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INCLUSIONS FOR COMPOSITION AND BOX PRODUCTS OF RELATIONS

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Abstract. Motivated by some important classes and continuities of relations, we give some intrinsic characterizations of the relational inclusions

$$G^{-1} \circ S \circ F \subset R \qquad \text{and} \qquad R \subset G^{-1} \circ S \circ F\,.$$

Our main tool is the box product $F \boxtimes G$ which has the crucial property that

$$G^{-1} \circ S \circ F = \big(F^{-1} \boxtimes G^{-1} \big) [\, S\,] = (\, F \boxtimes G\,)^{-1} [\, S\,] = \operatorname{cl}_{F \boxtimes G} (\, S\,)\,.$$

For any fixed relations F, G, and R, by considering the extremal relation

$$S_{(F,\, G,\, R)} = \operatorname{int}_{(\, F \boxtimes G)^{-1}} (R) = \big\{ \, (\, y\, ,\, w\,) : \quad (\, F \boxtimes G\,)^{-1} (\, y\, ,\, w\,) \subset R \, \big\}\,,$$

we give some easily applicable necessary and sufficient conditions in order that the equality $R = G^{-1} \circ S \circ F$ could hold for some relation S.

The results obtained extend and unify several former results on transitive, idempotent, non-mingled-valued, regular, normal, and conjugative relations, for instance. Moreover, they can also be well used to briefly treat proper and uniform, mild, upper, and lower continuity properties of a pair (F, G) of relations on one relator space $(\, X\, ,\, Z\,) (\mathcal{R})$ to another $(\, Y\, ,\, W\,) (\mathcal{S})$.

Introduction

According to [27], a relation R on a set X may, for instance, be called

(1) reflexive if $\Delta_X \subset R$;
(2) symmetric if $R^{-1} \subset R$;
(3) involutive if $R \circ R = \Delta_X$;
(4) transitive if $R \circ R \subset R$;
(5) idempotent if $R \circ R = R$;
(6) onto if $\Delta_X \subset R \circ R^{-1}$;
(7) single-valued if $R \circ R^{-1} \subset \Delta_X$;
(8) non-partial if $\Delta_X \subset R^{-1} \circ R$;
(9) disjunctive if $R^{-1} \circ R \subset \Delta_X$;
(10) semi-directive if $X^2 \subset R^{-1} \circ R$;
(11) exclusive if $\big(R^{-1} \circ R^c \big)^c \subset \Delta_X$;
(12) constant-like if $X^2 \subset \big(R^{-1} \circ R^c \big)^c$;

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(13) non-mingled-valued if $R \circ R^{-1} \circ R \subset R$.

Now, a reflexive and transitive relation may be called a preorder relation, and a symmetric preorder relation may be called an equivalence relation. Moreover, a reflexive and symmetric (anti-symmetric) relation may be called a tolerance (anti-tolerance) relation.

In [27], for any relation R on X, the relations $R^{-} = R^{-1} \circ R$ and $R^{\circ} = \left(R^{-1} \circ R^{c} \right)^{c}$ were called the pointwise self closure and interior of R, respectively. And, it was shown that a non-partial relation R on X is

(1) a preorder if and only if $R = R^{\circ}$;

(2) an equivalence if and only if $R = R^{-}$;

(3) non-mingled-valued if and only if $R^{\circ} = R^{-}$.

Actually, assertion (1) does not require R to be non-partial. Therefore, an arbitrary relation R on X is a preorder if and only if $R = \left(R^{-1} \circ R^{c} \right)^{c}$, or equivalently $R^{c} = R^{-1} \circ R^{c}$. (A dual of this statement was formerly proved by Bandelt [1].)

On the other hand, according to an unfortunate terminology, a relation R on X may, for instance, be called

(1) regular [45] if $R = R \circ S \circ R$;

(2) conjugative [7] if $R = R^{-1} \circ S \circ R$;

(3) dually conjugative [7] if $R = R \circ S \circ R^{-1}$;

(4) normal [9] if $R = R \circ S \circ \left(R^{c} \right)^{-1}$;

(5) dually normal [8] if $R = \left(R^{c} \right)^{-1} \circ S \circ R$;

(6) quasi-regular [19] if $R = R^{c} \circ S \circ R$;

(7) dually quasi-regular [19] if $R = R \circ S \circ R^{c}$;

(8) quasi-conjugative [20] if $R = R^{-1} \circ S \circ R^{c}$;

(9) dually quasi-conjugative [20] if $R = R^{c} \circ S \circ R^{-1}$;

(10) quasi-normal [21] if $R = R^{c} \circ S \circ \left(R^{c} \right)^{-1}$;

(11) dually quasi-normal [21] if $R = \left(R^{c} \right)^{-1} \circ S \circ R^{c}$;

(12) bi-normal [22] if $R = \left(R^{c} \right)^{-1} \circ S \circ \left(R^{c} \right)^{-1}$;

(13) bi-quasiregular [22] if $R = R^{c} \circ S \circ R^{c}$;

for some relation S on X.

Diverse characterizations of regularity were given in [46, 44, 14, 11, 23, 1, 2, 5, 42, 43]. For instance, it has been proved that, for a relation R on X, the following assertions are equivalent:

(1) R is a regular relation on X;

(2) $R \subset R \circ \widetilde{R} \circ R$ holds true for $\widetilde{R} = \left(R^{-1} \circ R^{c} \circ R^{-1} \right)^{c}$;

(3) $\left\{ R[A] \right\}_{A \subset X}$, with $\subset$, is a completely distributive lattice;

(4) $\forall\ (x, y) \in R: \quad \exists\ u, v \in X: \qquad (x, y) \in R^{-1}(v) \times R(u) \subset R$;

(5) $\forall\ (x, y) \in R: \quad \exists\ u, v \in X:$

(a) $(x, v) \in R, \quad (u, y) \in R,$

(b) $(s, v) \in R, \quad (u, t) \in R \quad \Longrightarrow \quad (s, t) \in R.$

Here, according to [42, 43], R may be allowed to be a relation on X to Y, and assertion (5) may be called the intrinsic characterization of regularity.

In this respect, it is also noteworthy that, by [28, Definition 4.1], a pair (F, G) of relations on one relator space $(X, Z)(\mathcal{R})$ to another $(Y, W)(\mathcal{S})$ may, for instance, be called properly (uniformly)

(1) upper continuous if $G \circ R = S \circ F \quad (G \circ R \subset S \circ F)$;

(2) mildly continuous if $R = G^{-1} \circ S \circ F \quad (R \subset G^{-1} \circ S \circ F)$;

(3) lower continuous if $R \circ F^{-1} = G^{-1} \circ S \quad (R \circ F^{-1} \subset G^{-1} \circ S)$;

for all $S \in \mathcal{S}$ and some $R \in \mathcal{R}$ depending on S.

Thus, by using the operation $\mathcal{R}^{-1} = \{ R^{-1} : \quad R \in \mathcal{R} \}$, we can, for instance, see that (F, G) is a properly (uniformly) lower continuous pair of relations on $(X, Z)(\mathcal{R})$ to $(Y, W)(\mathcal{S})$ if and only if (G, F) is a properly (uniformly) upper continuous pair of relations on $(Z, X)\big(\mathcal{R}^{-1}\big)$ to $(W, Y)\big(\mathcal{S}^{-1}\big)$.

Moreover, by using the operation $\mathcal{R}^{*} = \{ U \subset X \times Z : \quad \exists \ R \in \mathcal{R} : \quad R \subset U \}$, we can, for instance, also see that (F, G) is a uniformly upper (lower) continuous pair of relations on $(X, Z)(\mathcal{R})$ to $(Y, W)(\mathcal{S})$ if and only if (F, G) is a properly upper (lower) continuous pair of relations on $(X, Z)(\mathcal{R}^{*})$ to $(Y, W)(\mathcal{S}^{*})$.

The above definitions, and the fact that $V \circ U = V \circ \Delta_{Y} \circ U$ for any relations U on X to Y and V on Y to Z, motivate the investigations of the relational inclusions

$$G^{-1} \circ S \circ F \subset R \qquad \text{and} \qquad R \subset G^{-1} \circ S \circ F,$$

and for any given relations F, G, and R the existence of a relation S such that $R = G^{-1} \circ S \circ F$. (Such existence problems were also studied by Zareckiĭ [46].)

For this, for any relations F on X to Y and G on Z to W, we define a relation $F \boxtimes G$ such that

$$(F \boxtimes G)(x, z) = F(x) \times G(z)$$

for all $x \in X$ and $z \in Z$, and we show that

$$G^{-1} \circ S \circ F = \big(F^{-1} \boxtimes G^{-1}\big)[S] = (F \boxtimes G)^{-1}[S] = \mathrm{cl}_{F \boxtimes G}(S).$$

for any relation S on Y to W.

