inclusions

Notation 19.1. *In this and the next section, in addition to assumptions of Notation 18.1, we shall use the notation*

$$R = F \boxtimes G.$$

Remark 19.2. Thus, R is a relation on $X \times Y$ to $Z \times W$ such that

$$R(x, y) = (F \boxtimes G)(x, y) = F(x) \times G(y)$$

for all $x \in X$ and $y \in Y$.

Moreover, by Theorems 14.4, 18.4 and 18.9, we can at once state

Theorem 19.3. *We have*

(1) $F_R(A) = G \circ A \circ F^{-1}$ *for all* $A \subseteq X \times Y$,

(2) $G_R(B) = \big(G^{-1} \circ B^c \circ F\big)^c$ *for all* $B \subseteq Z \times W$.

Proof. To check (2), note that by Theorems 14.4 and 18.9 we have

$$G_R(B) = R^{-1}[B^c]^c = (F \boxtimes G)^{-1}[B^c]^c = \big(G^{-1} \circ B^c \circ F\big)^c$$

for all $B \subseteq Z \times W$.

Definition 19.4. For any $B \subseteq Z \times W$, we define

$$\Gamma_{(F, G, B)} = \big(G^{-1} \circ B^c \circ F\big)^c.$$

Remark 19.5. Thus, $\Gamma_{(F,\,G,\,B)}$ is a relation on X to Y. Moreover, by Theorem 19.3 and the corresponding definitions, we have

$$\begin{aligned}\Gamma_{(F,\,G,\,B)} = G_R(B) = \operatorname{int}_R(B) &= \big\{ (x,\,y) \in X \times Y : \quad R(x,\,y) \subseteq B \big\} \\ &= \big\{ (x,\,y) \in X \times Y : \quad (F \boxtimes G)(x,\,y) \subseteq B \big\}\end{aligned}$$

for all $B \subseteq Z \times W$.

Hence, by the corresponding definitions and Corollary 18.5, it is clear that we also have the following

Theorem 19.6. *For any $(x,\,y) \in X \times Y$ and $B \subseteq Z \times W$, the following assertions are equivalent:*

(1) $(x,\,y) \in \Gamma_{(F,\,G,\,B)}$, *(2)* $F(x) \times G(y) \subseteq B$,

(3) $(F \boxtimes G)(x,\,y) \subseteq B$, *(4)* $G \circ \{(x,\,y)\} \circ F^{-1} \subseteq B$,

(5) $(x,\,z) \in F, \quad (y,\,w) \in G \quad \Longrightarrow \quad (z,\,w) \in B$.

Proof. To see the equivalence of (3) and (4), recall that by Corollary 18.5 we have

$$(F \boxtimes G)(x,\,y) = G \circ \{(x,\,y)\} \circ F^{-1}$$

for all $(x,\,y) \in X \times Y$.

Moreover, by using Corollary 17.3 and Theorem 19.3, we can also easily prove

Theorem 19.7. *If $B \subseteq Z \times W$, then $A = \Gamma_{(F,\,G,\,B)}$ is the largest subset of $X \times Y$ such that*

$$G \circ A \circ F^{-1} \subseteq B.$$

Proof. Note that, by Corollary 17.3, $A = G_F(B)$ is the largest subset of X such that $F_R(A) \subseteq B$. Moreover, by Theorem 19.3 and Definition 19.4, we have $F_R(A) = G \circ A \circ F^{-1}$ and $G_F(B) = \Gamma_{(F,\,G,\,B)}$.

Now, as an immediate consequence of this theorem and the increasingness of composition, we can also state

Theorem 19.8. *For any $A \subseteq X \times Y$ and $B \subseteq Z \times W$, the following assertions are equivalent:*

(1) $A \subseteq \Gamma_{(F,\,G,\,B)}$, *(2)* $G \circ A \circ F^{-1} \subseteq B$.

Proof. To prove the implication (1) $\Longrightarrow$ (2), note that if (1) holds, then by the increasingness of composition and Theorem 19.7 we have

$$G \circ A \circ F^{-1} \subseteq G \circ \Gamma_{(F,\,G,\,B)} \circ F^{-1} \subseteq B,$$

and thus (2) also holds.

