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A PARTICULAR GALOIS CONNECTION BETWEEN RELATIONS AND SET FUNCTIONS

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Abstract. Motivated by a recent paper of U. Höhle and T. Kubiak, we investigate a Galois-type connection between relations on one set X to another Y and functions on the power set $\mathcal{P}(X)$ to $\mathcal{P}(Y)$.

Since relations can largely be identified with union-preserving set functions, the results obtained can be used to provide some natural generalizations of most of the former results on relations and relators (families of relations).

Introduction

In this paper, a subset R of a product set $X \times Y$ is called a relation on X to Y. And, a function U on the power set $\mathcal{P}(X)$ to $\mathcal{P}(Y)$ is called a corelation on X to Y.

Motivated by a recent paper of Höhle and Kubiak [8], for any relation R on X to Y, we define a corelation $R^\star$ on X to Y such that

$$R^\star(A) = R[A]$$

for all $A \subset X$.

Moreover, for any corelation U on X to Y, we define a relation U^* on X to Y such that

$$U^*(x) = U\big(\{x\}\big)$$

for all $x \in X$.

And, we show that the functions $\star$ and $*$ establish an interesting Galois-type connection between the family $\mathcal{P}(X \times Y)$ of all relations on X to Y and the family $\mathcal{Q}(X, Y)$ of all correlations on X to Y, whenever $\mathcal{P}(X \times Y)$ is considered to be partially ordered by the ordinary set inclusion and $\mathcal{Q}(X, Y)$ by the pointwise one.

Since relations can largely be identified with union-preserving corelations, the results obtained can be used to provide some natural generalizations of most of the former results on relations and relators. (The most relevant ones are in [15] and [11].) The results on inverse relations and relators seem to be the only exceptions.

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1. A few basic facts on relations

A subset F of a product set $X{\times}Y$ is called a relation on X to Y. If in particular $F \subset X^{\,2}$, with $X^{\,2} = X{\times}X$, then we may simply say that F is a relation on X. In particular, $\Delta_X = \{\,(x,\,x):\ \ x\in X\,\}$ is called the identity relation on X.

If F is a relation on X to Y, then for any $x \in X$ and $A \subset X$ the sets $F(x) = \{\,y \in Y:\ \ (x,\,y) \in F\,\}$ and $F\,[\,A\,] = \bigcup_{a \in A} F(a)$ are called the images of x and A under F, respectively.

Moreover, the sets $D_F = \{\,x \in X:\ \ F(x) \neq \emptyset\,\}$ and $R_F = F\,[\,D_F\,]$ are called the domain and range of F, respectively. If in particular $D_F = X$, then we say that F is a relation of X to Y, or that F is a non-partial relation on X to Y.

In particular, a relation f on X to Y is called a function if for each $x \in D_f$ there exists $y \in Y$ such that $f(x) = \{y\}$. In this case, by identifying singletons with their elements, we may simply write $f(x) = y$ in place of $f(x) = \{y\}$.

In particular, a function $\star$ of a set X to itself is called an unary operation on X. While, a function $*$ of $X^{\,2}$ to X is called a binary operation on X. And, for any $x,\,y \in X$, we write $x^{\star}$ and $x * y$ instead of $\star(x)$ and $*(x,\,y)$, respectively.

Concerning relations, one can easily establish the following

Theorem 1.1. *For any relation F on X to Y, we have*

$$F = \bigcup_{x \in X} \{\,x\,\}{\times}F(x)\,.$$

Hence, one can immediately derive the following

Corollary 1.2. *For any two relations F and G on X to Y, we have $F \subset G$ if and only if $F(x) \subset G(x)$ for all $x \in X$.*

Remark 1.3. Note that $F(x) = \emptyset$ if $x \in D_F^{\,c}$. Therefore, in the assertions of Theorem 1.1 and Corollary 1.2 we may write D_F in place of X.

Moreover, we can also note that $F = G$ if and only if $F(x) = G(x)$ for all $x \in X$, or equivalently $D_F = D_G$ and $F(x) = G(x)$ for all $x \in D_F$.

From Theorem 1.1, we can also at once see that a relation F on X to Y can be naturally defined by specifying $F(x)$ for all $x \in X$, or by specifying D_F and $F(x)$ for all $x \in D_F$.

However, the latter possibility will be of no importance for us. Namely, for instance, we may naturally have the following two definitions.

Definition 1.4. For any relation F on X to Y, we define a relation F^{-1} on Y to X such that

$$F^{-1}(y) = \big\{\,x \in X:\ \ y \in F(x)\,\big\}$$

for all $y \in Y$. The relation F^{-1} is called the inverse of F.