By using this box product, for instance, we can easily show that, for any relations F on X to Y, G on Z to W, R on X to Z, and S on Y to W, the following assertions are equivalent:

(1) $G^{-1} \circ S \circ F \subset R$;

(2) $\big(F \boxtimes G\big)^{-1}[S] \subset R$;

(3) $G(z) \cap S[F(x)] \neq \emptyset \quad \Longrightarrow \quad z \in R(x)$;

(4) $(x, y) \in F, \ (z, w) \in G, \ (y, w) \in S \quad \Longrightarrow \quad (x, z) \in R.$

Moreover, for any relations F on X to Y, G on Z to W, and R on X to Z, we define

$$S_{(F, G, R)} = \mathrm{int}_{(F \boxtimes G)^{-1}}(R) = \big\{ (y, w) \in Y \times W : \quad (F \boxtimes G)^{-1}(y, w) \subset R \big\},$$

and by using a simple basic fact on interiors induced by relations we show that $S = S_{(F,\,G,\,R)}$ is the largest relation on Y to W such that $\big(F \boxtimes G\big)^{-1}[\,S\,] \subset R$, or equivalently $G^{-1} \circ S \circ F \subset R$.

Concerning this extremal relation, we also show that

$$S_{(F,\,G,\,R)} = \big(F \boxtimes G\big)\,[\,R^c\,]^c = \big(G \circ R^c \circ F^{-1}\big)^c .$$

Thus, in particular $\widetilde{R} = \big(R^{-1} \circ R^c \circ R^{-1}\big)^c = S_{(R,\,R^{-1},\,R)}$ whenever $X = Z$.

However, the importance of the relation $S_{(F,\,G,\,R)}$ lies mow mainly in the fact that, for any relations F on X to Y, G on Z to W, and R on X to Z, the following assertions are equivalent:

(1) $R \subset \big(F \boxtimes G\big)^{-1}[\,S_{(F,\,G,\,R)}\,]$, or equivalently $R \subset G^{-1} \circ S_{(F,\,G,\,R)} \circ F$;

(2) $\forall\ (x,\,z) \in R: \quad \exists\ (y,\,w) \in Y \times W: \quad (x,\,z) \in F^{-1}(y) \times G^{-1}(w) \subset R$;

(3) $\forall\ (x,\,z) \in R: \quad \exists\ (y,\,w) \in Y \times W:$

(a) $(x,\,y) \in F, \quad (z,\,w) \in G$;

(b) $(s,\,y) \in F, \quad (t,\,w) \in G \quad \Longrightarrow \quad (s,\,t) \in R$;

(4) $R = \big(F \boxtimes G\big)^{-1}[\,S\,]$, or equivalently $R = G^{-1} \circ S \circ F$, for some relation S on Y to W.

Hence, the reader can already notice that our present results are natural generalizations of the corresponding statements of Schein [23], Bandelt [2], Hardy and Petrich [5], Xu, Liu and Lou [42, 43], Jiang, Xu, Cai, and Han [7, 8, 9], and Romano [19, 20, 21, 22].

1. A few basic facts on relations

A subset F of a product set $X \times Y$ is called a relation on X to Y. If in particular $F \subset X^2$, with $X^2 = X \times X$, then we may simply say that F is a relation on X. In particular, $\Delta_X = \{\,(x,\,x): \ x \in X\,\}$ is called the identity relation on X.

If F is a relation on X to Y, then for any $x \in X$ and $A \subset X$ the sets $F(x) = \{\,y \in Y: \ (x,\,y) \in F\,\}$ and $F[\,A\,] = \bigcup_{a \in A} F(a)$ are called the images of x and A under F, respectively.

Moreover, the sets $D_F = \{\,x \in X: \ F(x) \neq \emptyset\,\}$ and $R_F = F[\,D_F\,]$ are called the domain and range of F, respectively. If in particular $D_F = X$, then we say that F is a relation of X to Y, or that F is a non-partial relation on X to Y.

In particular, a relation f on X to Y is called a function if for each $x \in D_f$ there exists $y \in Y$ such that $f(x) = \{\,y\,\}$. In this case, by identifying singletons with their elements, we may simply write $f(x) = y$ in place of $f(x) = \{\,y\,\}$.

In particular, a function $\star$ of a set X to itself is called an unary operation on X. While, a function $*$ of X^2 to X is called a binary operation in X. And, for any $x,\,y \in X$, we write $x^\star$ and $x * y$ instead of $\star(x)$ and $*(x,\,y)$, respectively.

Concerning relations, we can easily establish the following

Theorem 1.1. *For any relation F on X to Y, we have*

$$F = \bigcup_{x \in X} \{\,x\,\} \times F(x) .$$

From this theorem, we can immediately derive

Corollary 1.2. *For any two relations F and G on X to Y, we have $F \subset G$ if and only if $F(x) \subset G(x)$ for all $x \in X$.*

Remark 1.3. Note that $F(x) = \emptyset$ if $x \in D_F^c$. Therefore, in the assertions of Theorem 1.1 and Corollary 1.2 we may write D_F in place of X.

Moreover, we can also note that $F = G$ if and only if $F(x) = G(x)$ for all $x \in X$, or equivalently $D_F = D_G$ and $F(x) = G(x)$ for all $x \in D_F$.

From Theorem 1.1, we can also at once see that a relation F on X to Y can be naturally defined by specifying $F(x)$ for all $x \in X$, or by specifying D_F and $F(x)$ for all $x \in D_F$. However, the latter possibility is of no importance for us.

Namely, for instance, we may naturally have the following two definitions.

Definition 1.4. For any relation F on X to Y, we define a relation F^{-1} on Y to X such that

$$F^{-1}(y) = \{ x \in X : \quad y \in F(x) \}$$

for all $y \in Y$. The relation F^{-1} is called the inverse of F.

Remark 1.5. Thus, for any $x \in X$ and $y \in Y$, we have $(y, x) \in F^{-1}$ if and only if $(x, y) \in F$.

Definition 1.6. For any relations F on X to Y and G on Y to Z, we define a relation $G \circ F$ on X to Z such that

$$(G \circ F)(x) = G[F(x)]$$

for all $x \in X$. The relation $G \circ F$ is called the composition of G and F.

Remark 1.7. Thus, for any $x \in X$ and $z \in Z$, we have $(x, z) \in G \circ F$ if and only if $(x, y) \in F$ and $(y, z) \in G$ for some $y \in Y$.

Moreover, we can also easily prove the following two theorems.

Theorem 1.8. *For any relations F on X to Y and G on Y to Z, we have*

$$(G \circ F)^{-1} = F^{-1} \circ G^{-1}.$$

Theorem 1.9. *If F is a relation on X to Y and G is a relation on Y to Z, then for any $A \subset X$ we have*

$$(G \circ F)[A] = G\big[F[A]\big].$$

2. Some further useful theorems on relations

Concerning relations, we can also easily prove the following theorems.

Theorem 2.1. *If F is a relation on X to Y, then for any family $\mathcal{A}$ of subsets of X we have*

(1) $F\Big[\bigcup \mathcal{A}\Big] = \bigcup_{A \in \mathcal{A}} F[A]$; *(2)* $F\Big[\bigcap \mathcal{A}\Big] \subset \bigcap_{A \in \mathcal{A}} F[A]$.

Theorem 2.2. *If F is a relation on X to Y, then for any $A, B \subset X$ we have*

(1) $F[A] \setminus F[B] \subset F[A \setminus B]$; *(2)* $F[A]^c \subset F[A^c]$ *if* $Y = R_F$.

Remark 2.3. If in particular F^{-1} is a function, then the corresponding equalities are also true in the above two theorems.

Theorem 2.4. *If $\mathcal{F}$ is a family of relations on X to Y, then for any $A \subset X$ we have*

(1) $\left(\bigcup \mathcal{F}\right)[A] = \bigcup_{F \in \mathcal{F}} F[A]$; *(2)* $\left(\bigcap \mathcal{F}\right)[A] \subset \bigcap_{F \in \mathcal{F}} F[A]$.

Theorem 2.5. *If F and G are relations on X to Y, then for any $A \subset X$ we have*

(1) $F[A] \setminus G[A] \subset (F \setminus G)[A]$; *(2)* $F[A]^c \subset F^c[A]$ *if* $A \neq \emptyset$.

Remark 2.6. If in particular A is a singleton, then the corresponding equalities are also true in the above two theorems.

Concerning the complement relation F^c we can also easily prove the following two theorems.

Theorem 2.7. *For any relation F on X to Y, we have*

$$\left(F^c\right)^{-1} = \left(F^{-1}\right)^c.$$

Theorem 2.8. *If F is a relation on X to Y, the for any $A \subset X$ we have*

$$F^c[A]^c = \bigcap_{x \in A} F(x).$$

Proof. Namely, by Remark 2.6 and DeMorgan's law, we have

$$F^c[A]^c = \left(\bigcup_{x \in A} F^c(x)\right)^c = \left(\bigcup_{x \in A} F(x)^c\right)^c = \bigcap_{x \in A} \left(F(x)^c\right)^c = \bigcap_{x \in A} F(x).$$

From the latter theorem, we can easily derive

Corollary 2.9. *If F is a relation on X to Y and G is a relation on Y to Z, then for any $x \in X$ we have*

$$\left(G \circ F\right)^c(x) = \bigcap_{y \in F(x)} G^c(y).$$

Proof. By using Remark 2.6 and Theorem 2.8, we can at once see that

$$\left(G \circ F\right)^c(x) = \left(G \circ F\right)(x)^c = G[F(x)]^c = \left(G^c\right)^c[F(x)]^c = \bigcap_{y \in F(x)} G^c(y).$$

Remark 2.10. In addition to this corollary, it is also worth proving that

(1) $(G \circ F)^c \subset G^c \circ F$ if $X = D_F$; (2) $(G \circ F)^c \subset G \circ F^c$ if $Z = R_G$.