Remark 19.9. Note that, by Theorems 18.4 and 18.9, we have

(1) $G \circ A \circ F^{-1} = (F \boxtimes G)\,[A]$ for all $A \subseteq X \times Y$,

(2) $\Gamma_{(F,\,G,\,B)} = (F \boxtimes G)^{-1}\,[B^c]^c$ for all $B \subseteq Z \times W$.

Therefore, Theorem 19.8 can also be reformulated in the following form.

Theorem 19.10. *For any $A \subseteq X \times Y$ and $B \subseteq Z \times W$, the following assertions are equivalent:*

(1) $(F \boxtimes G)\,[A] \subseteq B$, *(2)* $(F \boxtimes G)^{-1}\,[B^c] \subseteq A^c$.

Hence, by Definition 8.2 and Theorem 3.7, it is clear that we can also state

Theorem 19.11. *For any* $A \subseteq X \times Y$ *and* $B \subseteq Z \times W$, *the following assertions are equivalent:*

(1) $A \in \operatorname{Int}_{F \boxtimes G}(B)$, *(2)* $\operatorname{cl}_{F \boxtimes G}[B^c] \subseteq A^c$.

Definition 19.12. Now, for instance, we may also define

$$G^* = \left(G^{-1} \circ G^c \circ G^{-1}\right)^c.$$

Remark 19.13. Thus, G^* is a relation on W to Y. Moreover, by Definition 19.4, we have

$$G^* = \Gamma_{(G^{-1},\, G,\, G)}.$$

Therefore, as some immediate consequence of Theorems 19.6, 19.7 and 19.8, we can state the following three theorems.

Theorem 19.14. *For any* $(w, y) \in W \times Y$, *the following assertions are equivalent:*

(1) $(w, y) \in G^*$, *(2)* $G^{-1}(w) \times G(y) \subseteq G$,

(3) $(G^{-1} \boxtimes G)(w, y) \subseteq G$, *(4)* $G \circ \{(w, y)\} \circ G \subseteq G$,

(5) $(v, w) \in G, \ (y, \omega) \in G \implies (v, \omega) \in G$.

Theorem 19.15. G^* *is the largest subset of* $W \times Y$ *such that*

$$G \circ G^* \circ G \subseteq G.$$

Theorem 19.16. *For any* $\Omega \subseteq W \times Y$, *the following assertions are equivalent:*

(1) $\Omega \subseteq G^*$, *(2)* $G \circ \Omega \circ G \subseteq G$.

Remark 19.17. Note that, by Remark 19.9 and Theorem 18.8, we have

(1) $G^* = \left(G \boxtimes G^{-1}\right)[G^c]^c$,

(2) $G \circ \Omega \circ G = \left(G^{-1} \boxtimes G\right)[\Omega]$ for all $\Omega \subseteq W \times Y$,

Moreover, as some immediate consequences of Theorems 19.10 and 19.11, we can also state the following two theorems.

Theorem 19.18. *For any* $\Omega \subseteq W \times Y$, *the following assertions are equivalent:*

(1) $(G^{-1} \boxtimes G)[\Omega] \subseteq G$, *(2)* $(G \boxtimes G^{-1})[G^c] \subseteq A^c$.

Theorem 19.19. *For any* $\Omega \subseteq W \times Y$, *the following assertions are equivalent:*

(1) $\Omega \in \operatorname{Int}_{G^{-1} \boxtimes G}(G)$, *(2)* $\operatorname{cl}_{G^{-1} \boxtimes G}[G^c] \subseteq \Omega^c$.

Remark 19.20. For a relation ρ, the notation $\rho^* = \{(a, b) : \ \rho \circ \{a, b)\} \circ \rho \subseteq \rho\}$ was already used by Zareckiĭ [68, p. 299].

However, the equality $\rho^* = \left(\rho^{-1} \circ \rho^c \circ \rho\right)^c$ and the $Y = W$ particular cases of Theorems 19.15 and 19.16 were first proved by Schein [33].

