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REMARKS AND PROBLEMS AT THE CONFERENCE ON INEQUALITIES AND APPLICATIONS, HAJDÚSZOBOSZLÓ, HUNGARY, 2014

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Abstract. This paper contains improved forms of most of the remarks and problems of the author presented at Conference on Inequalities and Applications, Hajdúszoboszló, Hungary, 2014.

In particular, the following interesting subjects are included:

1. Two Galois connections derived from a single relation.
2. Galois connections can be used to establish some properties of recession cones.
3. Galois connections can be used to solve some equations and extremum problems.
4. Which continuity properties force an additive function of the real line to be linear?
5. A functional equationist motivation for the investigation of equations and inclusions for compositions of relations.
6. An instructive reformulation of the definition of quasi-contractions of M. Bessenyei.

1. Two Galois connections derived from a single relation

Definition 1.1. Let R be a relation on one set X to another Y. Then, according to [36], for any $A \subseteq X$ and $B \subseteq Y$ we define

(1) $A \in \operatorname{Int}_R(B)$ if $R[A] \subseteq B$,

(2) $A \in \operatorname{Cl}_R(B)$ if $R[A] \cap B \neq \emptyset$,

(3) $A \in \operatorname{Lb}_R(B)$ and $B \in \operatorname{Ub}_{\mathcal{R}}(A)$ if $A \times B \subseteq R$.

Remark 1.2. The above relations are not independent of each other, since by [36] we have

(1) $\operatorname{Ub}_R = \operatorname{Lb}_{R^{-1}} = \operatorname{Lb}_R^{-1}$,

(2) $\operatorname{Lb}_R = \left(\operatorname{Cl}_{\mathcal{R}^c}\right)^c$, (3) $\operatorname{Int}_R = \left(\operatorname{Cl}_R \circ \mathcal{C}\right)^c$,

where $\mathcal{C}$ is the complement function defined by $\mathcal{C}(B) = B^c = Y \setminus B$ for all $B \subset Y$.

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Definition 1.3. By identifying singletons with their elements, we may naturally consider X as a subset of $\mathcal{P}(X)$, and we may briefly define

(1) $\operatorname{lb}_R(B) = X \cap \operatorname{Lb}_R(B)$, (2) $\operatorname{cl}_R(B) = X \cap \operatorname{Cl}_R(B)$,

(3) $\operatorname{ub}_R(A) = Y \cap \operatorname{Ub}_R(A)$, (4) $\operatorname{int}_R(B) = X \cap \operatorname{Int}_R(B)$,

for all $A \subseteq X$ and $B \subseteq Y$.

Remark 1.4. Concerning the latter relations, by [36], we have

(1) $\operatorname{ub}_R(A) = R^c[A]^c = \bigcap_{x \in A} R(x)$,

(2) $\operatorname{cl}_R(B) = R^{-1}[B]$, (3) $\operatorname{int}_R(B) = R^{-1}[B^c]^c$,

for all $A \subseteq X$ and $B \subseteq Y$.

However, it is now more important to note that, under the above notations, we also have the following two closely related theorems.

Theorem 1.5. *If*

$$F_R(A) = \operatorname{cl}_{R^{-1}}(A) \qquad \text{and} \qquad G_R(B) = \operatorname{int}_R(B)$$

for all $A \subseteq X$ *and* $B \subseteq Y$, *then* F_R *and* G_R *establish a Galois connection between the posets* $\mathcal{P}(X)$ *and* $\mathcal{P}(Y)$.

Theorem 1.6. *If*

$$F_R(A) = \operatorname{ub}_R(A) \qquad \text{and} \qquad G_R(B) = \operatorname{lb}_R(B)$$

for all $A \subseteq X$ *and* $B \subseteq Y$, *then* F_R *and* G_R *establish a Galois connection between* $\mathcal{P}(X)$ *and the dual of* $\mathcal{P}(Y)$.