Remark 1.5. Thus, for any $x \in X$ and $y \in Y$, we have $(y,\,x) \in F^{-1}$ if and only if $(x,\,y) \in F$.

Definition 1.6. For any relations F on X to Y and G on Y to Z, we define a relation $G \circ F$ on X to Z such that

$$(G \circ F)(x) = G\,[\,F(x)\,]$$

for all $x \in X$. The relation $G \circ F$ is called the composition of G and F.

Remark 1.7. Thus, for any $x \in X$ and $z \in Z$, we have $(x, z) \in G \circ F$ if and only if $(x, y) \in F$ and $(y, z) \in G$ for some $y \in Y$.

2. Some further results on relations

Concerning the above two basic operations on relations, one can easily prove the following two theorems.

Theorem 2.1. *For any relations F on X to Y and G on Y to Z, we have*

$$(G \circ F)^{-1} = F^{-1} \circ G^{-1}.$$

Theorem 2.2. *If F is a relation on X to Y and G is a relation on Y to Z, then for any $A \subset X$ we have*

$$\big(G \circ F\big)[A] = G\big[F[A]\big].$$

In addition to the above theorems, it is also worth mentioning that the following theorems are also true.

Theorem 2.3. *If F is a relation on X to Y, then for any family $\mathcal{A}$ of subsets of X we have*

(1) $F\Big[\bigcup \mathcal{A}\Big] = \bigcup_{A \in \mathcal{A}} F[A]$; *(2)* $F\Big[\bigcap \mathcal{A}\Big] \subset \bigcap_{A \in \mathcal{A}} F[A]$.

Theorem 2.4. *If F is a relation on X to Y, then for any $A, B \subset X$ we have*

(1) $F[A] \setminus F[B] \subset F[A \setminus B]$; *(2)* $F[A]^c \subset F[A^c]$ *if* $Y = R_F$.

Remark 2.5. If in particular F^{-1} is a function, then the corresponding equalities are also true in the above two theorems.

Theorem 2.6. *If $\mathcal{F}$ is a family of relations on X to Y, then for any $A \subset X$ we have*

(1) $\Big(\bigcup \mathcal{F}\Big)[A] = \bigcup_{F \in \mathcal{F}} F[A]$; *(2)* $\Big(\bigcap \mathcal{F}\Big)[A] \subset \bigcap_{F \in \mathcal{F}} F[A]$.

Theorem 2.7. *If F and G are relations on X to Y, then for any $A \subset X$ we have*

(1) $F[A] \setminus G[A] \subset (F \setminus G)[A]$; *(2)* $F[A]^c \subset F^c[A]$ *if* $A \neq \emptyset$.

Remark 2.8. If in particular A is a singleton, then the corresponding equalities are also true in the above two theorems.

Concerning the complement relation F^c, one can also easily prove the following two theorems.

Theorem 2.9. *For any relation F on X to Y, we have*

$$\big(F^c\big)^{-1} = \big(F^{-1}\big)^c.$$

Theorem 2.10. *If F is a relation on X to Y, then for any $A \subset X$, we have*

$$F^c[A]^c = \bigcap_{x \in A} F(x).$$

From this theorem, by using Remark 2.8, one can easily derive

Corollary 2.11. *If* F *is a relation on* X *to* Y *and* G *is a relation on* Y *to* Z, *then for any* $x \in X$, *we have*

$$\big(G \circ F\big)^c(x) = \bigcap_{y \in F(x)} G^c(y).$$

Remark 2.12. In addition to this corollary, it is also worth proving that

(1) $(G \circ F)^c \subset G^c \circ F$ if $X = D_F$; (2) $(G \circ F)^c \subset G \circ F^c$ if $Z = R_G$.

3. Functions on one power set to another

Definition 3.1. If U is a function on one power set $\mathcal{P}(X)$ to another $\mathcal{P}(Y)$, then we simply say that U is a corelation on X to Y.

Remark 3.2. According to Birkhoff [1, p. 111], the term "operation" could also be used. However, this may cause some confusions because of the ordinary use of this term.

Definition 3.3. A corelation U on X to Y, is called

(1) increasing if $U(A) \subset U(B)$ for all $A \subset B \subset X$;

(2) quasi-increasing if $U\big(\{x\}\big) \subset U(A)$ for all $x \in A \subset X$

(3) union-preserving if $U\Big(\bigcup \mathcal{A}\Big) = \bigcup_{A \in \mathcal{A}} U(A)$ for all $\mathcal{A} \subset \mathcal{P}(X)$.

Remark 3.4. In the $X = Y$ particular case, U may also be called extensive, intensive, involutive, and idempotent if $A \subset U(A)$, $U(A) \subset A$, $U\big(U(A)\big) = A$, and $U\big(U(A)\big) = U(A)$ for all $A \subset X$, respectively.