20. Applications of the box product and Theorem 16.1 to relational equations

Now, by using Theorems 16.1 and 19.3, we can also easily prove the following

Theorem 20.1. *For any* $B \subseteq Z \times W$, *the following assertions are equivalent:*

(1) $B = G \circ A \circ F^{-1}$ *for some* $A \subseteq X \times Y$,

(2) $B = G \circ \Gamma_{(F,\,G,\,B)} \circ F^{-1}$, (3) $B \subseteq G \circ \Gamma_{(F,\,G,\,B)} \circ F^{-1}$.

Proof. From Theorem 16.1, we know that, under the notation $\Psi_R = F_{\mathcal{R}} \circ G_R$, the following assertions are equivalent:

(a) $B = F_R(A)$ for some $A \subseteq X \times Y$,

(b) $B = \Psi_R(B)$, (c) $B \subseteq \Psi_R(B)$.

Moreover, from Theorem 19.3 and Definition 19.4, we can see that

$$F_R(A) = G \circ A \circ F^{-1} \qquad \text{and} \qquad G_R(B) = \Gamma_{(F,\,G,\,B)},$$

and thus

$$\Psi_R(B) = F_R\big(G_R(B)\big) = F_R\big(\Gamma_{(F,\,G,\,B)}\big) = G \circ \Gamma_{(F,\,G,\,B)} \circ F^{-1}.$$

Therefore, assertions (1), (2) and (3) are also equivalent.

Remark 20.2. Note that now, by Theorem 15.9, Ψ_R is an interior on $\mathcal{P}(Z \times W)$ such that $G_R = G_R \circ \Psi_R$.

Moreover, by Theorem 19.3, we have

$$\Psi_R(B) = F_{\mathcal{R}}\big(G_R(B)\big) = G \circ \big(G^{-1} \circ B^c \circ F\big)^c \circ F^{-1}$$

for all $B \subseteq Z \times W$.

Moreover, by using Remark 19.9, the above theorem can be reformulated in the following form.

Theorem 20.3. *For any* $B \subseteq Z \times W$, *the following assertions are equivalent:*

(1) $B = (F \boxtimes G)[A]$ *for some* $A \subseteq X \times Y$,

(2) $B = \big(F \boxtimes G\big)\big[(F \boxtimes G)^{-1}[B^c]^c\big]$, (3) $B \subseteq \big(F \boxtimes G\big)\big[(F \boxtimes G)^{-1}[B^c]^c\big]$.

However, it is now more important to note that, by using Theorem 20.1 and 19.6, we can also prove the following

Theorem 20.4. *For any* $B \subseteq Z \times W$, *the following assertions are equivalent:*

(1) $B = G \circ A \circ F^{-1}$ *for some* $A \subseteq X \times Y$,

(2) $\forall\ (z,\,w) \in B: \quad \exists\ (x,\,y) \in X \times Y: \quad (z,\,w) \in F(x) \times G(y) \subseteq B$,

(3) $\forall\ (z,\,w) \in B: \quad \exists\ (x,\,y) \in X \times Y:$

(a) $(x,\,z) \in F, \quad (y,\,w) \in G$,

(b) $(x,\,s) \in F, \quad (y,\,t) \in G \implies (s,\,t) \in B$.

Proof. If (1) holds, then by Theorems 20.1 and 18.4 we have

$$B \subseteq G \circ \Gamma_{(F,\,G,\,B)} \circ F^{-1} = \big(F \boxtimes G\big)\big[\Gamma_{(F,\,G,\,B)}\big].$$

Therefore, for every $(z,\,w) \in B$, there exists $(x,\,y) \in X \times Y$ such that

$$(x,\,y) \in \Gamma_{(F,\,G,\,B)} \qquad \text{and} \qquad (z,\,w) \in (F \boxtimes G)(x,\,y).$$

Thus, in particular $(z, w) \in F(x) \times G(y)$. Moreover, from Theorem 19.6, we can see that $F(x) \times G(y) \subseteq B$. Therefore, (2) also holds.

While, if (2) holds, then from $(z, w) \in F(x) \times G(y)$ we can see that $z \in F(x)$ and $w \in G(y)$, and thus $(x, z) \in F$ and $(y, w) \in G$. Moreover, from $F(x) \times G(y) \subseteq B$, we can see that, for every $s \in F(x)$ and $t \in G(y)$, we have $(s, t) \in B$. Hence, by noticing that $(x, s) \in F$ and $(y, t) \in G$, we can infer that (3) also holds.