Proof. For this, by the definitions of [7] and [43], we need only show that, for any $A \subseteq X$ and $B \subseteq Y$, we have

$$F_R(A) \subseteq^{-1} B \iff A \subseteq G_R(B),$$

However, by the corresponding definitions, it is clear that

$$\begin{aligned} F_R(A) \subseteq^{-1} B &\iff B \subseteq F_R(A) \iff B \subseteq \operatorname{ub}_R(A) \\ &\iff \forall\ b \in B: \ A \times \{b\} \subset R \iff \forall\ a \in A: \ \{a\} \times B \subset R \\ &\iff A \subset \operatorname{lb}_R(B) \iff A \subseteq G_R(B). \end{aligned}$$

Remark 1.7. The upper and lower bound Galois connection, described in Theorem 2, was first studied by Birkhoff [2, p. 122] under the name polarities.

While, the closure–interior Galois connection, described in Theorem 1, seems to have been only mentioned in Exercise 7.18 of Davey and Priestly [7, p. 172].

The above mentioned authors, and Ganter and Wille [10, p. 17] used quite different notations. The novelty of our treatment lies mainly in the use of the techniques of simple relator spaces $(X, Y)(R)$.

2. Galois connections can be used to establish some properties of recession cones

Definition 2.1. Let X be a vector space over K. For any $A \subseteq K$ and $U, V \subseteq X$, define

$$A\,U = \big\{ \alpha\, x : \quad \alpha \in A, \quad x \in U \big\} \qquad \text{and} \qquad U + V = \big\{ x + y : \quad x \in U, \;\; y \in V \big\} .$$

Remark 2.2. Now, in particular, for any $\alpha \in K$ and $x \in X$, we may also define

$$\alpha\, U = \{\alpha\}\, U\,, \qquad A\, x = A\,\{x\}\,, \qquad x + U = \{x\} + U\,, \qquad U + x = U + \{x\} .$$

Moreover, we may also naturally define $-U = (-1)\,U$ and $U - V = U + (-V\,)$.

Thus, the above elementwise linear operations have several useful properties. For instance, one can easily see that only two axioms of a vector space may fail to hold for the family $\mathcal{P}(X)$ of all subsets of X.

Remark 2.3. In the sequel, by identifying singletons with their elements, we shall consider X as a subset of $\mathcal{P}(X)$.

Thus, if F is a function of $\mathcal{P}(X)$ to itself, then we may simply write

$$F(x) = F\big(\{x\}\big) \qquad \text{and} \qquad F\,[\,A\,] = \big\{ F(a) : \quad a \in A \big\}$$

for all $x \in X$ and $A \subseteq X$.

Moreover, we can easily see that the function F is union-preserving if and only if $F(A) = \bigcup F\,[\,A\,]$ for all $A \subseteq X$.

The following definition has been mainly motivated by a recent lecture of González [13] about certain applications of recession cones.

Definition 2.4. Let X be a vector space over K, $A \subseteq K$ and $B \subseteq X$. For any $U, V \subseteq X$, define

$$F\,(\,U\,) = F_{(A,\,B)}(\,U\,) = A\,U + B$$

and

$$G(\,V\,) = G_{(A,\,B)}(\,V\,) = \operatorname{int}_F(\,V\,) = \big\{ x \in X : \quad F\,(x) \subseteq V \big\} .$$

Remark 2.5. If in particular $K = \mathbb{R}$ and $A = \mathbb{R}_+$, with $\mathbb{R}_+ = [\,0, +\infty\,[$, then

$$G(\,B\,) = \big\{ x \in X : \quad \mathbb{R}_+\, x + B \subseteq B \big\} = \big\{ x \in X : \quad \forall\; \alpha \geq 0,\; y \in B : \quad \alpha\, x + y \in B \big\}$$

is just the recession cone $\operatorname{rec}(\,B\,)$ originally introduced by Rockafellar [22, p. 61] only for convex subsets of $\mathbb{R}^{\,n}$.