3. Interiors and closures induced by relations

The following definition is a very particular case of a basic definition from the theory of relator spaces. (See, for instance, [30] .)

Definition 3.1. If R is a relation on X to Y, then for any $B \subset Y$ we define

$$B^{\circ} = \operatorname{int}_R(B) = \big\{ x \in X : \quad R(x) \subset B \big\}$$

and

$$B^{-} = \operatorname{cl}_R(B) = \big\{ x \in X : \quad R(x) \cap B \neq \emptyset \big\} .$$

The sets B° and B^{-} are called the R–interior and R–closure of B, respectively.

Concerning the relations cl_R and int_R, we can easily prove the following two theorems.

Theorem 3.2. *If R is a relation on X to Y, then for any $B \subset Y$ we have*

(1) $\operatorname{cl}_R(B) = R^{-1}[B]$; *(2)* $\operatorname{int}_R(B) = R^{-1}[B^c]^c$.

Proof. To check (2), note that for any $x \in X$ we have

$$x \in \operatorname{int}_R(B) \iff R(x) \subset B \iff R(x) \cap B^c = \emptyset \iff x \notin \operatorname{cl}_R(B^c) \iff x \in \operatorname{cl}_R(B^c)^c.$$

Therefore, $\operatorname{int}_R(B) = \operatorname{cl}_R(B^c)^c = R^{-1}[B^c]^c$.

Theorem 3.3. *If R is a relation on X to Y and $B \subset Y$, then $A = \operatorname{int}_R(B)$ is the largest subset of X such that $R[A] \subset B$.*

Proof. If $U \subset X$ such that $R[U] \subset B$, then

$$R(x) \subset R[U] \subset B$$

for all $x \in U$. Therefore, $U \subset \operatorname{int}_R(B) = A$.

Moreover, since $x \in A$ implies $R(x) \subset B$, it is clear that

$$R[A] = \bigcup_{x \in A} R(x) \subset B .$$

Therefore, the required assertion is also true.

From the latter theorem, we can immediately get

Corollary 3.4. *For any relation R on X to Y and sets $A \subset X$ and $B \subset Y$, the following assertions are equivalent:*

(1) $R[A] \subset B$; *(2)* $A \subset \operatorname{int}_R(B)]$.

Moreover, by using Theorem 3.3, we can also easily prove the following

Theorem 3.5. *For any relation R on X to Y and subset B of Y, the following assertions are equivalent:*

(1) $B = R[A]$ *for some* $A \subset X$.

(2) $B = R[\operatorname{int}_R(B)]$; *(3)* $B \subset R[\operatorname{int}_R(B)]$.

Proof. Assertion (2) trivially implies both (1) and (3). Moreover, by Theorem 3.3, we always have

$$R\,[\,\mathrm{int}_R(B)\,] \subset B\,.$$

Therefore, (3) also implies (2).

On the other hand, if (1) holds, then again by Theorem 3.3 we necessarily have $A \subset \mathrm{int}_R(B)$. Hence, by using (1), we can already infer that

$$B = R\,[\,A\,] \subset R\,[\,\mathrm{int}_R(B)\,] \subset B\,.$$

Therefore, $B = R\,[\,\mathrm{int}_R(B)\,]$, and thus (2) also holds.

Finally, we note that the following partial extension of [4, Theorem 1] of G. Birkhoff is also true. (For some more general ideas, see also [25, 32] .)

Theorem 3.6. *For any relation* cl *on* $\mathcal{P}(Y)$ *to* X, *the following assertions are equivalent:*

(1) $\mathrm{cl}\,(B) = \bigcup_{y \in B} \mathrm{cl}\big(\{y\}\big)$ *for any* $B \subset Y$;

(2) $\mathrm{cl} = \mathrm{cl}_R$ *for some relation* R *on* X *to* Y;

(3) $\mathrm{cl}\big(\bigcup \mathcal{B}\big) = \bigcup_{B \in \mathcal{B}} \mathrm{cl}(B)$ *for any* $\mathcal{B} \subset \mathcal{P}(Y)$.

Proof. To prove that (1) implies (2), assume (1) and define a relation R on X to Y such that

$$R(x) = \big\{\, y \in Y : \quad x \in \mathrm{cl}\big(\{y\}\big)\big\}$$

for all $x \in X$. Then, for any $x \in X$ and $y \in Y$, we have

$$x \in R^{-1}(y) \iff y \in R(x) \iff x \in \mathrm{cl}\big(\{y\}\big)$$

Therefore, $R^{-1}(y) = \mathrm{cl}\big(\{y\}\big)$ for all $y \in Y$. Hence, by using (1) and Theorem 3.2, we can see that

$$\mathrm{cl}\,(B) = \bigcup_{y \in B} \mathrm{cl}\big(\{y\}\big) = \bigcup_{y \in B} R^{-1}(y) = R^{-1}\,[\,B\,] = \mathrm{cl}_R(B)$$

for all $B \subset Y$. Therefore, $\mathrm{cl} = \mathrm{cl}_R$, and thus (2) also holds.

Remark 3.7. Properly simple relator spaces $(X, Y)(R)$, under the name context spaces, were also intensively studied in Ganter and Wille [6] .

In general, for some operation $\square$ on relators, a relator $\mathcal{R}$ on X to Y is called $\square$–simple if $\mathcal{R}^{\square} = \big\{R\big\}^{\square}$ holds for some relation R on X to Y.

$\square$–simple relators were first investigated by the second author in an unfished doctoral thesis "Relators, Nets and Integrals" in 1991. However, the most remarkable results on them have been achieved by G. Pataki whose paper [13] was formerly, roughly refused by Á. Császár at the Acta Math. Hungarica.

It was a serious problem, posed at some conference, to find a non-paratopologically simple preorder relator in order that the study of generalized nets could be justified. Surprisingly enough, Pataki [13, Example 5.11] showed that even a three element equivalence relator on a four element set need not be paratopologically simple.

4. The box product of relations

Definition 4.1. For any relations F on X to Y and G on Z to W, we define a relation $F \boxtimes G$ on $X \times Z$ to $Y \times W$ such that

$$(F \boxtimes G)(x, z) = F(x) \times G(z)$$

for all $x \in X$ and $z \in Z$. The relation $F \boxtimes G$ will be called the box product of the relations F and G.

Remark 4.2. Note that thus the box product $F \boxtimes G$ is actually a relation of $D_F \times D_G$ onto $R_F \times R_G$.

Namely, if $(y, w) \in R_F \times R_G$, then $y \in R_F$ and $w \in R_G$. Thus, there exist $x \in X$ and $z \in Z$ such that $y \in F(x)$ and $w \in G(z)$. Therefore, we also have $(y, w) \in F(x) \times G(z) = (F \boxtimes G)(x, z)$, and hence $(y, w) \in R_{F \boxtimes G}$.

Remark 4.3. While, the Cartesian product

$$F \times G = \big\{ \big((x, y), (z, w) \big) : \quad (x, y) \in F , \quad (z, w) \in G \big\}$$

is a relation on $X \times Y$ to $Z \times W$ such that

$$\big(F \times G)(x, y) = G \quad \text{if} \quad (x, y) \in F \quad \text{and} \quad \big(F \times G)(x, y) = \emptyset \quad \text{if} \quad (x, y) \in F^c.$$

Therefore, $F \times G$ is actually a relation of F onto G.

Remark 4.4. Hence, it is clear that the box product $F \boxtimes G$ greatly differs from the box relation $F \times G$.

The more general box relations $\Gamma_{(A, B)} = A \times B$, with $A \subset X$ and $B \subset Y$ have been intensively investigated in [33, 35, 34].

Concerning the box products of relations, we can also easily prove the following

Theorem 4.5. *For any relations F on X to Y and G on Z to W, we have*

$$(F \boxtimes G)^{-1} = F^{-1} \boxtimes G^{-1}.$$

Proof. For any $(x, z) \in X \times Z$ and $(y, w) \in Y \times W$, we have

$$\begin{gathered}(x, z) \in (F \boxtimes G)^{-1}(y, w) \iff (y, w) \in (F \boxtimes G)(x, z) \\ \iff (y, w) \in F(x) \times G(z) \iff y \in F(x), \quad w \in G(z) \iff x \in F^{-1}(y), \quad z \in G^{-1}(w) \\ \iff (x, z) \in F^{-1}(y) \times G^{-1}(w) \iff (x, z) \in \big(F^{-1} \boxtimes G^{-1} \big)(y, w).\end{gathered}$$

Therefore, $(F \boxtimes G)^{-1}(y, w) = \big(F^{-1} \boxtimes G^{-1} \big)(y, w)$ for all $(y, w) \in Y \times W$, and thus the required equality is also true.