The converse implications (3) $\Longrightarrow$ (2) and (2) $\Longrightarrow$ (1) can be proved quite similarly.

Now, by using Remark 19.13, we can easily establish the following particular cases of Theorems 20.1, 20.3 and 20.4.

Theorem 20.5. *The following assertions are equivalent:*

(1) $G = G \circ \Omega \circ G$ *for some* $\Omega \subseteq W \times Y$,

(2) $G = G \circ G^* \circ G$, *(3)* $G \subseteq G \circ G^* \circ G$.

Theorem 20.6. *The following assertions are equivalent:*

(1) $G = \big(G^{-1} \boxtimes G\big)\big[\,\big(G \boxtimes G^{-1}\big)[\,G^c\,]^c\,\big]$,

(2) $G \subseteq \big(G^{-1} \boxtimes G\big)\big[\,\big(G \boxtimes G^{-1}\big)[\,G^c\,]^c\,\big]$,

(3) $G = \big(G^{-1} \boxtimes G\big)[\,\Omega\,]$ *for some* $\Omega \subseteq W \times Y$.

Theorem 20.7. *The following assertions are equivalent:*

(1) $G = G \circ \Omega \circ G$ *for some* $\Omega \subseteq W \times Y$,

(2) $\forall\ (y, w) \in G: \quad \exists\ (\omega, v) \in W \times Y: \quad (y, w) \in G^{-1}(\omega) \times G(v) \subseteq G$,

(3) $\forall\ (y, w) \in G: \quad \exists\ (\omega, v) \in W \times Y$:

(a) $(y, \omega) \in G, \quad (v, w) \in G$,

(b) $(\omega, s) \in G, \quad (v, t) \in G \implies (s, t) \in G$.

Remark 20.8. Certain forms of Theorems 20.5 and 20.7, for a relation ρ on a set X, were first proved by Schein [33] and Xu and Liu [63].

Some more difficult characterizations of regular relations were formerly discovered by Zareckiĭ [68], Markowsky [22], Bandelt [1] and Hardy and Petrich [16].

References

[1] H.-J. Bandelt, *Regulatity and complete distributivity*, Semigroup Forum **19** (1980), 123–126.

[2] H.-J. Bandelt, *On regulatity classes of binary relations*, In: Universal Algebra and and Applications, Banach Center Publications (PWN, Varsawa) **9** (1982), 329–333.

[3] G. Birkhoff, *Lattice Theory*, Amer. Math. Soc. Colloq. Publ. **25**, Providence, RI, 1967.

[4] T. S. Blyth and M. F. Janowitz, *Residuation Theory*, Pergamon Press, Oxford, 1972.

[5] Z. Boros and Á. Száz, *Finite and conditional completeness properties of generalized ordered sets*, Rostock. Math. Kolloq. **59** (2005), 75–86.

[6] Z. Boros and Á. Száz, *Infimum and supremum completeness properties of ordered sets without axioms*, An. St. Univ. Ovid. Constanta **16** (2008), 1–7.

[7] A. Brøndsted, *Fixed points and partial orders*, Proc. Amer. Math. Soc. **60** (1976), 365–366.

[8] S. Buglyó and Á. Száz, *A more important Galois connection between distance functions and inequality relations*, Sci. Ser. A Math. Sci. (N.S.) **18** (2009), 17–38.