The appropriateness of the above definitions is apparent from the following

Theorem 2.6. *The functions F and G establish a Galois connection between the poset $\mathcal{P}(X)$ and itself.*

Proof. For this, by the definitions of [7] and [43], we need only show that, for any $U, V \subseteq X$, we have

$$F\,(\,U\,) \subset V \quad \Longleftrightarrow \quad U \subseteq G(\,V\,) .$$

However, by the corresponding definitions, it is clear that

$$\begin{aligned} F\,(\,U\,) \subset V \quad &\Longleftrightarrow \quad A\,U + B \subseteq V \quad \Longleftrightarrow \quad \forall\; x \in U : \quad A\,x + B \subseteq V \\ &\Longleftrightarrow \quad \forall\; x \in U : \quad F\,(x) \subseteq V \quad \Longleftrightarrow \quad U \subseteq G(\,V\,). \end{aligned}$$

From this theorem, by using the theory of Galois connections, we can immediately derive several theorems on the functions F and G, and their compositions.

For instance, from Theorem 2.6, by using a particular case of [43, Theorem 5.6], we can immediately derive

Corollary 2.7. *For any* $V \subset X$, $U = G(V)$ *is the largest subset of* X *such that* $F(U) \subseteq V$.

Hence, it is clear that in particular we also have

Corollary 2.8. $U = G(B)$ *is the largest subset of* X *such that* $F(U) \subseteq B$.

Remark 2.9. Thus, under the notation of Remark 2.5, $U = \operatorname{rec}(B)$ is the largest subset of X such that $\mathbb{R}_+ U + B \subseteq B$.

3. Galois connections can be used to solve some equations and extremum problems

In [37], having in mind a terminology of Birkhoff [2], we have introduced the following

Definition 3.1. If X is a set and $\leq$ is a relation on X, then the ordered pair $X(\leq) = (X, \leq)$ is called a goset (generalized ordered set), and we usually write X instead of $X(\leq)$

Remark 3.2. Thus, the goset $X(\leq)$ may, for instance, be called reflexive if the relation $\leq$ is reflexive on X.

Moreover, a reflexive and transitive goset may be called a proset (preordered set). And, an antisymmetric proset may be called a poset (partially ordered set).

Remark 3.3. If $X(\leq)$ is a goset, and $X' = X$ and $\leq' = \leq^{-1}$, then the goset $X'(\leq')$ will be called the dual of $X(\leq)$. The dual goset inherits several properties of the original goset.

In [49], slightly extending the ideas of Ore [19], Schmidt [24, p. 209], Blyth and Janowitz [3, p. 11], and the present author [43] on Galois connections, residuated mappings, and increasingly normal functions, we have introduced the following

Definition 3.4. Let X and Y be gosets. Then, for any functions f of X to Y and g of Y, we say that:

(1) f is increasingly upper g–seminormal if $f(x) \leq y$ implies $x \leq g(y)$ for all $x \in X$ and $y \in Y$,

(2) f is increasingly lower g–seminormal if $x \leq g(y)$ implies $f(x) \leq y$ for all $x \in X$ and $y \in Y$.

Remark 3.5. Now, the function f may be naturally called increasingly g-normal if it is both increasingly upper and lower g–seminormal.

Moreover, a function f of X to Y may, for instance, be naturally called increasingly normal if it is increasingly g-normal for some function g of Y to X.

Remark 3.6. By [49, Theorem 8.7], an increasingly normal function of a transitive goset to a reflexive one is already increasing.

Therefore, a function f of X to Y may, for instance, be naturally called decreasingly normal if it is increasing normal as a function of X to Y'.

In this respect, it is also worth mentioning that, by using the above definitions, we can easily prove the following dualization principle.

Theorem 3.7. *If f is an increasingly upper (lower) g–seminormal function of one goset X to another Y, then g is an increasingly lower (upper) f–seminormal function of Y' to X'.*

Proof. If f is increasingly upper g–seminormal, then by the corresponding definitions it is clear that $y \leq' f(x) \implies f(x) \leq y \implies x \leq g(y) \implies g(y) \leq' x$ for all $y \in Y$ and $x \in X$ Therefore, g is increasingly lower f–seminormal as a function of Y' to X'.

Remark 3.8. By using this priciple, the properties of the functions g and $f \circ g$ can be immediately derived from those of f and $g \circ f$.

For instance, from Remark 3.6, we can at once see that if f is an increasingly g–normal function of a reflexive goset to a transitive one, then g is increasing. (Thus, condition (i) in Definition 3.1 of [12, p. 18] is superfluous.)