Moreover, in particular an increasing and idempotent corelation may be called a projection or modification operation. And an extensive (intensive) projection operation may be called a closure (interior) operation.

Simple reformulations of properties (1) and (2) in Definition 3.3 give the following two theorems.

Theorem 3.5. *For a corelation* U *on* X *to* Y, *the following assertions are equivalent:*

(1) U *is quasi-increasing;*

(2) $\bigcup_{x \in A} U\big(\{x\}\big) \subset U(A)$ *for all* $A \subset X$.

Theorem 3.6. *For a corelation operation* U *on* X *to* Y, *the following assertions are equivalent:*

(1) U *is increasing;*

(2) $\bigcup_{A \in \mathcal{A}} U(A) \subset U\Big(\bigcup \mathcal{A}\Big)$ *for all* $\mathcal{A} \subset \mathcal{P}(X)$;

(3) $U(A) \cup U(B) \subset U(A \cup B)$ *for all* $A, B \subset X$.

Hence, it is clear that in particular we also have

Corollary 3.7. *A corelation U on X to Y is union-preserving if and only if U is increasing and $U\Big(\bigcup\mathcal{A}\Big)\subset\bigcup_{A\in\mathcal{A}}U(A)$ for all $\mathcal{A}\subset\mathcal{P}(X)$.*

However, it now more important to note that now we also have the following theorem which has also been proved, in a different way, by Pataki [9].

Theorem 3.8. *For a corelation U on X to Y, the following assertions are equivalent:*

(1) U is uninon-preserving;

(2) $U(A)=\bigcup\limits_{x\in A}U\big(\{x\}\big)$ for all $A\subset X$.

Proof. Since $A=\bigcup_{x\in A}\{x\}$ for all $A\subset X$, it is clear that (1) implies (2).

On the other hand, if (2) holds, then we can note that U is already increasing. Therefore, by Theorem 3.6, we have $\bigcup_{A\in\mathcal{A}}U(A)\subset U\big(\bigcup\mathcal{A}\big)$ for any $\mathcal{A}\subset\mathcal{P}(X)$. Thus, to get (1), we need only prove the converse inclusion.

For this, note that if $\mathcal{A}\subset\mathcal{P}(X)$, then by (2) we have

$$U\Big(\bigcup\mathcal{A}\Big)=\bigcup_{x\in\bigcup\mathcal{A}}U(\{x\}).$$

Therefore, if $y\in U\Big(\bigcup\mathcal{A}\Big)$, then there exists $x\in\bigcup\mathcal{A}$ such that $y\in U\big(\{x\}\big)$. Thus, in particular there exists $A_o\in\mathcal{A}$ such that $x\in A_o$, and so $\{x\}\subset A_o$. Hence, we can already see that

$$y\in U\big(\{x\}\big)\subset U(A_o)\subset\bigcup_{A\in\mathcal{A}}U(A).$$

Therefore, $\Big(\bigcup\mathcal{A}\Big)\subset\bigcup_{A\in\mathcal{A}}U(A)$ also holds.

From this theorem, by Theorem 3.5, it is clear that in particular we also have

Corollary 3.9. *A corelation U on X to Y is union-preserving if and only if U is quasi-increasing and $U(A)\subset\bigcup\limits_{x\in A}U\big(\{x\}\big)$ for all $A\subset X$.*

Definition 3.10. For any two corelations U and V on X to Y, we write

$$U\leq V\iff U(A)\subset V(A)\quad\text{for all}\quad A\subset X.$$

Remark 3.11. Note that if in particular $U\subset V$, then $U(A)=V(A)$ for all $A\in D_U$ and $U(A)=\emptyset\subset V(A)$ for all $A\subset X$ with $A\notin D_U$. Therefore, we have $U(A)\subset V(A)$ for all $A\subset X$, and thus $U\leq V$.

Theorem 3.12. *With the inequality considered in Definition 3.10, the family $\mathcal{Q}(X,Y)$ of all corelations on X to Y, forms a complete poset.*

Proof. It can be easily seen that if $\mathcal{U}$ is a family of corelations on X to Y and

$$V(A)=\bigcup_{U\in\mathcal{U}}U(A)$$

for all $A\subset X$, then $V\in\mathcal{Q}(X,Y)$ such that $V=\sup(\mathcal{U})$.

Thus, $\mathcal{Q}(X,Y)$ is sup-complete, and hence it is also inf-complete by [1, Theorem 3, p. 112]. (See also [3, Theorem 4.1] for an immediate extension.)