[9] B. A. Davey and H. A. Priestley, *Introduction to Lattices and Order*, Cambridge University Press, Cambridge, 2002.
[10] K. Denecke, M. Erné and S. L. Wismath (Eds.) *Galois Connections and Applications*, Kluwer Academic Publisher, Dordrecht, 2004.
[11] C. J. Everett, *Closure operators and Galois theory in lattices*, Trans. Amer. Math. Soc. **55** (1944), 514–525.
[12] B. Ganter and R. Wille, *Formal Concept Analysis*, Springer-Verlag, Berlin, 1999.
[13] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove, and D. S. Scott, *A Compendium of Continuous Lattices*, Springer-Verlag, Berlin, 1980.
[14] T. Glavosits, *Generated preorders and equivalences*, Acta Acad. Paed. Agriensis, Sect. Math. **29** (2002), 95–103.
[15] R. L. Graham, D. E. Knuth and T. S. Motzkin, *Complements and transitive closures*, Discrete Math. **2** (1972), 17–29.
[16] D. Hardy and M. Petrich, *Binary relations as lattice isomorphisms*, Ann. Mat. Pura Appl. **177** (1999), 195–224.
[17] U. Höhle and T. Kubiak, *On regularity of sup-preserving maps: generalizing Zareckiĭ's theorem*, Semigroup Forum **83** (2011), 313–319.
[18] G. Jiang and L. Xu, *Conjugative relations and applications*, Semigroup Forum **80** (2010), 85–91.
[19] G. Jiang, L. Xu, J. Cai and G. Han, *Normal relations on sets and applications*, Int. J. Contemp. Math. Sci. **6** (2011), 721–726.
[20] G. Jiang and L. Xu, *Dually normal relations on sets*, Semigroup Forum **85** (2012), 75–80.
[21] J. Lambek, *Some Galois connections in elementary number theory*, J. Number Theory **47** (1994), 371–377.
[22] G. Markowsky, *Idempotents and product representations with applications to the semigroup of binary relations*, Semigroup Forum **5** (1972), 95–119.
[23] O. Ore, *Galois connexions*, Trans. Amer. Math. Soc. **55** (1944), 493–513.
[24] G. Pataki, *On the extensions, refinements and modifications of relators*, Math. Balk. **15** (2001), 155–186.
[25] R. J. Plemmons and M.T. West, *On the semigroup of binary relations*, Pacific J. Math. **35** (1970), 743–753.
[26] G. N. Raney, *A subdirect-union representation for completely distributive lattices*, Proc. Amer. Math. Soc. **4** (1953), 518–522.
[27] M. J. Riguet, *Relations binaires, fermetures et correspondances de Galois*, Bull. Soc. Math. France **76** (1948), 114–155.
[28] D. A. Romano, *Quasi-regular relations – a new class of relations on sets*, Publ. Inst. Math. **93** (2013), 127–132.
[29] D. A. Romano, *Quasi-conjugative relations on sets*, Mat.-Kol. (Banja Luka) **19** (2013), 5–10.
[30] D. A. Romano, *Two new classes of relations*, In: Mateljevic, Stanimirovic, Maric and Svetlik (eds.), Symposium Mathematics and Applications, Fac. Math., Univ. Belgrade, Belgrade 2014, 34–39.
[31] D. A. Romano, *Bi-conjugative relations*, Romaian J. Math. Comp. Science **4** (2014), 203–208.
[32] D. A. Romano, *Quasi-normal relations*, Kyungpook Math. J. **55** (2015), 541–548.
[33] B. M. Shein, *Regular elements of the semigroup of all binary relations*, Semigroup Forum **13** (1976), 95–102.
[34] J. Schmidt, *Bieträge zur Filtertheorie II*, Math. Nachr. **10** (1953), 197–232.
[35] Á. Száz, *Basic tools and mild continuities in relator spaces*, Acta Math. Hungar. **50** (1987), 177–201.
[36] Á. Száz, *Projective and inductive generations of relator spaces*, Acta Math. Hungar. **53** (1989), 407–430.
[37] Á. Száz, *Structures derivable from relators*, Singularité **3** (1992), 14–30.
[38] Á. Száz, *Refinements of relators*, Tech. Rep., Inst. Math., Univ. Debrecen **76** (1993), 19 pp.
[39] Á. Száz *Topological characterizations of relational properties*, Grazer Math. Ber. **327** (1996), 37–52.
[40] Á. Száz, *A Galois connection between distance functions and inequality relations*, Math. Bohem. **127** (2002), 437–448.
[41] Á. Száz, *Somewhat continuity in a unified framework for continuities of relations*, Tatra Mt. Math. Publ. **24** (2002), 41–56.

[42] Á. Száz, *Upper and lower bounds in relator spaces*, Serdica Math. J. **29** (2003), 239–270.

[43] Á. Száz, *Lower and upper bounds in ordered sets without axioms*, Tech. Rep., Inst. Math., Univ. Debrecen **2004/1**, 11 pp.