However, it now more important to note that, by using the results of [49], we can also easily prove the following two theorems.

Theorem 3.9. *If f is an increasingly lower g–seminormal function of a reflexive goset to an antisymmetric one Y, $\psi = f \circ g$, and $y \in Y$ such that $y \leq \psi(y)$, then $y = \psi(y)$ and $y \in f[X]$.*

Proof. By [49, Theorem 8.14], we have $\psi(y) \leq y$. Hence, because of the assumption $y \leq \psi(y)$ and the antisymmetry of Y, we can infer that $y = \psi(y)$. Therefore, $y = f\big(g(y)\big)$, and thus $y \in f[X]$ also holds.

Theorem 3.10. *If f is an increasingly g–normal function of a transitive goset X to a reflexive one Y, $\psi = f \circ g$, and $y \in f[X]$, then $y \leq \psi(y)$.*

Proof. Since $y \in f[X]$, there exists $x \in X$ such that $y = f(y)$. Hence, by using the reflexivity of Y, we can infer that $f(y) \leq y$. Hence, by using the upper g–seminormality of f, we can infer that $x \leq g(y)$. Hence, by using Remark 3.6, we can infer that $f(x) \leq f\big(g(y)\big)$, and thus $y \leq \psi(y)$.

Now, as an immediate consequence of this theorem, we can also state

Corollary 3.11. *If f is an increasingly g–normal function of a transitive goset to a reflexive and antisymmetric one Y, $\psi = f \circ g$, and y is a maximal element of $f[X]$, then $\psi(y) = y$.*

Moreover, as an immediate consequence of Theorems 3.9 and 3.10, we can state

Theorem 3.12. *If f is an increasingly g–normal function of a proset X to a reflexive and antisymmetric goset Y and $\psi = f \circ g$, then for any $y \in Y$ the following assertions are equivalent:*

(1) $y \in f[X]$, *(2)* $y = \psi(y)$, *(3)* $y \leq \psi(y)$.

Remark 3.13. Note that if f is only an increasingly g–normal function of a proset X to a reflexive goset Y, then by [49, Corollary 8.19] $\psi = f \circ g$ is already a semiinterior operation on Y.

While, in order that ψ could be an interior operation on Y, by [49, Theorem 8.24] it seems necessary to assume that f be an increasingly g–normal function of one proset X to another Y.

For instance, from Theorem 3.12, by using Theorem 1.5, we can at once drive

Corollary 3.14. *If R is a relation on X to Y, then for any $B \subseteq Y$ the following assertions are equivalent:*

(1) $B = \mathrm{cl}_{R^{-1}}(A)$ *for some* $A \subseteq X$,

(2) $B = \mathrm{cl}_{R^{-1}}\big(\mathrm{int}_R(B)\big)$, *(3)* $B \subseteq \mathrm{cl}_{R^{-1}}\big(\mathrm{int}_R(B)\big)$.

Hence, by Remark 1.4, it is clear that equivalently we also have

Corollary 3.15. *If R is a relation on X to Y, then for any $B \subseteq Y$ the following assertions are equivalent:*

(1) $B = R[A]$ *for some* $A \subseteq X$,

(2) $B = R\big[R^{-1}[B^c]^c\big]$, *(3)* $B \subseteq R\big[R^{-1}[B^c]^c\big]$.

Remark 3.16. Moreover, it can also be easily seen that (1) is also equivalent to the assertion that, for each $y \in B$, there exists $x \in X$ such that $y \in R(x) \subseteq B$.

Theorem 3.12 gives a necessary and sufficient condition in order that, for some $y \in Y$, the equation $y = f(x)$ could have a solution. For this, it says that y should be a ψ–open element of Y.

Therefore, Galois connection can be used to decide on the solvability of certain equations. In this respect, it is also worth mentioning that they can also be used to solve some extremal problems. Namely, by [49, Theorem 9.9], we have the following

Theorem 3.17. *Let X and Y be prosets. Then, for a function f of X to Y and a function g of Y to X, the following assertions are equivalent:*

(1) f *is increasingly* g*–normal,*

(2) f *is increasing and* $g(y) \in \max\big(\mathrm{Int}_f(y)\big)$ *for all* $y \in Y$.