Remark 3.13. Note that if in particular each member of $\mathcal{U}$ is increasing (quasi-increasing), then V is also increasing (quasi-increasing).

Therefore, with the inequality given in Definition 3.10, the family $\mathcal{Q}_1(X, Y)$ of all quasi-increasing corelations on X to Y is also a complete poset.

4. A particular Galois connection between relations and corelations

According to the corresponding definitions of Höhle and Kubiak [8], we may also naturally have the following

Definition 4.1. For any relation R on X to Y, we define a corelation $R^\star$ on X to Y such that

$$R^\star(A) = R[A]$$

for all $A \subset X$.

Conversely, for any corelation U on X to Y, we define a relation U^* on X to Y such that

$$U^*(x) = U\big(\{x\}\big)$$

for all $x \in X$.

Now, by using the corresponding definitions, we can easily prove the following two theorems.

Theorem 4.2. *If U is a corelation on X to Y, then $R^\star \leq U$ implies $R \subset U^*$ for any relation R on X to Y.*

Proof. If $R^\star \leq U$, then by the corresponding definitions

$$R(x) = R\big[\{x\}\big] = R^\star\big(\{x\}\big) \subset U\big(\{x\}\big) = U^*(x)$$

for all $x \in X$. Therefore, by Corollary 1.2, $R \subset U^*$ also holds.

Theorem 4.3. *For a corelation U on X to Y, the following assertions are equivalent:*

(1) U is quasi-increasing;

(2) $R \subset U^*$ implies $R^\star \leq U$ for any relation R on X to Y.

Proof. If (1) holds and $R \subset U^*$, then

$$R^\star(A) = R[A] = \bigcup_{x \in A} R(x) \subset \bigcup_{x \in A} U^*(x) = \bigcup_{x \in A} U\big(\{x\}\big) \subset U(A)$$

for all $A \subset X$. Therefore, $R^\star \leq U$, and thus (2) also holds.

Conversely, if (2) holds, then because of $U^* \subset U^*$ we have $U^{*\star} = \big(U^*\big)^\star \leq U$. Therefore, for any $A \subset X$, we have

$$U^{*\star}(A) \subset U(A).$$

Moreover, by using the corresponding definitions, we can see that

$$U^{*\star}(A) = \big(U^*\big)^\star(A) = U^*[A] = \bigcup_{x \in A} U^*(x) = \bigcup_{x \in A} U\big(\{x\}\big).$$

Therefore, $\bigcup_{x \in A} U\big(\{x\}\big) \subset U(A)$, and thus (1) also holds.

Now, as an immediate consequence of the above two theorems, we can also state

Corollary 4.4. *For an arbitrary relation* R *and a quasi-increasing corelation* U *on* X *to* Y, *we have*

$$R^{\star} \leq U \quad \Longleftrightarrow \quad R \subset U^{*}.$$

Remark 4.5. This corollary shows that the operation $\star$ and the restriction of $*$ to $\mathcal{Q}_1(X, Y)$ establish an increasing Galois connection. (For the relevant definition, see [4, p. 155] and [13, 14].)

Therefore, the extensive theory of Galois connections (see [2, 7, 4, 6, 5]) could be applied here. However, because of the simplicity of Definition 4.1, it seems now more convenient to use some elementary, direct proofs.

5. Some further properties of the operations $\star$ and $*$

By the corresponding definitions, we evidently have the following

Theorem 5.1. *For any two relations* R, S *and corelations* U, V *on* X *to* Y,

(1) $R \subset S$ *implies* $R^{\star} \leq S^{\star}$;

(2) $U \leq V$ *implies* $U^{*} \subset V^{*}$.

Remark 5.2. Note that, by using Corollary 4.4, instead of (2) we could only prove that the restriction of the operation $*$ to $\mathcal{Q}_1(X, Y)$ is increasing.

From (2), by using Remark 3.11, we can immediately get

Corollary 5.3. *For any two corelations* U *and* V *on* X *to* Y, $U \subset V$ *also implies* $U^{*} \subset V^{*}$.

Moreover, we can also easily prove the following theorem whose first statement has also been established by Höhle and Kubiak [8].

Theorem 5.4. *For any two relations* R *and* S *on* X *to* Y,

(1) $R^{\star *} = R$;

(2) $R^{\star} \leq S^{\star}$ *implies* $R \subset S$.

Proof. By the corresponding definitions, we have

$$R^{\star *}(x) = \big(R^{\star}\big)^{*}(x) = R^{\star}\big(\{x\}\big) = R\left[\{x\}\right] = R(x)$$

for all $x \in X$. Therefore, by Theorem 1.1, (1) is also true.