[44] Á. Száz, *The importance of reflexivity, transitivity, antisymmetry and totality in generalized ordered sets*, Tech. Rep., Inst. Math., Univ. Debrecen **2004/2**, 15 pp.

[45] Á. Száz, *Galois-type connections on power sets and their applications to relators*, Tech. Rep., Inst. Math., Univ. Debrecen **2005/2**, 38 pp.

[46] Á. Száz, *Supremum properties of Galois–type connections*, Comment. Math. Univ. Carolin. **47** (2006), 569–583.

[47] Á. Száz, *Minimal structures, generalized topologies, and ascending systems should not be studied without generalized uniformities*, Filomat **21** (2007), 87–97.

[48] Á. Száz, *Galois type connections and closure operations on preordered sets*, Acta Math. Univ. Comen. **78** (2009), 1–21.

[49] Á. Száz, *Galois-type connections and continuities of pairs of relations*, J. Int. Math. Virt. Inst. **2** (2012), 39–66.

[50] Á. Száz, *Inclusions for compositions and box products of relations*, J. Int. Math. Virt. Inst. **3** (2013), 97–125.

[51] Á. Száz, *Galois and Pataki connections revisited*, Tech. Rep., Inst. Math., Univ. Debrecen **2013/3**, 20 pp.

[52] Á. Száz, *A particular Galois connection between relations and set functions*, Acta Univ. Sapientiae Math. **6** (2014), 73–91.

[53] Á. Száz, *Galois and Pataki connections on generalized ordered sets*, Tech. Rep., Inst. Math., Univ. Debrecen **403** (2014), 27 pp.

[54] Á. Száz, *Generalizations of Galois and Pataki connections to relator spaces*, J. Int. Math. Virt. Inst. **4** (2014), 43–75.

[55] Á. Száz, *Equations for compositions of relations can also be solved*, Conference on Inequalities and Applications, Hajdúszoboszó, Hungary, 2014.

[56] Á. Száz, *Remarks and problems at the Conference on Inequalities and Applications, Hajdúszoboszló, Hungary, 2014*, Tech. Rep., Inst. Math., Univ. Debrecen **2004/5**, 12 pp.

[57] Á. Száz, *Basic tools, increasing functions, and closure operations in generalized ordered sets*, In: P. M. Pardalos and Th. M. Rassias (eds.), Contributions in Mathematics and Engineering: In Honor of Constantion Caratheodory, to appear.

[58] Á. Száz, *A unifying framework for studying continuity, increasingness, and Galois connections*, manuscript.

[59] Á. Száz and J. Túri, *Comparisons and compositions of Galois–type connections*, Miskolc Math. Notes **7** (2006), 189–203.

[60] Á. Száz and A. Zakaria, *Mild continuity properties of relations and relators in relator spaces*, In: P. M. Pardalos and Th. M. Rassias (eds.), Essays in Mathematics and its Applications, In Honor of Vladimir Arnold, to appear.

[61] E.S. Wolk, *A characterization of partial order*, Bull. Acad. Polon. **17** (1969), 207–208.

[62] W. Xia, *Galois connections and formal concept analysis*, Demonstratio Math. **27** (1994), 751–767.

[63] X.-Q. Xu and Y. M. Liu, *Relational representations of hypercontinuous lattices*, In: Domain Theory, Logic. and Computation, Kluwer, Dordrecht, 2003, 65–74.

[64] X.-Q. Xu and Y. M. Liu, *Regular relations and strictly complete regular ordered spaces*, Topology Applications, **135** (2004), 1–12.

[65] X.-Q. Xu and M. K. Luo, *Regular relations and normality of topologies*, Semigroup Forum **72** (2006), 477–480.

[66] J.-Ch. Yang, *A theorem on the semigroup of binary relations*, Proc. Amer. Math. Soc. **22** (1969),134–135.

[67] K. A. Zareckiĭ, *Regular element of the semigroup of binary relations*, Uspehi Mat. Nauk. **3** (1962), 177–179. (Russian.)

[68] K. A. Zareckiĭ, *The semigroup of binary relations*, Mat. Sb. **61** (1963), 291–305. (Russian.)

Árpád Száz, Department of Mathematics, University of Debrecen, H–4010 Debrecen, Pf. 12, Hungary

E-mail address: `szaz@science.unideb.hu`