Remark 4.6. In this respect, it is also worth noticing that

$$(F \times G)^{-1} = G \times F.$$

However, the importance of the box product lies mainly in the following

Theorem 4.7. *If F is a relation on X to Y and G is a relation on Z to W, then for any $R \subset X \times Z$ we have*

$$(F \boxtimes G) [R] = G \circ R \circ F^{-1}.$$

Proof. If $(y, w) \in (F \boxtimes G)[R]$, then there exists $(x, z) \in R$ such that

$$(y, w) \in (F \boxtimes G)(x, z) = F(x) \times G(z),$$

and thus $y \in F(x)$ and $w \in G(z)$. Hence, by noticing that $x \in F^{-1}(y)$, we can already see that

$$z \in R(x) \subset R\left[F^{-1}(y)\right] = \left(R \circ F^{-1}\right)(y),$$

and thus

$$w \in G(z) \subset G\left[\left(R \circ F^{-1}\right)(y)\right] = \left(G \circ \left(R \circ F^{-1}\right)\right)(y).$$

Therefore, $(y, w) \in G \circ \left(R \circ F^{-1}\right) = G \circ R \circ F^{-1}$ also holds.

Thus, we have proved that $(F \boxtimes G)[R] \subset G \circ R \circ F^{-1}$. The converse inclusion can be proved quite similarly.

Remark 4.8. In contrast to the above theorem, for any $R \subset X \times Y$, we have

$$\left(F \times G\right)[R] = \emptyset \quad \text{if} \quad R \subset F^c \qquad \text{and} \qquad \left(F \times G\right)[R] = G \quad \text{if} \quad R \not\subset F^c.$$

From Theorem 4.7, by taking $R = \{(x, z)\}$, we can immediately get

Corollary 4.9. *If F is a relation on X to Y and G is a relation on Z to W, then for any $x \in X$ and $z \in Z$, we have*

$$(F \boxtimes G)(x, z) = G \circ \{(x, z)\} \circ F^{-1}.$$

Moreover, by using Theorem 4.7, we can also easily establish the following

Corollary 4.10. *For any relations F on X to Y and G on Y to Z, we have*

$$G \circ F = \left(F^{-1} \boxtimes G\right)[\Delta_Y].$$

Proof. By the corresponding definitions and Theorem 4.7, it is clear that

$$G \circ F = G \circ \Delta_Y \circ \left(F^{-1}\right)^{-1} = \left(F^{-1} \boxtimes G\right)[\Delta_Y].$$

Remark 4.11. The above corollaries show that the box and composition products of relations are actually equivalent tools.

However, in contrast to the composition product, the box product of relations can be immediately defined for arbitrary families of relations.

Now, by using Theorems 4.7, 4.5, and 3.2, we can also easily prove the following

Theorem 4.12. *If F is a relation on X to Y and G is a relation on Z to W, then for any $S \subset Y \times W$ we have*

$$G^{-1} \circ S \circ F = \left(F^{-1} \boxtimes G^{-1}\right)[S] = (F \boxtimes G)^{-1}[S] = \mathrm{cl}_{F \boxtimes G}(S).$$

Proof. By Theorems 4.7, 4.5 and 3.2, it is clear that

$$G^{-1} \circ S \circ F = G^{-1} \circ S \circ \left(F^{-1}\right)^{-1}$$
$$= \left(F^{-1} \boxtimes G^{-1}\right)[S] = (F \boxtimes G)^{-1}[S] = \mathrm{cl}_{F \boxtimes G}(S).$$

From this theorem, by letting $Y = W$ and $S = \Delta_Y$, we can immediately get

Corollary 4.13. *For any relations F on X to Y and G on Z to Y, we have*
$$G^{-1} \circ F = \big(F^{-1} \boxtimes G^{-1} \big)[\, \Delta_Y \,] = (\, F \boxtimes G \,)^{-1} [\, \Delta_Y \,] = \mathrm{cl}_{F \boxtimes G} (\Delta_Y) \,.$$

Remark 4.14. Note that, for any $x \in X$, we have
$$\big(G^{-1} \circ F \,\big)(x) = G^{-1} \big[\, F(x) \big] = \mathrm{cl}_G \big(F(x) \big) \,.$$
Thus, $G^{-1} \circ F$ is just the pointwise G–closure F^{cl_R} of F considered in [38, Definition 8.9].

From Corollary 4.13, by letting $X = Y = Z$ and $F = G = R$ for some relation R on X, we can get

Corollary 4.15. *For any relation R on X, we have*
$$R^{-1} \circ R = \big(R^{-1} \boxtimes R^{-1} \big)[\, \Delta_X \,] = (\, R \boxtimes R \,)^{-1} [\, \Delta_X \,] = \mathrm{cl}_{R \boxtimes R} (\Delta_X) \,.$$

Remark 4.16. By Remark 4.14, for any $x \in X$, we have
$$\big(R^{-1} \circ R \big)(x) = \mathrm{cl}_R \big(R(x) \big) \,.$$
Thus, $R^{-1} \circ R$ is just the pointwise self-closure R^- of R investigated in [27].

Remark 4.17. Hence, by Theorem 1.8, we can at once see that
$$\big(R^- \big)^{-1} = \big(R^{-1} \circ R \big)^{-1} = R^{-1} \circ \big(R^{-1} \big)^{-1} = R^{-1} \circ R = R^- \,.$$
Therefore, R^- is always a symmetric relation on X. While, R° is, in general, only a reflexive relation on X.

In [27], it was also shown that R^- is a reflexive (non-partial) relation on X if and only if $R^\circ \subset R^-$, or equivalently R is non-partial relation on X. While, $R^- \subset R^\circ$ if and only if R is non-mingled-valued.

5. Inclusions for composition and box products

Theorem 5.1. *For any relations F on X to Y, G on Z to W, R on X to Z, and S on Y to W, the following assertions are equivalent:*

(1) $G^{-1} \circ S \circ F \subset R\,$;

(2) $\big(F \boxtimes G \big)^{-1} [\, S \,] \subset R\,$;

(3) $G(z) \cap S\, [\, F(x) \,] \neq \emptyset \quad \Longrightarrow \quad z \in R(x)\,$;

(4) $(\, x ,\, y \,) \in F \,, \;\; (\, z ,\, w \,) \in G \,, \;\; (\, y ,\, w \,) \in S \quad \Longrightarrow \quad (\, x ,\, z \,) \in R \,.$

Proof. By Theorem 4.12, it is clear that (1) and (2) are equivalent.

Moreover, if (2) holds, then for any $(\, y ,\, w \,) \in S$ we have
$$F^{-1}(y) \times G^{-1}(w) = \big(F^{-1} \boxtimes G^{-1} \big)(\, y ,\, w \,) = \big(F \boxtimes G \big)^{-1} [\, S \,] \subset R \,.$$
Therefore, for any $x \in F^{-1}(y)$ and $z \in G^{-1}(w)$ we have $(\, x ,\, z \,) \in R$. Hence, since $(\, x ,\, y \,) \in F$ and $(\, z ,\, w \,) \in G$ imply $x \in F^{-1}(y)$ and $z \in G^{-1}(w)$, it is clear that (4) also holds.

The proofs of the implications $(4) \Longrightarrow (3) \Longrightarrow (1)$ are left to the reader.

From this theorem, by letting $Y = W$ and $S = \Delta_Y$, we can immediately get

Corollary 5.2. *For any relations F on X to Y, G on Z to Y, and R on X to Z, the following assertions are equivalent:*

(1) $G^{-1} \circ F \subset R$;

(2) $\big(F \boxtimes G \big)^{-1} [\, \Delta_Y \,] \subset R$;

(3) $F(x) \cap G(z) \neq \emptyset \quad \Longrightarrow \quad z \in R(x)$;

(4) $(x,\, y) \in F \,, \ (z,\, y) \in G \quad \Longrightarrow \quad (x,\, z) \in R$.

Hence, by letting $X = Z$, and $R = \Delta_X$, we can immediately get

Corollary 5.3. *For any two relations F and G on X to Y, the following assertions are equivalent:*

(1) $G^{-1} \circ F \subset \Delta_X$;

(2) $\big(F \boxtimes G \big)^{-1} [\, \Delta_Y \,] \subset \Delta_X$;

(3) $F(x) \cap G(z) \neq \emptyset \quad \Longrightarrow \quad x = z$;

(5) $(x,\, y) \in F \,, \ (z,\, y) \in G \quad \Longrightarrow \quad x = z$.

Moreover, from Corollary 5.2 we can also immediately get the following

Theorem 5.4. *For any two relations R and S on X, the following assertions are equivalent:*

(1) $R^{-1} \circ R \subset S$;

(2) $\big(R \boxtimes R \big)^{-1} [\, \Delta_X \,] \subset S$;

(3) $R(x) \cap R(z) \neq \emptyset \quad \Longrightarrow \quad z \in S(x)$;

(4) $(x,\, y) \in R \,, \ (z,\, y) \in R \quad \Longrightarrow \quad (x,\, z) \in S$.

Hence, it is clear that in particular we also have the following two corollaries.

Corollary 5.5. *For any relation R on X, the following assertions are equivalent:*

(1) $R^{-1} \circ R \subset R$;

(2) $\big(R \boxtimes R \big)^{-1} [\, \Delta_X \,] \subset R$;

(3) $R(x) \cap R(z) \neq \emptyset \quad \Longrightarrow \quad z \in R(x)$;

(4) $(x,\, y) \in R \,, \ (z,\, y) \in R \quad \Longrightarrow \quad (x,\, z) \in R$.

