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AN INSTRUCTIVE TREATMENT OF SINGLETONS, DOUBLETONS AND ORDERED PAIRS

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ABSTRACT. By using the usual definitions, we give some necessary and sufficient conditions for various inclusions among singletons, doubletons, and ordered pairs. Thus, we can prove some criteria for equalities of these fundamental objects in a more convenient way.

1. INTRODUCTION

Throughout this paper, X will denote an arbitrary set. And, the reader will only be assumed to be familiar with the most primitive notions and notations of set theory.

As is customary, for any $a, b \in X$, the set $\{a\} = \{x \in X : \ x = a\}$,

$$\{a, b\} = \{a\} \cup \{b\} \qquad \text{and} \qquad (a, b) = \big\{\{a\}, \{a, b\}\big\}$$

is called a singlelon, doubleton and ordered pair, respectively.

By giving some necessary and sufficient conditions for various inclusions among these fundamental objects, we shall prove some criteria for equalities of these objects in a more convenient way.

In particular, by using some preliminary results on inclusions, we shall show that, for any $a, b, c, d \in X$, the following assertions are equivalent:

(1) $(a, b) = (c, d)$, (2) $a = c$ and $b = d$.

A nice direct proof of the implication (1) $\Longrightarrow$ (2) can be found at a recent page of Wikipedia [https://proofwiki.org/wiki/Equality_of_Ordered_Pairs].

Similar direct proofs of the above implication were formerly also included in the classical books of Halmos [2, p. 23] and Suppes [4, p. 32], for instance.

While, Bourbaki [1, p. 72] took the implication (1) $\Longrightarrow$ (2) as an axiomatic definition of an ordered pair with a remark that (2) also implies (1).

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The present ingenious set theoretic definition of ordered pairs was first introduced by Kazimierz Kuratowski in 1921.

It simplifies some former similar definitions given by Norbert Wiener and Felix Hausdorff in 1914. (See [http://en.wikipedia.org/wiki/Ordered_pair].)

The importance of ordered pairs lies mainly in the fact that Cartesian products, and hence also binary relations and functions, can only be defined precisely in terms of ordered pairs.

If $A, B \subseteq X$, then an arbitrary subset R of the product set

$$A \times B = \big\{ (a, b) : \quad a \in A, \quad b \in B \big\}$$

is called a relation between the sets A and B.

This precise definition of relations is due to Charles Sanders Peirce by Kelley [3, p. 7]. In contrast to this definition, Bourbaki [1, p. 76] would call the ordered triple (R, A, B) to be a correspondence between A and B.

2. Inclusions between singletons and doubletons

Definition 2.1. For any $a \in X$, the set

$$\{a\} = \{ x \in X : \quad x = a \}.$$

is called the singleton constructed from the element a.

Thus, we evidently have the following

Theorem 2.2. *For any $a, b \in X$, the following assertion are equivalent:*

(1) $a = b$, *(2)* $a \in \{b\}$, *(3)* $b \in \{a\}$.

Remark 2.3. Thus, in particular we have $a \in \{a\}$ for all $a \in X$.

Remark 2.4. Moreover, we can note that, for any $a \in X$ and $A \subseteq X$, the following assertions are equivalent:

(1) $a \in A$, (2) $\{a\} \subseteq A$.

Hence, by Theorem 2.2, it is clear that we also have the following

Theorem 2.5. *For any $a, b \in X$, the following assertions are equivalent:*

(1) $a = b$, *(2)* $\{a\} = \{b\}$, *(3)* $\{a\} \subseteq \{b\}$, *(4)* $\{b\} \subseteq \{a\}$.

Remark 2.6. Since, (2) implies (1), the singleton $\{a\}$ can usually be identified with the element a.

Thus, the set X can usually be considered as a subset of its power set

$$\mathcal{P}(X) = \big\{ A : \ A \subseteq X \big\}.$$

Definition 2.7. For any $a, b \in X$, the set

$$\{a, b\} = \{a\} \cup \{b\}$$

is called the doubleton constructed from the elements a and b.

Remark 2.8. Thus, in particular we have

$$\{a, a\} = \{a\} \cup \{a\} = \{a\}$$

for all $a \in X$.

Therefore, by Remark 2.6, the doubleton $\{a, a\}$ can also be identified with the element a.

Remark 2.9. Moreover, we also have

$$\{a, b\} = \{a\} \cup \{b\} = \{b\} \cup \{a\} = \{b, a\}$$

for all $a, b \in X$.

By using the corresponding definitions, Remark 2.4 and Theorem 2.2, we can easily establish the following

Theorem 2.10. *For any $a, b, c \in X$, the following assertions are equivalent:*

(1) $c \in \{a, b\}$, $\{c\} \subseteq \{a, b\}$, *(2) either* $c = a$ *or* $c = b$.