Remark 3.18. Here, by [49, Definition 4.1], we have

$$\mathrm{Int}_f(y) = \{x \in X : \ f(x) \leq y\}$$

for all $y \in Y$.

Therefore, if (1) holds, then for any $y \in Y$ we can state that $x = g(y)$ is a largest element of X such that $f(x) \leq y$.

For instance, from Theorem 3.17, by using Theorem 1.5, we can at once drive

Corollary 3.19. *If R is a relation on X to Y and $B \subseteq Y$, then $A = \mathrm{int}_R(B)$ is the largest subset of X such that $\mathrm{cl}_{R^{-1}}[A] \subseteq B$.*

Hence, by Remark 1.4, it is clear that equivalently we also have

Corollary 3.20. *If R is a relation on X to Y and $B \subseteq Y$, then $A = R^{-1}[B^c]^c$ is the largest subset of X such that $R[A] \subseteq B$.*

Remark 3.21. Finally, we note that, from a specialization of Theorem 3.17 to regular functions [49, Theorem 10.9], by using Theorem 1.5 we can easily get two quite similar maximality results.

However, it is now important to note that, from Corollary 3.20, by using the box product $F \boxtimes G$ of relations F on X to Z and G on Y to W, defined such that $(F \boxtimes G)(x, y) = F(x) \times G(y)$ for all $x \in X$ and $y \in Y$, we can immediately derive some results of [48].

4. Which continuity properties force an additive function of the real line to be linear?

Some particular cases of the following problem have formerly been also considered in a talk by Zoltán Boros.

If f is an additive function of $\mathbb{R}$, then having in mind some classical results [23] we may naturally ask the question that: Which one of the following basic continuity (or monotonicity) properties of f forces f to be linear?

For this, let $\mathcal{R}$ and $\mathcal{S}$ be relators (arbitrary families of relations) on $\mathbb{R}$. That is, $\mathcal{R}$, $\mathcal{S} \subseteq \mathcal{P}(\mathbb{R}^2)$. Moreover, suppose that $\square = (\square_i)_{i=1}^4$ is a family of unary operations on the family $\mathcal{P}^2(\mathbb{R}^2) = \mathcal{P}\big(\mathcal{P}(\mathbb{R}^2)\big)$ of all relators on $\mathbb{R}$.

Then, analogously to [33] (or [52]), the function f may be naturally called $\square$–continuous (or $\square$–monotonic) with respect to the relators $\mathcal{R}$ and $\mathcal{S}$ if

$$\Big(\mathcal{S}^{\square_1} \circ f\Big)^{\square_2} \subseteq \Big(f \circ \mathcal{R}^{\square_3}\Big)^{\square_4},$$

where the two compositions are to be taken elementwise.

Here, $\mathcal{R}$ may, for instance, be $\mathcal{R}_{\leq} = \{\leq\}$, or the family $\mathcal{R}_d$ of all surroundings

$$B_r^d = \big\{(x, y) \in \mathbb{R}^2 : \quad d(x, y) < r\big\}$$

with $r > 0$, and thus also $\mathcal{R}_{(d, \leq)} = \mathcal{R}_d \cup R_{\leq}$.

Moreover, if $\mathcal{A}$ is the family of all open, fat, or measurable subsets of $\mathbb{R}$, then the family $\mathcal{R}_{\mathcal{A}}$ of all Pervin relations [42] $R_A = A^2 \cup A^c {\times} X$, with $A \in \mathcal{A}$, is also an important relator on $\mathbb{R}$. Note that $\mathcal{R}_{\mathcal{A}}$ is a preorder, while $\mathcal{R}_d$ is a tolerance relator on $\mathbb{R}$.