Corollary 5.6. *For any relation R on X, the following assertions are equivalent:*

(1) $R^{-1} \circ R \subset \Delta_X$;

(2) $\big(R \boxtimes R \big)^{-1} [\, \Delta_X \,] \subset \Delta_X$;

(3) $R(x) \cap R(z) \neq \emptyset \quad \Longrightarrow \quad x = z$;

(4) $(x,\, y) \in R \,, \ (z,\, y) \in R \quad \Longrightarrow \quad x = z$.

Remark 5.7. Note that (5) means only that R^{-1} is a function. Thus, R preserves all set theoretic operations.

Therefore, the above corollary gives some useful characterizations of a quite important class of relations.

Furthermore, from Corollary 5.2 we can also immediately get the following

Theorem 5.8. *For any two relations* R *and* S *on* X*, the following assertions are equivalent:*

(1) $R \circ R \subset S$;

(2) $\big(R \boxtimes R^{-1}\big)^{-1}[\Delta_X] \subset S$;

(3) $R(x) \cap R^{-1}(z) \neq \emptyset \implies z \in S(x)$;

(4) $(x, y) \in R,\ (y, z) \in R \implies (x, z) \in S$.

Hence, it is clear that in particular we also have the following two corollaries.

Corollary 5.9. *For any relation* R *on* X*, the following assertions are equivalent:*

(1) $R \circ R \subset R$;

(2) $\big(R \boxtimes R^{-1}\big)^{-1}[\Delta_X] \subset R$;

(3) $R(x) \cap R^{-1}(z) \neq \emptyset \implies z \in R(x)$.

(4) $(x, y) \in R,\ (y, z) \in R \implies (x, z) \in R$.

Remark 5.10. Note that (5) means only that R is transitive. Therefore, the above corollary gives some useful characterizations of transitive relations.

Corollary 5.11. *For any relation* R *on* X*, the following assertions are equivalent:*

(1) $R \circ R \subset \Delta_X$;

(2) $\big(R \boxtimes R^{-1}\big)^{-1}[\Delta_X] \subset \Delta_X$;

(3) $R(x) \cap R^{-1}(z) \neq \emptyset \implies x = z$.

(4) $(x, y) \in R,\ (y, z) \in R \implies x = z$.

6. The converse inclusions for composition and box products

Theorem 6.1. *For any relations* F *on* X *to* Y*,* G *on* Z *to* W*,* R *on* X *to* Z*, and* S *on* Y *to* W*, the following assertions are equivalent:*

(1) $R \subset G^{-1} \circ S \circ F$;

(2) $R \subset \big(F \boxtimes G\big)^{-1}[S]$;

(3) $z \in R(x) \implies G(z) \cap S[F(x)] \neq \emptyset$

(4) $\forall\ (x, z) \in R :\ \exists\ (y, w) \in S :\quad (x, y) \in F,\quad (z, w) \in G$.

Proof. By Theorem 4.12, it is clear that (1) and (2) are equivalent.

Moreover, if (2) holds, then for any $(x, z) \in R$ there exists $(y, w) \in S$ such that

$$(x, z) \in \big(F \boxtimes G\big)^{-1}(y, w) = \big(F^{-1} \boxtimes G^{-1}\big)(y, w) = F^{-1}(y) \times G^{-1}(w) .$$

Hence, we can see that $x \in F^{-1}(y)$ and $z \in G^{-1}(w)$, and thus $y \in F(x)$ and $w \in G(z)$. Therefore, $(x, y) \in F$ and $(z, w) \in G$, and thus (4) also holds.

The proofs of the implications $(4) \implies (3) \implies (1)$ are left to the reader.

From this theorem, by letting $Y = W$ and $S = \Delta_Y$, we can immediately get

Corollary 6.2. *For any relations F on X to Y, G on Z to Y, and R on X to Z, the following assertions are equivalent:*

(1) $R \subset G^{-1} \circ F$;

(2) $R \subset \big(F \boxtimes G \big)^{-1} [\, \Delta_Y \,]$;

(3) $z \in R(x) \quad \Longrightarrow \quad F(x) \cap G(z) \neq \emptyset$;

(4) $\forall \ (x, z) \in R: \ \exists \ y \in Y: \quad (x, y) \in F, \quad (z, y) \in G$.

Hence, by letting $X = Z$, and $R = \Delta_X$, we can immediately get

Corollary 6.3. *For any two relations F and G on X to Y, the following assertions are equivalent:*

(1) $\Delta_X \subset G^{-1} \circ F$;

(2) $\Delta_X \subset \big(F \boxtimes G \big)^{-1} [\, \Delta_Y \,]$;

(3) $\forall \ x \in X: \quad F(x) \cap G(x) \neq \emptyset$;

(4) $\forall \ x \in X: \ \exists \ y \in Y: \quad (x, y) \in F, \quad (x, y) \in G$.

Remark 6.4. Note that (1) means only that $G^{-1} \circ F$ is reflexive. Therefore, this corollary gives some useful characterizations of reflexivity of the relation $G^{-1} \circ F$.

Moreover, from Corollary 6.2 we can also immediately get the following

Theorem 6.5. *For any two relations R and S on X, the following assertions are equivalent:*

(1) $S \subset R^{-1} \circ R$;

(2) $S \subset \big(R \boxtimes R \big)^{-1} [\, \Delta_X \,]$;

(3) $z \in S(x) \quad \Longrightarrow \quad R(x) \cap R(z) \neq \emptyset$;

(4) $\forall \ (x, z) \in S: \quad \exists \ y \in Y: \quad (x, y) \in R, \quad (z, y) \in R$.

Hence, it is clear that in particular we also have the following two corollaries.

Corollary 6.6. *For any relation R on X, the following assertions are equivalent:*

(1) $R \subset R^{-1} \circ R$;

(2) $R \subset \big(R \boxtimes R \big)^{-1} [\, \Delta_X \,]$;

(3) $z \in R(x) \quad \Longrightarrow \quad R(x) \cap R(z) \neq \emptyset$;

(4) $\forall \ (x, z) \in R: \quad \exists \ y \in Y: \quad (x, y) \in R, \quad (z, y) \in R$.

Remark 6.7. Note that (4) trivially holds if R is reflexive. Therefore, $R \subset R^{-1} \circ R = R^{-}$ if R is reflexive.

Corollary 6.8. *For any relation R on X, the following assertions are equivalent:*

(1) $\Delta_X \subset R^{-1} \circ R$;

(2) $\Delta_X \subset \big(R \boxtimes R \big)^{-1} [\, \Delta_X \,]$;

(3) $\forall \ x \in X: \quad R(x) \neq \emptyset$;

(4) $\forall \ x \in X: \quad \exists \ y \in X: \quad (x, y) \in R$.

Remark 6.9. Hence, in accordance with Remark 4.17, we can also at once see $R^{-} = R^{-1} \circ R$ is a reflexive (tolerance) relation if and only if R is non-partial.

Furthermore, from Corollary 6.2 we can also immediately get the following

Theorem 6.10. *For any two relations F and S on X, the following assertions are equivalent:*

(1) $S \subset R \circ R\,;$

(2) $S \subset \big(R \boxtimes R^{-1} \big)^{-1} [\, \Delta_X \,]\,;$

(3) $z \in S(x) \quad \Longrightarrow \quad R(x) \cap R^{-1}(z) \neq \emptyset\,.$

(4) $\forall\ (x,\, z) \in S: \quad \exists\ y \in X: \quad (x,\, y) \in R,\ \ (y,\, z) \in R\,.$

Hence, it is clear that in particular we also have the following two corollaries.

Corollary 6.11. *For any relation R on X, the following assertions are equivalent:*

(1) $R \subset R \circ R\,;$

(2) $R \subset \big(R \boxtimes R^{-1} \big)^{-1} [\, \Delta_X \,]\,;$

(3) $z \in R(x) \quad \Longrightarrow \quad R(x) \cap R^{-1}(z) \neq \emptyset\,.$

(4) $\forall\ (x,\, z) \in R: \quad \exists\ y \in X: \quad (x,\, y) \in R, \quad (y,\, z) \in R\,.$

Remark 6.12. Note that if in particular R is reflexive then (4) trivially holds. Therefore, by the above corollary, (1) also holds.

Corollary 6.13. *For any relation R on X, the following assertions are equivalent:*

(1) $\Delta_X \subset R \circ R\,;$

(2) $\Delta_X \subset \big(R \boxtimes R^{-1} \big)^{-1} [\, \Delta_X \,]\,;$

(3) $\forall\ x \in X: \quad R(x) \cap R^{-1}(x) \neq \emptyset\,;$

(4) $\forall\ x \in X: \quad \exists\ y \in X: \quad (x,\, y) \in R, \quad (y,\, x) \in R\,.$

7. Equalities for composition and box products

Combining Theorems 5.1 and 6.1, we can at once state the following

Theorem 7.1. *For any relations F on X to Y, G on Z to W, R on X to Z, and S on Y to W, the following assertions are equivalent:*

(1) $R = G^{-1} \circ S \circ F\,;$

(2) $R = \big(F \boxtimes G \big)^{-1} [\, S \,]\,;$

(3) $z \in R(x) \quad \Longleftrightarrow \quad G(z) \cap S\,[\, F(x) \,] \neq \emptyset\,;$

(4) $(x,\, z) \in R \quad \Longleftrightarrow \quad \exists\ (y,\, w) \in S: \quad (x, y) \in F, \quad (z,\, w) \in G\,;$

Hence, by letting $Y = W$ and $S = \Delta_Y$, we can immediately get

Corollary 7.2. *For any relations F on X to Y, G on Z to Y, and R on X to Z, the following assertions are equivalent:*

(1) $R = G^{-1} \circ F$;

(2) $R = \big(F \boxtimes G \big)^{-1} [\Delta_Y]$;

(3) $z \in R(x) \iff F(x) \cap G(z) \neq \emptyset$;

(4) $(x, z) \in R \iff \exists\ y \in Y : \quad (x, y) \in F, \quad (z, y) \in G$.