To prove (2), note that if $R^{\star} \leq S^{\star}$ holds, then by Theorem 5.1 we also have $R^{\star *} \subset S^{\star *}$. Hence, by using (1), we can see that $R \subset S$ also holds.

Remark 5.5. The above theorem shows that the function $\star$ is injective, $*$ is onto $\mathcal{P}(X, Y)$, and $\star *$ is the identity function of $\mathcal{P}(X \times Y)$.

Moreover, by Theorems 5.1 and 5.4, we can also at once state

Corollary 5.6. *For any two relations* R *and* S *on* X *to* Y, *we have* $R \subset S$ *if and only if* $R^{\star} \leq S^{\star}$.

Concerning the dual operation $\ast\star$, we can only prove the following theorem which, to some extent, has also been established by Höhle and Kubiak [8] and Pataki [9].

Theorem 5.7. *For a corelation U on X to Y, the following assertions are equivalent:*

(1) $U^{\ast\star} = U$;

(2) U *is union-preserving;*

(3) $U = R^{\star}$ *for some relation R on X to Y.*

Proof. By the proof of Theorem 4.3 and assertion (2), we have

$$U^{\ast\star}(A) = \bigcup_{x \in A} U\big(\{x\}\big) = U(A)$$

for all $A \subset X$. Therefore, (2) implies (1).

Now, since (1) trivially implies (3), we need only note that if (3) holds, then

$$U(A) = R^{\star}(A) = R[A] = \bigcup_{x \in A} R(x) = \bigcup_{x \in A} R[\{x\}] = \bigcup_{x \in A} R^{\star}\big(\{x\}\big) = \bigcup_{x \in A} U\big(\{x\}\big)$$

for all $A \subset X$. Therefore, by Theorem 3.8, (2) also holds.

Remark 5.8. The above theorem, together with Theorem 2.3, shows that the function $\star$ maps $\mathcal{P}(X \times Y)$ onto the family $\mathcal{Q}_3(X, Y)$ of all union-preserving corelations on X to Y.

Moreover, the restriction of $\ast$ to $\mathcal{Q}_3(X, Y)$ is injective and that of $\ast\star$ is the identity function of $\mathcal{Q}_3(X, Y)$. Therefore, the Galois connection mentioned in Remark 4.5 is rather particular.

Now, as an immediate consequence of Theorems 5.1 and 5.7, we can also state

Corollary 5.9. *For any two union-preserving corelations U and V on X to Y, we have $U \leq V$ if and only if $U^{\ast} \subset V^{\ast}$.*

Proof. Note that if $U^{\ast} \subset V^{\ast}$ holds, then by Theorem 5.1 we also have $U^{\ast\star} \leq V^{\ast\star}$. Hence, by Theorem 5.7, we can see that $U \leq V$ also holds.

Moreover, in addition to Theorem 5.7, we can also prove the following

Theorem 5.10. *Under the notation $\circ = \ast\star$, for any two corelations U and V on X to Y, we have*

(1) $U^{\circ\circ} = U^{\circ}$;

(2) $U \leq V$ *implies* $U^{\circ} \leq V^{\circ}$.

(3) $U^{\circ} \leq U$ *if and only if U is quasi-increasing.*

Proof. Assertion (2) is immediate from Theorem 5.1. While, from the proof of Theorem 4.3, we know that

$$U^{\circ}(A) = U^{\ast\star}(A) = \bigcup_{x \in A} U\big(\{x\}\big)$$

for all $A \subset X$. Hence, by Definition 3.3 and Theorem 3.5, it is clear that (3) is true.

Moreover, from the above equality, we can also see that

$$U^{\circ\circ}(A)=\bigcup_{x\in A}U^{\circ}\big(\{x\}\big)=\bigcup_{x\in A}U\big(\{x\}\big)=U^{\circ}(A)$$

for all $A\subset X$. Therefore, (1) is also true.

Remark 5.11. The above theorem shows that the function $\circ$ is a projection operation operation on $\mathcal{Q}(X,Y)$ such that its restriction to $\mathcal{Q}_1(X,Y)$ is already an interior operation.

Moreover, from Theorem 5.7, we can see that, for any corelation U on X to Y, we have $U^{\circ}=U$ if and only if U is union-preserving. Therefore, $\mathcal{Q}_3(X,Y)$ is the family of all open elements of $\mathcal{Q}(X,Y)$.

Now, as some useful consequences of our former results, we can also easily prove the following two theorems.

Theorem 5.12. *If R is a relation on X to Y and $U=R^{\star}$, then*

(1) U is the smallest quasi-increasing corelation on X to Y such that $R\subset U^{*}$;

(2) U is the largest union-preserving corelation on X to Y such that $U^{*}\subset R$.