Moreover, for any relator $\mathcal{R}$ on $\mathbb{R}$, $\mathcal{R}^{\square}$ may, for instance, be

$$\mathcal{R}^{\infty} = \big\{R^{\infty} : \quad R \in \mathcal{R}\big\} \qquad \text{or} \qquad \mathcal{R}^{\partial} = \big\{S \subset \mathbb{R}^2 : \quad S^{\infty} \in \mathcal{R}\big\},$$

where $R^{\infty} = \bigcup_{n=0}^{\infty} R^n$ is the smallest preorder relation on $\mathbb{R}$ containing R.

Furthermore, $\mathcal{R}^{\square}$ may, for instance, be

$$\mathcal{R}^{\vartriangle} = \big\{S \subseteq \mathbb{R}^2 : \quad \forall\ x \in \mathbb{R} : \quad \exists\ R \in \mathcal{R} : \quad \exists\ y \in \mathbb{R} : \quad R(y) \subseteq S(x)\big\},$$

and thus also $\mathcal{R}^{\vartriangle\infty}$ or $\mathcal{R}^{\vartriangle\partial}$. But, the operation $\vartriangle \partial$ is already not idempotent.

Here, we can note that a relator $\mathcal{R}$ on $\mathbb{R}$ may be called properly well-chained [20] if $\mathcal{R}^{\infty} = \{\mathbb{R}^2\}$. Moreover, $\mathcal{R}$ may be called paratopologically compact [30] if for each $R \in \mathcal{R}^{\vartriangle}$ there exists a finite subset A of $\mathbb{R}$ such that $\mathbb{R} = R[A]$.

Furthermore, it is also noteworthy that $\mathcal{R}^{\vartriangle}$ is the largest relator on $\mathbb{R}$ inducing the same family of fat subsets of $\mathbb{R}$ as $\mathcal{R}$ does. However, the family of all open sets induced by $\mathcal{R}$ fails to have such a property. (See [47, Example 5.17].)

Note that, in our former definition of $\square$–continuity, f may be an arbitrary relation on $\mathbb{R}$. However, in that case the term "$\square$–continuous" has to be replaced by "upper $\square$–semicontinuous".

Finally, we note that one may also naturally consider quite similar questions in connection with subadditive and superadditive relations, and also with subadditive and superadditive functions (and their Pexiderizations).

However, it can be easily seen that if R is a superhomogeneous, superadditive relation of $\mathbb{R}$, then R is already linear. And thus, it is either a linear function or the whole space $\mathbb{R}^2$. (For some more general results, see [34].)

5. A functional equationist motivation for the investigation of equations and inclusions for compositions of relations

If X and Y are groupoids, and F, G, and H are relations on X to Y, then as a straightforward generalization of the Pexiderization of the classical Cauchy equation [23] we may naturally consider the relational equation

$$H(x+y) = F(x) + G(y) \tag{1}$$

for all $x, y \in X$.

Such an equation, for set-valued functions, has only been investigated by K. Nikodem [18], W. Smajdor [26], and A. Smajdor [25]. (See also [11, 5].) While, additive relations and set-valued functions have already been intensively studied by several algebraists and functional equationists. (See [45, p. 638].)

Now, by defining $\phi(x, y) = x + y$ for all $x, y \in X$, we can at once see that

$$H(x+y) = H\big(\phi(x, y)\big) = (H \circ \phi)(x, y)$$

for all $x, y \in X$.

Moreover, by defining

$$K(x, y) = (F \boxtimes G)(x, y) = F(x) \times G(y)$$

for all $x, y \in X$, and $\psi(z, w) = z + w$ for all $z, w \in Y$, we can easily see that

$$\begin{aligned} F(x) + G(y) &= \big\{ z + w : \quad z \in F(x), \quad w \in G(y) \big\} \\ &= \big\{ z + w : \quad (z, w) \in F(x) \times G(y) \big\} = \big\{ \psi(z, w) : \quad (z, w) \in K(x, y) \big\} \\ &= \psi\,[K(x, y)] = (\psi \circ K)(x, y) \end{aligned}$$

for all $x, y \in X$.