Hence, by letting $X = Z$, and $R = \Delta_X$ and $R = X^2$, respectively, we can immediately get the following two corollaries.

Corollary 7.3. *For any two relations F and G on X to Y, the following assertions are equivalent:*

(1) $\Delta_X = G^{-1} \circ F$;

(2) $\Delta_X = \big(F \boxtimes G \big)^{-1} [\Delta_Y]$;

(3) $\forall\ x, z \in X : \quad \big(x = z \iff F(x) \cap G(z) \neq \emptyset \big)$;

(4) $\forall\ x, z \in X : \quad \big(x = z \iff \exists\ y \in Y : \quad (x, y) \in F, \quad (z, y) \in G \big)$.

Corollary 7.4. *For any two relations F and G on X to Y, the following assertions are equivalent:*

(1) $X^2 = G^{-1} \circ F$;

(2) $X^2 = \big(F \boxtimes G \big)^{-1} [\Delta_Y]$;

(3) $\forall\ x, z \in X : \quad F(x) \cap G(z) \neq \emptyset$;

(4) $\forall\ x, z \in X : \quad \exists\ y \in Y : \quad (x, y) \in F, \quad (z, y) \in G$.

Remark 7.5. Note that, by the corresponding definitions, (1) means only that

$$X = X^2(x) = \big(G^{-1} \circ F \big)(x) = G^{-1} [F(x)] = \mathrm{cl}_G \big(F(x) \big)$$

for all $x \in X$. That is, $F(x)$ is a G–dense subset of Y for all $x \in X$.

While, by Theorem 4.12, (2) means only $X^2 = \mathrm{cl}_{F \boxtimes G} (\Delta_Y)$. That is, Δ_Y is a $F \boxtimes G$–dense subset of Y^2. (Fat and dense subsets of relator spaces were mainly utilized in [31].)

From Corollary 7.2, we can also immediately get the following

Theorem 7.6. *For any two relations R and S on X, the following assertions are equivalent:*

(1) $S = R^{-1} \circ R$;

(2) $S = \big(R \boxtimes R \big)^{-1} [\Delta_X]$;

(3) $z \in S(x) \iff R(x) \cap R(z) \neq \emptyset$,

(4) $(x, z) \in S \iff \exists\ y \in Y : \quad (x, y) \in R, \quad (z, y) \in R$.

Hence, it is clear that in particular we also have the following three corollaries.

Corollary 7.7. *For any relation R on X, the following assertions are equivalent:*

(1) $R = R^{-1} \circ R$;

(2) $R = \big(R \boxtimes R \big)^{-1} [\Delta_X]$;

(3) $z \in R(x) \iff R(x) \cap R(z) \neq \emptyset$.

(4) $(x, z) \in R \iff \exists\ y \in Y: \quad (x, y) \in R, \quad (z, y) \in R$.

Corollary 7.8. *For any relation R on X, the following assertions are equivalent:*

(1) $\Delta_X = R^{-1} \circ R$;

(2) $\Delta_X = \big(R \boxtimes R \big)^{-1} [\Delta_X]$;

(3) $\forall\ x, z \in X: \quad \big(x = z \iff R(x) \cap R(z) \neq \emptyset \big)$;

(4) $\forall\ x, z \in X: \quad \big(x = z \iff \exists\ y \in Y: \quad (x, y) \in R, \quad (z, y) \in R \big)$.

Corollary 7.9. *For any relation R on X, the following assertions are equivalent:*

(1) $X^2 = R^{-1} \circ R$;

(2) $X^2 = \big(R \boxtimes R \big)^{-1} [\Delta_X]$;

(3) $\forall\ x, z \in X: \quad R(x) \cap R(z) \neq \emptyset$;

(4) $\forall\ x, z \in X: \quad \exists\ y \in X: \quad (x, y) \in R, \quad (z, y) \in R$.

Remark 7.10. Note that, by Remark 7.5, (1) means only that $R(x)$ is an R-dense subset of X for all $x \in X$. While, (2) means only that Δ_X is an $R \boxtimes R$–dense subset of X^2.

From Corollary 7.2, we can also immediately get the following

Theorem 7.11. *For any relations F and S on X, the following assertions are equivalent:*

(1) $S = R \circ R$;

(2) $S = \big(R \boxtimes R^{-1} \big)^{-1} [\Delta_X]$;

(3) $z \in S(x) \iff R(x) \cap R^{-1}(z) \neq \emptyset$;

(4) $(x, z) \in S \iff \exists\ y \in Y: \quad (x, y) \in R, \quad (y, z) \in R$.

Hence, it is clear that in particular we also have the following three corollaries.

Corollary 7.12. *For any relation R on X, the following assertions are equivalent:*

(1) $R = R \circ R$;

(2) $R = \big(R \boxtimes R^{-1} \big)^{-1} [\Delta_X]$;

(3) $z \in R(x) \iff R(x) \cap R^{-1}(z) \neq \emptyset$;

(4) $(x, z) \in R \iff \exists\ y \in Y: \quad (x, y) \in R, \quad (y, z) \in R$.

Remark 7.13. Note that if in particular R is a preorder relation on X, then by Remarks 5.10 and 6.12 the equality (1) already holds.

Corollary 7.14. *For any relation R on X, the following assertions are equivalent:*

(1) $\Delta_X = R \circ R$;

(2) $\Delta_X = \big(R \boxtimes R^{-1} \big)^{-1} [\, \Delta_X \,]$;

(3) $\forall \ x, z \in X : \ \big(x = z \iff R(x) \cap R^{-1}(z) \neq \emptyset \big)$;

(4) $\forall \ x, z \in X : \ \big(x = z \iff \exists \ y \in X : \ (x, y) \in R, \ (y, z) \in R \big)$.

Corollary 7.15. *For any relation R on X, the following assertions are equivalent:*

(1) $X^2 = R \circ R$;

(2) $X^2 = \big(R \boxtimes R^{-1} \big)^{-1} [\, \Delta_X \,]$;

(3) $\forall \ x, z \in X : \ R(x) \cap R^{-1}(z) \neq \emptyset$;

(4) $\forall \ x, z \in X : \ \exists \ y \in X : \ (x, y) \in R, \ (y, z) \in R$.

Remark 7.16. Note that, Remark 7.5, (1) means only that $R(x)$ is an R^{-1}–dense subset of X for all $x \in X$. While, (2) means only that Δ_X is an $R \boxtimes R^{-1}$–dense subset of X^2.

8. Applications to non-mingled-valued relations

By using Theorems 5.1 and 6.1, we can easily get the following two theorems.

Theorem 8.1. *For any relation F on X to Y, the following assertions are equivalent:*

(1) $F \circ F^{-1} \circ F \subset F$;

(2) $\big(F \boxtimes F^{-1} \big)^{-1} [\, F^{-1} \,] \subset F$;

(3) $F(x) \cap F(y) \neq \emptyset \implies F(x) \subset F(y)$;

(4) $F^{-1}(z) \cap F^{-1} [\, F(x) \,] \neq \emptyset \implies z \in F(x)$;

(5) $(x, y) \in F, \ (w, y) \in F, \ (w, z) \in F \implies (x, z) \in F$.

Proof. By Theorem 5.1, it is clear that assertions (1), (2) and (4) are equivalent.

Moreover, we can note that (5) is equivalent to the statement that

$$(x, y) \in F, \quad (y, w) \in F^{-1}, \quad (z, w) \in F^{-1} \implies (x, z) \in F.$$

Hence, again by Theorem 5.1, it is clear that (1) and (5) are also equivalent.

Therefore, we need only show that (3) and (5) are also equivalent. For this, note that if (5) holds, then

$$y \in F(x), \quad y \in F(w), \quad z \in F(w) \implies z \in F(x).$$

Hence, it is clear that

$$F(x) \cap F(w) \neq \emptyset \implies F(w) \subset F(x),$$

and thus (3) also holds. While, if (3) holds, then we can quite similarly see that (5) also holds.

Theorem 8.2. *For any relation F on X to Y, the following assertions hold:*

(1) $F \subset F \circ F^{-1} \circ F$;

(2) $F \subset \left(F \boxtimes F^{-1} \right)^{-1} [F^{-1}]$.

(3) $z \in F(x) \quad \Longrightarrow \quad F^{-1}(z) \cap F^{-1} [F(x)] \neq \emptyset$

(4) $\forall \ (x, z) \in F : \quad \exists \ (w, y) \in F : \quad (x, y) \in F , \quad (w, z) \in F$.