Proof. From Theorems 5.7 and 5.4, we can see that U is union-preserving and $U^{*}=R^{\star *}=R$.

Moreover, if V is a quasi-increasing corelation on X to Y such that $R\subset V^{*}$, then by Theorem 4.3 we also have $R^{\star}\leq V$, and thus $U\leq V$. Therefore, (1) is true.

On the other hand, if V is a corelation on X to Y such that $V^{*}\subset R$, then by Theorem 5.1 we also have $V^{*\star}\leq R^{\star}$, and thus $V^{*\star}\leq U$. Hence, if in particular V is union-preserving, then by Theorem 5.7 we can see that $V\leq U$. Therefore, (2) is also true.

Theorem 5.13. *If U is a corelation on X to Y and $R=U^{*}$, then*

(1) R is the largest relation on X to Y such that $R^{\star}\leq U$ whenever U is quasi-increasing;

(2) R is the smallest relation on X to Y such that $U\leq R^{\star}$ whenever U is union-preserving.

Proof. If U is quasi-increasing, then by Theorem 5.10 we have $R^{\star}=U^{*\star}=U^{\circ}\leq U$. While, if U is union-preserving, then by Theorem 5.7 we have $R^{\star}=U^{*\star}=U$.

Moreover, if S is a relation on X to Y such that $S^{\star}\leq U$, then by Theorem 4.2 we also have $S\subset U^{*}$, and thus $S\subset R$ even if U is not supposed to be quasi-increasing. Thus, in particular (1) is true.

While, if S is a relation on X to Y such that $U\leq S^{\star}$, then by Theorem 5.1, we also have $U^{*}\subset S^{\star *}$. Hence, by the definition of R and Theorem 5.4, we can see that $R\subset S$ even if U is not supposed to be union-preserving. Thus, in particular (2) is also true.

Remark 5.14. Concerning the operations $\star$ and $*$, it is also worth noticing that if R is relation and U is a corelation on X to Y, then by the corresponding definitions we have

(1) $R^{\star}(A)=\mathrm{cl}_{R^{-1}}(A)$ for all $A\subset X$;

(2) $R^\star \leq U \iff A \in \operatorname{Int}_R\big(U(A)\big)$ for all $A \subset X$.

Moreover, if U is quasi-increasing, then under the notation

$$\operatorname{Int}_\star(U) = \big\{ S \subset X \times Y : \quad S^\star \leq U \big\},$$

we have $U^* = \bigcup \operatorname{Int}_\star(U)$ by the assertion (1) of Theorem 5.13.

6. Compatibility of the operation $\star$ with some set and relation theoretic ones

Now, as some immediate consequence of the corresponding results of Section 2, we can also state the following theorems.

Theorem 6.1. *If R is a relation on X to Y, then for any family $\mathcal{A}$ of subsets of X we have*

(1) $R^\star\Big(\bigcup \mathcal{A}\Big) = \bigcup\limits_{A \in \mathcal{A}} R^\star(A)\,;$ *(2)* $R^\star\Big(\bigcap \mathcal{A}\Big) \subset \bigcap\limits_{A \in \mathcal{A}} R^\star(A)\,.$

Theorem 6.2. *If R is a relation on X to Y, then for any $A, B \subset X$ we have*

(1) $R^\star(A) \setminus R^\star(B) \subset R^\star(A \setminus B)\,;$ *(2)* $R^\star(A)^c \subset R^\star(A^c)$ *if* $Y = R[X]$.

Remark 6.3. If in particular R^{-1} is a function, then the corresponding equalities are also true in the above two theorems.

Theorem 6.4. *If $\mathcal{R}$ is a family of relations on X to Y, then for any $A \subset X$ we have*

(1) $\Big(\bigcup \mathcal{R}\Big)^\star(A) = \bigcup\limits_{R \in \mathcal{R}} R^\star(A)\,;$ *(2)* $\Big(\bigcup \mathcal{R}\Big)^\star(A) \subset \bigcap\limits_{R \in \mathcal{R}} R^\star(A)\,.$

Theorem 6.5. *If R and S are relations on X to Y, then for any $A \subset X$ we have*

(1) $R^\star(A) \setminus S^\star(A) \subset (R \setminus S)^\star(A)\,;$ *(2)* $R^\star(A)^c \subset R^{c\star}(A)$ *if* $A \neq \emptyset$.

Theorem 6.6. *If R is a relation on X to Y, then for any $A \subset X$ we have*

$$R^{c\star}(A)^c = \bigcap_{x \in A} R(x)\,.$$

Moreover, we can also easily prove the following theorem which has also been established by Höhle and Kubiak [8].