Thus, our original equation (1) can be written in the form that

$$(H \circ \phi)(x, y) = (\psi \circ K)(x, y)$$

for all $x, y \in X$. Therefore, we actually have

$$H \circ \phi = \psi \circ K. \tag{2}$$

Now, the ordered triple (F, G, H) may be naturally called additive, subadditive and superadditive if

$$H \circ \phi = \psi \circ K, \qquad H \circ \phi \subseteq \psi \circ K, \qquad \psi \circ K \subseteq H \circ \phi.$$

However, since we have

$$(F \boxtimes G)[A] = G \circ A \circ F^{-1}$$

for all $A \subseteq X \times Y$, instead of equation (2) and the corresponding inclusions, it is more convenient to investigate first the more general equation

$$H \circ R \circ \Phi = \Psi \circ S \circ K \tag{3}$$

and the corresponding inclusions, with some relations Φ on X^2 to X, Ψ on Y^2 to Y, R on X to itself, and S on Y^2 to itself.

Note that by taking $R = \Delta_X$ and $S = \Delta_{Y^2}$ equation (2) can be obtained from (3). Moreover, before studying equation (3), it is convenient to investigate first the inclusions

$$H \circ R \circ \Phi \subseteq L \qquad \text{and} \qquad L \subseteq \Psi \circ S \circ K,$$

with a relation L of X^2 to Y.

This was actually the subject of our former paper [48], refused by the Semigroup Forum, where the reader can find three further substantial reasons and several references for studying compositional equations and inclusions for relations.

In the light of our present note, it would be instructive to solve the classical Cauchy equation $f \circ \varphi = \varphi \circ f$, where f is a function of $\mathbb{R}$ to itself and $\varphi(x+y) = x+y$ for all $x, y \in \mathbb{R}$, by using the notations and techniques of the theory of relations and relators (families of relations).

6. An instructive reformulation of the definition of quasi-contractions of M. Bessenyei

In his talk [1], to get a common generalization of the corresponding definitions of Ćirić [6] and Matkowski [17], M. Bessenyei introduced the following

Definition 6.1. Let $X(d)$ be a metric space, $\mathbb{R}_+ = [0, +\infty[$, and φ a function of $\mathbb{R}_+$ to itself.

Then, a function f of X to itself is called a φ–quasi-contraction if

$$d\big(f(x), f(y)\big) \leq \varphi\left(\operatorname{diam}\big\{x, y, f(x), f(y)\big\}\right)$$

holds true for all $x, y \in X$.

Remark 6.2. Hence, by noticing that $\{f(x), f(y)\} = f\,[\{x, y\}]$ and

$$d\big(f(x), f(y)\big) = \operatorname{diam}\big\{f(x), f(y)\big\} = \operatorname{diam}\,(f\,[\{x, y\}])\,,$$

and thus

$$\operatorname{diam}\,(f\,[\{x, y\}]) \leq \varphi\left(\operatorname{diam}\big(\{x, y\} \cup f\,[\{x, y\}]\big)\right)$$

for all $x, y \in X$, we can naturally arrive at the following straightforward generalization of Definition 6.1.

Definition 6.3. Let X and φ be as in Definition 6.1, and moreover $\mathcal{A}$ a family of nonvoid, bounded subsets of X.

Then, a relation F on X is called a $(\varphi, \mathcal{A})$–quasi-contraction if, for each $A \in \mathcal{A}$, the image $F[A]$ is also bounded and

$$\operatorname{diam}\,(F\,[A]) \leq \varphi\left(\operatorname{diam}\big(A \cup F\,[A]\big)\right).$$

Note that here the first diameter can be $-\infty$ if $F[A] = \emptyset$. But, the second one is in $\mathbb{R}_+$ since A is nonvoid, and both A and $F[A]$ are bounded. Therefore, the first one cannot also be $+\infty$.

Remark 6.4. Hence, by Remark 6.2, we can see that the function f, considered in Definition 6.1, is a φ–quasi-contraction if and only if it is a $(\varphi, \mathcal{A})$–quasi-contraction with $\mathcal{A}$ being the family of all two-point subsets of X.