Proof. Note that if (4) holds, then

$$\forall \ (x, z) \in F : \quad \exists \ (y, w) \in F^{-1} : \quad (x, y) \in F , \quad (z, w) \in F^{-1} .$$

Therefore, by Theorem 6.1, assertions (1), (2), and (3) also hold.

Therefore, we need only show that (4) holds. For this, note that if $(x, z) \in F$, then by taking $y = z$ and $w = x$, we have $(w, y) \in F$, $(x, y) \in F$, and $(w, z) \in F$.

Now, as an immediate consequence of Theorems 8.1 and 8.2, we can also state

Corollary 8.3. *For any relation F on X to Y, the following assertions are equivalent:*

(1) $F = F \circ F^{-1} \circ F$;

(2) $F = \left(F \boxtimes F^{-1} \right)^{-1} [F^{-1}]$;

(3) $F(x) \cap F(y) \neq \emptyset \quad \Longrightarrow \quad F(x) = F(y)$;

(4) $z \in F(x) \quad \Longleftrightarrow \quad F^{-1}(z) \cap F^{-1} [F(x)] \neq \emptyset$;

(5) $(x, z) \in F \quad \Longleftrightarrow \quad \exists \ (w, y) \in F : \quad (x, y) \in F , \quad (w, z) \in F$.

Remark 8.4. Relations F having property (3) have been called non-mingled-valued by the second author in [12, 24, 26, 27].

They were formerly called "semi-singled-valued" by Berge [3, p. 20] and others. The term "non-mingled" was taken from Whyburn [39].

The importance of non-mingled-valued relations is already apparent from the next simple examples.

Example 8.5. By the corresponding definitions, we can at once see that every function f on X to Y is a non-mingled-valued relation.

Example 8.6. Also by the corresponding definitions, it is clear that if f is a function on X to Y and R is a non-mingled-valued relation on Y to Z, then $R \circ f$ is a non-mingled-valued relation on X to Z.

Example 8.7. By using (1) in Corollary 8.3, we can at once see that each symmetric, idempotent relation R on X is non-mingled-valued.

Example 8.8. By using (1) in Corollary 8.3 and Theorem 1.8, we can also easily see that if F is a non-mingled-valued relation on X to Y, then F^{-1} is a non-mingled-valued relation on Y to X.

Remark 8.9. Since preorders are idempotents, from Example 8.7 it is clear that in particular every equivalence relation is non-mingled-valued.

Hence, by Example 8.6, we can at once state that if f is a function on X to Y and R is an equivalence relation on Y, then $R \circ f$ is a non-mingled-valued relation on X to Y.

In [12], it was actually proved that a relation is non-mingled-valued if and only if it is a composition of an equivalence relation and a function. (For some further characterizations, see [24].)

However, it is now more important to note that we also have the following

Example 8.10. If F is a homomorphic relation on one group X to another Y in the sense that

$$-F(x) \subset F(-x) \qquad \text{and} \qquad F(x) + F(y) \subset F(x+y)$$

for all $x, y \in X$, then F is non-mingled-valued.

Namely, if $x, y \in X$ such that $F(x) \cap F(y) \neq \emptyset$, then there exists $z \in Y$ such that $z \in F(x)$ and $z \in F(y)$. Therefore, for any $w \in F(x)$, we also have

$$w = w - z + z \in F(x) - F(x) + F(y) \subset F(x) + F(-x) + F(y) \subset F(x - x + y) = F(y)$$

Consequently, $F(x) \subset F(y)$ also holds. Thus, by Theorem 8.1, F is non-mingled-valued.

Remark 8.11. This fact was already proved by Lambek [10, Proposition 2] in 1958 by using the terminology of Riguet [17, 18] who called a relation F on X to Y difunctional if $F \circ F^{-1} \circ F \subset F$.

Some finer results in the linear case can be found in [29]. Moreover, in [37] it is proved that if F is a nonvoid, homomorphic relation on one group X to another Y, then D_F is a subgroup of X, $0 \in F(0)$, and F is quasi-additive and $\mathbb{Z}$–superhomogeneous. (This statement has actually been partly generalized in [37] by using quasi-odd relations.)

9. The associated extremal relations

Definition 9.1. For any relations F on X to Y, G on Z to W, and R on X to Z, we define

$$S_{(F,G,R)} = \operatorname{int}_{(F \boxtimes G)^{-1}}(R).$$

Hence, by using the corresponding definitions and Theorems 4.5 and 5.1, we can immediately derive the following

Theorem 9.2. *For any relations F on X to Y, G on Z to W, and R on X to Z, and any elements $y \in Y$ and $w \in W$, the following assertions are equivalent:*

(1) $(y, w) \in S_{(F,G,R)}$;

(2) $\big(F \boxtimes G\big)^{-1}(y, w) \subset R$;

(3) $F^{-1}(y) \times G^{-1}(w) \subset R$;

(4) $G^{-1} \circ \{(y, w)\} \circ F \subset R$;

(5) $(x, y) \in F, \ (z, w) \in G \implies (x, z) \in R$.

From this theorem, for instance, we can immediately get

Corollary 9.3. *For any relation F on X to Y, the following assertions are equivalent:*

(1) $(y, x) \in S_{(F, F^{-1}, F)}$;

(2) $\big(F \boxtimes F^{-1}\big)^{-1}(y, x) \subset F$;

(3) $F^{-1}(y) \times F(x) \subset F$;

(4) $F \circ \{(y, x)\} \circ F \subset F$;

(5) $(s, y) \in F, \ (x, t) \in F \implies (s, t) \in F$.

Moreover, from Definition 9.1, by Theorems 3.2, 4.7, 3.3, 4.12, and 3.5, it is clear that the following theorems are also true.

Theorem 9.4. *For any relations F on X to Y, G on Z to W, and R on X to Z, we have*

$$S_{(F, G, R)} = \big(F \boxtimes G\big)\,[R^c\,]^c = \big(G \circ R^c \circ F^{-1}\big)^c .$$

Theorem 9.5. *For any relations F on X to Y, G on Z to W, and R on X to Z, $S = S_{(F, G, R)}$ is the largest relation on Y to W such that*

$$\big(F \boxtimes G\big)^{-1}[\,S\,] \subset R, \qquad \text{or equivalently} \qquad G^{-1} \circ S \circ F \subset R .$$

Corollary 9.6. *For any relations F on X to Y, G on Z to W, R on X to Z, and S on Y to W, the following assertions are equivalent:*

(1) $S \subset S_{(F, G, R)}$; *(2)* $\big(F \boxtimes G\big)^{-1}[\,S\,] \subset R$; *(3)* $G^{-1} \circ S \circ F \subset R$.

Theorem 9.7. *For any relations F on X to Y, G on Z to W, and R on X to Z, the following assertions are equivalent:*

(1) $R = \big(F \boxtimes G\big)^{-1}[\,S_{(F, G, R)}\,]$, *or equivalently* $R = G^{-1} \circ S_{(F, G, R)} \circ F$;

(2) $R \subset \big(F \boxtimes G\big)^{-1}[\,S_{(F, G, R)}\,]$, *or equivalently* $R \subset G^{-1} \circ S_{(F, G, R)} \circ F$;

(3) $R = \big(F \boxtimes G\big)^{-1}[\,S\,]$, *or equivalently* $R = G^{-1} \circ S \circ F$, *for some relation S on Y to W.*

Remark 9.8. Note that, for any relation F on X to Y, we have

$$\widetilde{F} = \big(F^{-1} \circ F^c \circ F^{-1}\big)^c = S_{(F, F^{-1}, F)} .$$

Therefore, the above theorems, and Corollary 9.6 as well, are natural generalizations of the corresponding statements of Shein [23, Theorems 1 and 2, p. 96]. (See also Bandelt [2, Lemmas 1 and 2] for some generalizations.)

From Theorem 9.7, by using Theorems 6.1 and 9.2, we can immediately derive the following generalization the corresponding statements of Hardy and Petrich [5, Theorem 7.2], Xu and Liu [42, Theorem 2.4], and Xu and Luo [43, Theorem]. (Unfortunately, the paper [41] which is likely contain the same statement, is still not available to the author.)

Theorem 9.9. *For any relation F on X to Y, G on Z to W, and R on X to Z, the following assertions are equivalent:*

(1) $R = G^{-1} \circ S \circ F$ *for some relation S on Y to W;*

(2) $\forall\ (x, z) \in R: \quad \exists\ (y, w) \in Y \times W: \quad (x, z) \in F^{-1}(y) \times G^{-1}(w) \subset R$;

(3) $\forall\ (x, z) \in R: \quad \exists\ (y, w) \in Y \times W:$

(a) $(x, y) \in F, \quad (z, w) \in G$;

(b) $(s, y) \in F, \quad (t, w) \in G \quad \Longrightarrow \quad (s, t) \in R$.

Proof. To prove the equivalence of (1) and (2), note that, by Theorems 9.7 and 6.1, assertion (1) is equivalent to the statement that

(4) $\forall\ (x, z) \in R: \ \exists\ (y, w) \in S_{(F, G, R)}: \quad (x, y) \in F, \quad (z, w) \in G$.