Theorem 6.7. *For any two relations R on X to Y and S on Y to Z, we have*

$$(S \circ R)^\star = S^\star \circ R^\star\,.$$

Proof. By the corresponding definitions and Theorem 2.2, we have

$$(S \circ R)^\star(A) = (S \circ R)[A] = S\big[R[A]\big] = S^\star\big(R^\star(A)\big) = \big(S^\star \circ R^\star\big)(A)$$

for all $A \subset X$. Therefore, the required equality is also true.

From this theorem, by using Theorem 5.7, we can immediately get

Corollary 6.8. *For an arbitrary relation on R on X to Y and a union-preserving corelation V on Y to Z, we have*

$$(V^{*} \circ R)^{\star} = V \circ R^{\star}.$$

In addition to Theorem 6.7, we can also easily prove the following correction of a false statement of Höhle and Kubiak [8].

Theorem 6.9. *For an arbitrary corelation U on X to Y and a union-preserving corelation V on Y to Z, we have*

$$(V \circ U)^{*} = V^{*} \circ U^{*}.$$

Proof. By the corresponding definitions and Theorem 5.7, we have

$$\begin{aligned}(V \circ U)^{*}(x) &= (V \circ U)\big(\{x\}\big) = V\,\big(U\big(\{x\}\big)\big) \\ &= V\,\big(U^{*}(x)\big) = V^{**}\big(U^{*}(x)\big) = V^{*}\big[\,U^{*}(x)\big] = \big(V^{*} \circ U^{*}\big)(x)\end{aligned}$$

for all $x \in X$. Therefore, the required equality is also true.

From this theorem, by using Theorems 5.7 and 5.4, we can immediately get

Corollary 6.10. *For a corelation U on X to Y and a relation S on Y to Z, we have*

$$(S^{\star} \circ U)^{*} = S \circ U^{*}.$$

Remark 6.11. If R is a relation on X to Y and S is a relation on Z to W, then by defining a relation $R \boxtimes S$ on $X \times Z$ to $Y \times W$ such that

$$(R \boxtimes S)(x, z) = R(x) \times S(z)$$

for all $x \in X$ and $z \in Z$, we can also prove that

$$(R \boxtimes S)^{\star}(\Omega) = S \circ \Omega \circ R^{-1}.$$

for any relation Ω on X to Z. (The box product of relations has been mainly investigated in [15].)

From the above equality, by taking $\Omega = \{(x, z)\}$, and $\Omega = \Delta_Y$ in the $Y = Z$ particular case, we can see that the box and composition products of relations are equivalent tools. However, in contrast to the composition, the box product can be immediately extended to arbitrary families of relations.

7. Partial compatibility of the operation $\star$ with the relation theoretic inversion

Theorem 7.1. *For a relation R on X to Y, the following assertions are equivalent:*

(1) $R^{-1} \circ R = \Delta_X$ *;*

(2) $\big(R^{\star}\big)^{-1} \subset \big(R^{-1}\big)^{\star}$ *;*

(3) R^{-1} *is a function on Y onto X.*

Proof. For any $x \in X$, we have

$$R^\star(\{x\}) = R[\{x\}] = R(x), \qquad \text{and thus} \qquad \{x\} \in \big(R^\star\big)^{-1}\big(R(x)\big).$$

Hence, if (2) holds, we can infer that

$$\{x\} \in \big(R^{-1}\big)^\star\big(R(x)\big), \qquad \text{and thus} \qquad \big(R^{-1}\big)^\star\big(R(x)\big) = \{x\}.$$

Therefore,

$$R^{-1}\big[R(x)\big] = \{x\}, \qquad \text{and thus} \qquad \big(R^{-1} \circ R\big)(x) = \Delta_X(x).$$

Hence, we can see that (1) also holds.

To prove the converse implication, note that if $A \subset X$ and $B \subset Y$ such that $A \in \big(R^\star\big)^{-1}(B)$, then we also have

$$R^\star(A) = B, \qquad \text{and thus} \qquad R[A] = B.$$

Hence, we can infer that

$$R^{-1}\big[R[A]\big] = R^{-1}[B], \qquad \text{and thus} \qquad \big(R^{-1} \circ R\big)[A] = R^{-1}[B].$$

Therefore, if (1) holds, then

$$\Delta_X[A] = R^{-1}[B], \qquad \text{and thus} \qquad A = \big(R^{-1}\big)^\star(B).$$

Hence, it is clear that (2) also holds.

Therefore, (1) and (2) are equivalent. The proof of the equivalence of (1) and (3) will be left to the reader.