However, the functional particular case of Definition 6.3 is not a genuine generalization of Definition 6.1, since we have the following

Theorem 6.5. *Let $\mathcal{A}$ be the family of all one- or two-point subsets of X, and $\mathcal{B}$ an arbitrary family of nonvoid, bounded subsets of X. Moreover, assume that φ is an increasing function of $\mathbb{R}_+$ and F is a $(\varphi, \mathcal{A})$–quasi-contraction relation on X such that $F[B]$ is bounded for all $B \in \mathcal{B}$. Then, F is a also $(\varphi, \mathcal{B})$–quasi-contraction relation on X.*

Proof. If $B \in \mathcal{B}$ and $z, w \in F[B]$, then there exist $x, y \in B$ such that $z \in F(x)$ and $w \in F(y)$. Hence, by defining $A = \{x, y\}$, we can at once see that $A \in \mathcal{A}$ such that $A \subset B$ and $\{z, w\} \subseteq F[A]$.

Now, by the increasingness of the functions diam and φ, and the assumed contractivity of F, it is clear that

$$\begin{aligned} d(z, w) = \operatorname{diam}\big(\{z, w\}\big) \leq \operatorname{diam}\,(F[A]) \\ \leq \varphi\,\big(\operatorname{diam}\big(A \cup F[A]\big)\big) \leq \varphi\,\big(\operatorname{diam}\big(B \cup F[B]\big)\big)\,. \end{aligned}$$

Therefore, we also have

$$\operatorname{diam}\,(F[B]) = \sup\big\{\, d(z, w): \quad z, w \in F[B] \,\big\} \leq \varphi\,\big(\operatorname{diam}\big(B \cup F[B]\big)\big)$$

for all $B \in \mathcal{B}$, and thus the required assertion is also true.

Remark 6.6. Note that if in particular F is a function of X to itself in Theorem 6.5, then $\mathcal{A}$ may be only the family of all two-point subsets of X. Namely, in this case, we have $\operatorname{diam}\,(F[A]) = 0$ for all one-point subset A of X.

Now, analogously to Definition 6.3, we may also naturally introduce the following

Definition 6.7. Let X, φ, and $\mathcal{A}$ be as in Definitions 6.1 and 6.3. Then, a relation F of X to itself is called a $(\varphi, \mathcal{A})$–semi-contraction if, for each $A \in \mathcal{A}$, the image $F[A]$ is also bounded and

$$\operatorname{diam}\,(F[A]) \leq \varphi\,\big(\operatorname{diam}(A) + d\,\big(A, F[A]\big) + \operatorname{diam}\,\big(F[A]\big)\big)\,.$$

Note that here the distance and all the diameters are finite since now we also have $F[A] \neq \emptyset$ since $A \neq \emptyset$ and $F(x) \neq \emptyset$ for all $x \in X$.

This terminology can partly be justified by the following

Theorem 6.8. *Let X, φ, and $\mathcal{A}$ be as in Definitions 6.1 and 6.3. Moreover, assume that φ is an increasing function of $\mathbb{R}_+$ and F is a $(\varphi, \mathcal{A})$–quasi-contraction relation of X. Then, F is also a $(\varphi, \mathcal{A})$–semi-contraction relation of X*

Proof. By using a well-known property of the diameter, we can see that

$$\operatorname{diam}\big(A \cup F[A]\big) \leq \operatorname{diam}(A) + d\big(A, F[A]\big) + \operatorname{diam}\,\big(F[A]\big)\,.$$

Therefore, by the assumed contractivity property of F and the increasingness of φ, we have

$$\begin{aligned} \operatorname{diam}\,(F[A]) \leq \varphi\,\big(\operatorname{diam}\big(A \cup F[A]\big)\big) \\ \leq \varphi\,\big(\operatorname{diam}(A) + d\,\big(A, F[A]\big) + \operatorname{diam}\,\big(F[A]\big)\big)\,. \end{aligned}$$

Therefore, the required assertion is also true.

Remark 6.9. Now, to prove some analogues of the theorems of Bessenyei [1] for semi-contraction functions and relations, one has certainly require some much stronger regularity properties of the control function φ.

However, it would be more interesting to generalize the results of Bessenyei [1] to relator spaces analogously to [46]. Unfortunately, we have been planning to prove such an extension of a theorem of Kupka [14] for more than twenty years.

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