Moreover, by Theorem 9.2,

$$(y, w) \in S_{(F, G, R)} \quad \Longleftrightarrow \quad F^{-1}(y) \times G^{-1}(w) \subset R.$$

And, by the corresponding definitions,

$$(x, y) \in F, \quad (z, w) \in G \quad \Longleftrightarrow \quad (x, z) \in F^{-1}(y) \times G^{-1}(w).$$

From this theorem, for instance we can immediately get the following well-known characterizations of regular relations.

Corollary 9.10. *For any relation F on X to Y, the following assertions are equivalent:*

(1) $F = F \circ S \circ F$ *for some relation S on Y to X;*

(2) $\forall\ (x, y) \in F: \quad \exists\ (u, v) \in X \times Y: \quad (x, y) \in F^{-1}(v) \times F(u) \subset F$;

(3) $\forall\ (x, y) \in F: \quad \exists\ (u, v) \in X \times Y:$

(a) $(x, v) \in F, \quad (u, y) \in F$;

(b) $(s, v) \in F, \quad (u, t) \in F \quad \Longrightarrow \quad (s, t) \in F$.

Remark 9.11. Note that our present theorems are also common generalization of the corresponding statements Jiang and Xu [7, Lemma 2.1 and Theorem 2.3], Jiang, Xu, Cai, and Han [9, Proposition 2.2 and Theorem 2.3], Jiang and Xu [8, Proposition 2.3 and Theorem 2.4], Romano [19, 20, Lemma 2.1 and Theorem 2.1] and Romano [21, Lemmas 2.1 and 2.2 and Theorems 2.1 and 2.2].

10. Some further results on the relation $S_{(F, G, R)}$

By using Theorem 9.2, we can also easily establish the following

Theorem 10.1. *For any relations F on X to Y, G on Z to W, and R on X to Z, we have*

$$S^{-1}_{(F, G, R)} = S_{(G, F, R^{-1})}\,.$$

Proof. Namely, for any $w \in W$ and $y \in Y$, we have

$$(w, y) \in S^{-1}_{(F, G, R)} \iff (y, w) \in S_{(F, G, R)} \iff F^{-1}(y) \times G^{-1}(w) \subset R$$
$$\iff G^{-1}(w) \times F^{-1}(y) \subset R^{-1} \iff (w, y) \in S_{(G, F, R^{-1})}\,.$$

From this theorem, it is clear that in particular we have

Corollary 10.2. *If F is a relation on X to Y and R is a symmetric relation on X, then $S_{(F, F, R)}$ is a symmetric relation on Y.*

Moreover, by using Corollary 9.6, we can easily establish the following

Theorem 10.3. *For any relations F on X to Y, G on Z to Y, and R on X to Z, the following assertions are equivalent:*

(1) $S_{(F, G, R)}$ *is symmetric relation on* Y;

(2) $\big(F \boxtimes G\big)^{-1}\big[\,S^{-1}_{(F, G, R)}\,\big] \subset R$; *(3)* $G^{-1} \circ S^{-1}_{(F, G, R)} \circ F \subset R$.

Remark 10.4. Note that, by Theorem 9.5, we always have

$$G^{-1} \circ S_{(F, G, R)} \circ F \subset R, \qquad \text{and hence} \qquad F^{-1} \circ S^{-1}_{(F, G, R)} \circ G \subset R^{-1}\,.$$

Therefore, Corollary 10.2 can be also easily derived from the above theorem.

By using Corollary 9.6, we can also easily establish the following

Theorem 10.5. *For any relations F on X to Y, G on Z to Y, and R on X to Z, the following assertions are equivalent:*

(1) $S_{(F, G, R)}$ *is a reflexive relation on* Y;

(2) $G^{-1} \circ F \subset R$; *(3)* $\big(F \boxtimes G\big)^{-1}\big[\,\Delta_Y\,\big] \subset R$.

Hence, it is clear that in particular we also have the following

Corollary 10.6. *For any relation R on X, the following assertions are equivalent:*

(1) *R is a transitive relation on X;*

(2) $S_{(R, R^{-1}, R)}$ *is a reflexive relation on* X.

Remark 10.7. Note that, by Remark 9.8, we have $\widetilde{R} = S_{(R, R^{-1}, R)}$. Therefore, the above corollary coincides with a statement of Schein [23, p. 102].

Now, combining Theorems 10.5 and 10.3, we can also state

Theorem 10.8. *For any relations F on X to Y, G on Z to Y, and R on X to Z, the following assertions are equivalent:*

(1) $S_{(F,G,R)}$ *is a tolerance relation on Y;*

(2) $\big(F \boxtimes G\big)^{-1}[\,\Delta_Y \cup S_{(F,G,R)}^{-1}\,] \subset R\,;$ *(3)* $G^{-1} \circ \big(\Delta_Y \cup S_{(F,G,R)}^{-1}\big) \circ F \subset R\,.$

Proof. Note that, by Theorem 2.1, we have

$$\big(F \boxtimes G\big)^{-1}[\,\Delta_Y \cup S_{(F,G,R)}^{-1}\,] = \big(F \boxtimes G\big)^{-1}[\,\Delta_Y\,] \cup \big(F \boxtimes G\big)^{-1}[\,S_{(F,G,R)}^{-1}\,]\,.$$

Therefore, by Theorem 4.12, we also have

$$\big(F \boxtimes G\big)^{-1}[\,\Delta_Y \cup S_{(F,G,R)}^{-1}\,] = \big(G^{-1} \circ \Delta_Y \circ F\big) \cup G^{-1} \circ S_{(F,G,R)}^{-1} \circ F\,.$$

Thus, since $G^{-1} \circ F = G^{-1} \circ \Delta_Y \circ F$, Theorems 10.5 and 10.3 can be applied to get the required equivalences.

From the above theorem, it is clear that in particular we have

Corollary 10.9. *For any relation R on X, the following assertions are equivalent:*

(1) $S_{(R,R^{-1},R)}$ *is a tolerance relation on X;*

(2) $\big(R \boxtimes R^{-1}\big)^{-1}[\,\Delta_X \cup S_{(R,R^{-1},R)}^{-1}\,] \subset R\,;$ *(3)* $R \circ \big(\Delta_X \cup S_{(R,R^{-1},R)}^{-1}\big) \circ R \subset R\,.$

Remark 10.10. If R is a tolerance relation on X, then in contrast to [23, Lemma] of B. Schein we can only prove that $S_{(R,R,R)} \subset R$.

Namely, if $(x, y) \in S_{(R,R,R)}$, then because of the assumption $R = R^{-1}$ we also have $(x, y) \in S_{(R,R^{-1},R)}$. Now, by Corollary 9.3, we can see $(s, x) \in R$ and $(y, t) \in R$ imply $(s, t) \in R$. Hence, by using that $(x, x) \in R$ and $(y, y) \in R$, we can infer that $(x, y) \in R$ also holds.

However, slightly improving an argument of [23, p. 102], we can give a shorter proof of the non-trivial part of [40, Theorem 1] of E. S. Wolk.

Theorem 10.11. *If R is a regular anti-tolerance relation on X, then R is already partial order relation on X.*

Proof. Since R is reflexive, for any $x \in X$, we have $(x, x) \in R$. Hence, by Corollary 9.10, we can see that there exist $u, v \in X$ such that

(a) $(x, v) \in R, \quad (u, x) \in R;$

(b) $(s, v) \in R, \quad (u, t) \in R \quad \Longrightarrow \quad (s, t) \in R.$

From (b), by using that $(v, v) \in R$ and $(u, x) \in R$, we can infer that $(v, x) \in R$. Hence, by the antisymmetry of R, it is clear that $x = v$.

Also from (b), by using that $(x, v) \in R$ and $(u, u) \in R$, we can infer that $(x, u) \in R$. Hence, by the antisymmetry of R, it is clear that $x = u$.

Now, again by (b), we can see that $(s, x) \in R$ and $(x, t) \in R$ implies $(s, t) \in R$. Hence, by using Corollary 9.3, we can infer that $(x, x) \in S_{(R,R^{-1},R)}$. Therefore, $S_{(R,R^{-1},R)}$ is reflexive, and thus by Corollary 10.6 R is transitive.

Remark 10.12. To establish a certain converse of the above theorem, note that if R is only a preorder relation on X, then we already have $R = R \circ R = R \circ \Delta_X \circ R$. Therefore, R is, in particular, a regular relation on X.

Finally, we note that the following theorem is also true.

Theorem 10.13. *If F is a non-mingled-valued relation of X onto Y, then*

$$F^{-1} = S_{(F,\,F^{-1},\,F)}\,.$$

Proof. Define $S = S_{(F,\,F^{-1},\,F)}$. Then by Theorem 9.5, S is the largest relation on Y to X such that $F \circ S \circ F \subset F$. Thus, by Theorem 8.1, we necessarily have $F^{-1} \subset S$.

Moreover, if $(y,\,w) \in S$, then by Theorem 9.2 we have $F^{-1}(y) \times F(w) \subset F$. Therefore, by choosing $x \in F^{-1}(y)$ and $z \in F(w)$, we can state that $(x,\,z) \in F$, and thus $z \in F(x)$. Hence, we can now infer that $y \in F(x) = F(w)$, and thus $(y,\,w) \in F^{-1}$. Therefore, $S \subset F^{-1}$ also holds.

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