Remark 2.11. Thus, in particular we have

$$a \in \{a, b\} \qquad \text{and} \qquad b \in \{a, b\}$$

for all $a, b \in X$.

Remark 2.12. Moreover, we can note that, for any $a, b \in X$ and $A \subseteq X$, the following assertions are equivalent:

(1) $a, b \in A$, (2) $\{a, b\} \subseteq A$.

Now, in addition to Theorem 2.10, we can also easily prove the following

Theorem 2.13. *If $a, b, c, d \in X$ such that $a \neq b$, then the following assertions are equivalent:*

(1) $\{a, b\} = \{c, d\}$, *(2)* $\{a, b\} \subseteq \{c, d\}$,

(3) either $a = c$, $b = d$ *or* $a = d$, $b = c$.

Proof. Clearly, (1) always implies (2). Moreover, by Remark 2.11, we have $a, b \in \{a, b\}$. Therefore, if (2) holds, then we also have $a, b \in \{c, d\}$. Hence, by using Theorem 2.10, we can infer that either $a = c$ or $a = d$, and either $b = c$ or $b = d$.

However, if $a = c$ holds, then because of $a \neq b$ we can only have $b = d$. While, if $a = d$ holds, then again by $a \neq b$ we can only have $b = c$. Therefore, (3) also holds. Moreover, from Remark 2.9, it is clear that (3) always implies (1).

From this theorem, we can immediately derive the following

Corollary 2.14. *If* $a, b, c \in X$ *such that* $a \neq b$, *then the following assertions are equivalent:*

(1) $b = c$, *(2)* $\{a, b\} = \{a, c\}$, *(3)* $\{a, b\} \subseteq \{a, c\}$.

Proof. Clearly, the implications (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) are always true. Moreover, if (3) holds, then by using the assumption $a \neq b$ and Theorem 2.13 we can see that (1) holds.

Now, by using our former observations, we can also easily prove the following counterpart of Theorem 2.10.

Theorem 2.15. *For any* $a, b, c \in X$, *the following assertions are equivalent:*

(1) $\{a, b\} = \{c\}$, *(2)* $\{a, b\} \subseteq \{c\}$, *(3)* $a = c$, $b = c$.

Proof. Clearly, (1) implies (2). Moreover, if (2) holds, then by Remark 2.8 we also have $\{a, b\} \subseteq \{c, c\}$. Now, if $a \neq b$, then by Theorem 2.13 we can see that $a = c$ and $b = c$, and thus $a = b$. This contradiction proves that $a = b$.

Thus, by Remark 2.8, we now have

$$\{a\} = \{a, a\} = \{a, b\} \subseteq \{c, c\} = \{c\}.$$

Hence, by Theorem 2.5, we can already see that $a = c$. Thus, since $a = b$, assertion (3) also holds. On the other hand, if (3) holds, then by Remark 2.8 it is clear that (1) also holds.

From this theorem, it is clear that in particular we also have

Corollary 2.16. *If* $a, b \in X$ *such that* $a \neq b$, *then* $\{a, b\} \not\subseteq \{a\}$.

Proof. Namely, if $\{a, b\} \subseteq \{a\}$ holds, then by Theorem 2.15 we necessarily have $b = a$.

3. Inclusions between ordered pairs

Definition 3.1. For any $a, b \in X$, the set

$$(a, b) = \big\{\{a\}, \{a, b\}\big\}$$

is called the ordered pair constructed from the elements a and b.

Remark 3.2. Thus, in particular, for any $a \in X$, we have

$$(a, a) = \big\{ \{a\}, \{a, a\} \big\} = \big\{ \{a\}, \{a\} \big\} = \big\{ \{a\} \big\}.$$

Therefore, by Remark 2.6, the pair (a, a) could also be identified with the element a. However, if in particular X is the set of all real numbers, then each $a \in X$ is usually identified with the pair $(a, 0)$.

Now, by using Theorem 2.10, we can also prove the following

Theorem 3.3. *For any $a, b, c \in X$, the following assertions are equivalent:*

(1) $a = c$, *(2)* $(c, c) \subseteq (a, b)$.

Proof. If (1) holds, then by Theorem 2.5, Remark 2.11 and Definition 3.1 we have

$$\{c\} = \{a\} \in \big\{ \{a\}, \{a, b\} \big\} = (a, b).$$

Hence, by using Remarks 3.2 and 2.4, we can already infer that

$$(c, c) = \big\{ \{c\} \big\} \subseteq (a, b),$$

and thus (2) also holds.

While, if (2) holds, then by Remark 3.2 and Definition 3.1, we have

$$\big\{ \{c\} \big\} \subseteq \big\{ \{a\}, \{a, b\} \big\}.$$

Hence, by using Theorem 2.10, we can infer that either

$$\{c\} = \{a\} \qquad \text{or} \qquad \{c\} = \{a, b\}.$$

Hence, by using Theorems 2.5 and 2.15, we can already infer that either

$$c = a \qquad \text{or} \qquad c = a, \ \ c = b.$$

Thus, in particular (1) also holds.