From Theorem 7.1, by writing R^{-1} in place of R we can immediately derive the following

Theorem 7.2. *For a relation R on X to Y, the following assertions are equivalent:*

(1) $R \circ R^{-1} = \Delta_Y$ *;*

(2) $\big(R^{-1}\big)^\star \subset \big(R^\star\big)^{-1}$ *;*

(3) R is a function on X onto Y.

Proof. Note that now R^{-1} is a relation on Y to X. Therefore, by Theorem 7.1, the following assertions are equivalent:

(a) $\big(R^{-1}\big)^{-1} \circ R^{-1} = \Delta_Y$;

(b) $\Big(\big(R^{-1}\big)^\star\Big)^{-1} \subset \Big(\big(R^{-1}\big)^{-1}\Big)^\star$;

(c) $\big(R^{-1}\big)^{-1}$ is a function on X onto Y.

Hence, since $R = \big(R^{-1}\big)^{-1}$, and

$$\big(R^{-1}\big)^\star \subset \big(R^\star\big)^{-1} \iff \Big(\big(R^{-1}\big)^\star\Big)^{-1} \subset R^\star,$$

it is clear that assertions (1), (2) and (3) are also equivalent.

Now, as an immediate consequence of the above two theorems, we can also state

Corollary 7.3. *For a relation R on X to Y, the following assertions are equivalent:*

(1) $\big(R^{\star}\big)^{-1} = \big(R^{-1}\big)^{\star}$;

(2) $R^{-1} \circ R = \Delta_X$ *and* $R \circ R^{-1} = \Delta_Y$;

(3) *R is an injective function of X onto Y.*

8. Partial compatibility of the operation $*$ with the relation theoretic inversion

From Theorem 7.1, by writing U^* in place of R, we can easily derive

Theorem 8.1. *If U is a union-preserving corelation on X to Y such that $\big(U^*\big)^{-1}$ is a function on Y onto X, then*

$$\big(U^{-1}\big)^{*} \subset \big(U^{*}\big)^{-1}.$$

Proof. Now, by Theorems 5.7 and 7.1, we have

$$U^{-1} = \big(U^{*\star}\big)^{-1} = \Big(\big(U^{*}\big)^{\star}\Big)^{-1} \subset \Big(\big(U^{*}\big)^{-1}\Big)^{\star}.$$

Hence, by using Corollary 5.3 and Theorem 5.4, we can infer that

$$\big(U^{-1}\big)^{*} \subset \Big(\Big(\big(U^{*}\big)^{-1}\Big)^{\star}\Big)^{*} = \Big(\big(U^{*}\big)^{-1}\Big)^{\star *} = \big(U^{*}\big)^{-1}.$$

From Theorem 7.2, we can quite similarly derive the following

Theorem 8.2. *If U is a union-preserving corelation on X to Y such that U^* is a function on X onto Y, then*

$$\big(U^{*}\big)^{-1} \subset \big(U^{-1}\big)^{*}.$$

Now, as an immediate consequence of the above two theorems, we can also state

Corollary 8.3. *If U is a union-preserving corelation on X to Y such that U^* is an injective function of X onto Y, then*

$$\big(U^{*}\big)^{-1} = \big(U^{-1}\big)^{*}.$$

Moreover, by using Corollary 7.3, we can also easily prove the following

Theorem 8.4. *If U is an injective, union-preserving corelation on X to Y such that U^{-1} is also union-preserving, then the following assertions are equivalent:*

(1) $\big(U^{*}\big)^{-1} = \big(U^{-1}\big)^{*}.$;

(2) *U^* is an injective function of X onto Y.*

Proof. Now, since the implication (2) $\Longrightarrow$ (1) has already been established in Corollary 8.3, we need only prove that (1) also implies (2).

For this note that if (1) holds, then by Theorem 5.7 we also have

$$\Big(\big(U^{*}\big)^{\star}\Big)^{-1} = \big(U^{*\star}\big)^{-1} = U^{-1} = \big(U^{-1}\big)^{*\star} = \Big(\big(U^{-1}\big)^{*}\Big)^{\star} = \Big(\big(U^{*}\big)^{-1}\Big)^{\star}.$$

Therefore, by Corollary 7.3, assertion (2) also holds.

From Corollary 7.3, we can also immediately derive the following

Theorem 8.5. *For a symmetric relation R on X, the following assertions are equivalent:*

(1) $R^2 = \Delta_X$ *;*

(2) $R^\star$ *is an involution;*

(3) R *is an injective function of* X *onto* Y*.*

Remark 8.6. Moreover, by Theorem 6.7, we can at once see that, for an arbitrary relation R on X, the corelation $R^\star$ is an involution if and only if $R \circ R = \Delta_X$. That is, for any $x, y \in X$, we have $R(x) \cap R^{-1}(y) \neq \emptyset$ if and only if $x = y$.

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