Moreover, as a counterpart of Theorem 2.13, we can also prove the following

Theorem 3.4. *If $a, b, c, d \in X$ such that $a \neq b$, then the following assertions are equivalent:*

(1) $(a, b) = (c, d)$, *(2)* $(a, b) \subseteq (c, d)$, *(3)* $a = c$, $b = d$.

Proof. Clearly, (1) always implies (2). Moreover, if (2) holds, then by Definition 3.1 we have

$$\big\{ \{a\}, \{a, b\} \big\} \subseteq \big\{ \{c\}, \{c, d\} \big\}.$$

Moreover, by the assumption $a \neq b$ and Corollary 2.16, we necessarily have $\{a, b\} \not\subseteq \{a\}$, and thus $\{a\} \neq \{a, b\}$.

Hence, by using Theorem 2.13, we can see that either

$$\{a\}=\{c\}\,,\quad \{a,\,b\}=\{c,\,d\}\qquad\text{or}\qquad \{a\}=\{c,\,d\}\,,\quad \{a,\,b\}=\{c\}\,.$$

However, if $\{a,\,b\}=\{c\}$ holds, then by Theorem 2.15 we have $a=c$ and $b=c$, and thus also $a=b$. Therefore, because of the assumption $a\neq b$, we necessarily have

$$\{a\}=\{c\}\qquad\text{and}\qquad \{a,\,b\}=\{c,\,d\}\,.$$

Hence, by using Theorem 2.5, we can infer that $a=c$, and thus

$$\{a,\,b\}=\{a,\,d\}\,.$$

Hence, by using the assumption $a\neq b$ and Corollary 2.14 we can already infer that $b=d$. Therefore, (3) also holds. Moreover, it is clear that (3) always implies (1).

From this theorem, analogously to Corollary 2.14, we can immediately derive

Corollary 3.5. *If $a,b,c\in X$ such that $a\neq b$, then the following assertions are equivalent:*

(1) $b=c$, *(2)* $(a,\,b)=(a,\,c)$, *(3)* $(a,\,b)\subseteq(a,\,c)$.

Moreover, as a close analogue of Theorem 2.15, we can also prove

Theorem 3.6. *For any $a,\,b,\,c\in X$, the following assertions are equivalent:*

(1) $(a,\,b)=(c,\,c)$, *(2)* $(a,\,b)\subseteq(c,\,c)$, *(3)* $a=c$, $b=c$.

Proof. Clearly, (1) implies (2). Moreover, if (2) holds, then by Definition 3.1 and Remark 3.2, we have

$$\big\{\{a\},\,\{a,\,b\}\big\}\subseteq\big\{\,\{c\}\big\}\,.$$

Hence, by using Theorem 2.15, we can infer that

$$\{a\}=\{c\}\qquad\text{and}\qquad\{a,\,b\}=\{c\}\,.$$

Hence, by using Theorem 2.15, we can infer that $a=c$ and $b=c$, and thus (3) also holds. Moreover, if (3) holds, then it is clear that (1) also holds.

Now, as an immediate consequence of Theorems 3.4 and 3.6, we can state

Corollary 3.7. *For any $a,\,b,\,c,\,d\in X$, the following assertions are equivalent:*

(1) $(a,\,b)=(c,\,d)$, *(2)* $a=c$, $b=d$.

Proof. If $a \neq b$, then from Theorem 3.4 we can see that (1) implies (2). Moreover, if $a = b$, then (1) gives only that $(a, a) = (c, d)$. Hence, by using Theorem 3.6, we can already infer that $a = c$ and $a = d$. Thus, since $a = b$, assertion (2) also holds.

Concerning ordered pairs, it is also worth mentioning the following

Theorem 3.8. *For any* $a, b \in X$, *we have*

(1) $\bigcap (a, b) = \{a\}$, *(2)* $\bigcup (a, b) = \{a, b\}$.

Proof. Namely,

$$\bigcap (a, b) = \bigcap \big\{ \{a\}, \{a, b\} \big\} = \{a\} \cap \{a, b\} = \{a\}$$

and

$$\bigcup (a, b) = \bigcup \big\{ \{a\}, \{a, b\} \big\} = \{a\} \cup \{a, b\} = \{a, b\}.$$

Remark 3.9. In view of our present treatment, together with a family $\mathcal{S}$ of subsets of X, it seems also reasonable to investigate the families

$$\mathcal{D}_{\mathcal{S}} = \big\{ A \cup B : \quad A, B \in \mathcal{S} \big\}$$

and

$$\mathcal{P}_{\mathcal{S}} = \big\{ \{A, A \cup B\} : \quad A, B \in \mathcal{S} \big\}.$$

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