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DIVISIBLE AND CANCELLABLE SUBSETS OF GROUPOIDS

TAMÁS GLAVOSITS AND ÁRPÁD SZÁZ

Abstract. In this paper, after listing some basic facts on groupoids, we establish several simple consequences and equivalents of the following basic definitions and their obvious counterparts.

For some $n \in \mathbb{N}$, a subset U of a groupoid X is called

(1) n–cancellable if $n\,x = n\,y$ implies $x = y$ for all $x,\, y \in U$,

(2) n–divisible if for each $x \in U$ there exists $y \in U$ such that $x = n\,y$.

Moreover, for some $A \subset \mathbb{N}$, the set U is called A–divisible (A–cancellable) if it is n–divisible (n–cancellable) for all $n \in A$.

Our main tools here are the sets $n^{-1}x = \{\, y \in X : \quad x = n\,y \,\}$ satisfying $n\,(n^{-1}x) \subset \{x\} \subset n^{-1}(n\,x)$ for all $n \in \mathbb{N}$ and $x \in X$. They can be used to briefly reformulate properties (1) and (2), and naturally turn a uniquely $\mathbb{N}$–divisible commutative group into a vector space over $\mathbb{Q}$.

1. A few basic facts on groupoids

Definition 1.1. If X is a set and $+$ is a function of X^2 to X, then the function $+$ is called a *binary operation* on X, and the ordered pair $X(+) = (X, +)$ is called a *groupoid*.

Remark 1.2. In this case, we may simply write $x + y$ in place of $+(x, y)$ for all $x,\, y \in X$. Moreover, we may also simply write X in place of $X(+)$.

Instead of groupoids, it is more customary to consider only *semigroups* (associative grupoids) or even *monoids* (semigroups with zero). However, several definitions on semigroups can be naturally extended to groupoids.

Definition 1.3. If X is a groupoid, then for any $x \in X$ and $n \in \mathbb{N}$, we define

$$n\,x = x \quad \text{if} \quad n = 1 \qquad \text{and} \qquad n\,x = (n-1)\,x + x \quad \text{if} \quad n > 1\,.$$

Now, by induction, we can easily prove the following two basic theorems.

Theorem 1.4. *If X is a semigroup, then for any $x \in X$ and $m,\, n \in \mathbb{N}$ we have*

(1) $(m+n)\,x = m\,x + n\,x\,,$ *(2)* $(n\,m)\,x = n\,(m\,x)\,.$

Proof. To prove (2), note that if $(n\,m)\,x = n\,(m\,x)$ holds for some $n \in \mathbb{N}$, then by (1) we also have

$$((n+1)\,m)x = (n\,m + m)\,x = (n\,m)x + m\,x = n(m\,x) + m\,x = (n+1)(m\,x)\,.$$

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Theorem 1.5. *If X is a semigroup, then for any $m, n \in \mathbb{N}$ and $x, y \in X$, with $x + y = y + x$, we have*

(1) $m x + n y = n y + m x$, *(2)* $n (x + y) = n x + n y$.

Proof. To prove (1), note that if $x + n y = n y + x$ holds for some $n \in \mathbb{N}$, then we also have

$$x + (n + 1) y = x + n y + y = n y + x + y = n y + y + x = (n + 1) y + x .$$

While, to prove (2), note that if $n (x + y) = n x + n y$ holds for some $n \in \mathbb{N}$, then by (1) we also have

$$(n + 1) (x + y) = n (x + y) + x + y = n x + n y + x + y = \\ = n x + x + n y + y = (n + 1) x + (n + 1) y .$$

Definition 1.6. If in particular X is a groupoid with zero, then we also define $0 x = 0$ for all $x \in X$.

Moreover, if more specially X is a *group*, then we also define $(-n) x = n (-x)$ for all $x \in X$ and $n \in \mathbb{N}$.

Lemma 1.7. *If X is a group, then for any $x \in X$ and $n \in \mathbb{N}$ we also have $(-n) x = -(n x)$.*

Proof. By using $-x + x = 0 = x + (-x)$ and Theorem 1.5, we can at once see that $n (-x) + n x = n (-x + x) = n 0 = 0$. Therefore, $n (-x) = -(n x)$, and thus the required equality is also true.

Now, we can also easily prove the following counterparts of Theorems 1.4 and 1.5.

Theorem 1.8. *If X is a group, then for any $x \in X$ and $k, l \in \mathbb{Z}$ we have*

(1) $(k l) x = k (l x)$, *(2)* $(k + l) x = k x + l x$.

Theorem 1.9. *If X is a group, then for any $k, l \in \mathbb{Z}$ and $x, y \in X$, with $x + y = y + x$, we have*

(1) $k x + l y = l y + k x$, *(2)* $k (x + y) = k x + k y$.

Proof. To prove (2), note that by Lemma 1.7, Theorem 1.5 and assertion (1) we have

$$(-n) (x + y) = -\big(n (x + y) \big) = -(n x + n y) \\ = -(n y) + \big(-(n x) \big) = (-n) y + (-n) x = (-n) x + (-n) y$$

for all $n \in \mathbb{N}$. Moreover, $0 (x + y) = 0 = 0 x + 0 y$ also holds.

Remark 1.10. The latter two theorems show that a commutative group X is already a *module* over the ring $\mathbb{Z}$ of all integers.

2. Operations with subsets of groupoids

Definition 2.1. If X is a groupoid with zero, then for any $U \subset X$ we define

$$U_0 = U \cup \{0\} \quad \text{if} \quad 0 \notin U \qquad \text{and} \qquad U_0 = U \setminus \{0\} \quad \text{if} \quad 0 \in U\,.$$

Remark 2.2. In the sequel, this particular unary operation will mainly be applied to the subsets $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ of the additive group $\mathbb{R}$ of all real numbers.

Definition 2.3. If X is a groupoid, then for any $A \subset \mathbb{N}$, and $U, V \subset X$ we define

$$A\,U = \big\{\, n\,u : \quad n \in A\,, \quad u \in U\,\big\} \qquad \text{and} \qquad U + V = \big\{\, u + v : \quad u \in U\,, \quad v \in V\,\big\}\,.$$

Remark 2.4. Now, by identifying singletons with their elements, we may simply write $n\,U = \{n\}\,U$, $A\,u = A\{u\}$, $u + V = \{u\} + V$, and $U + v = U + \{u\}$ for all $n \in \mathbb{N}$ and $u, v \in X$.

The notation $n\,U$ may cause some confusions since in general we only have $n\,U \subset (\,n - 1\,)\,U + U$ for all $n > 1$. However, assertions 1.4 (1), (2) and 1.5 (1) can be generalized to sets.

Remark 2.5. If in particular, X is a group, then we may quite similarly define $A\,U$ for all $A \subset \mathbb{Z}$ and $U \subset X$.

Moreover, we may also naturally define $-U = (\,-1\,)\,U$ and $U - V = U + (\,-V\,)$ for all $V \subset X$. However, thus we have $U - U = \{\,0\,\}$ if and only if U is a singleton.

Remark 2.6. Moreover, if more specially if X is a *vector space* over K, then we may also quite similarly define $A\,U$ for all $A \subset K$ and $U \subset X$.

Thus, only two axioms of a vector space may fail to hold for $\mathcal{P}(\,X\,)$. Namely, in general, we only have $(\,\lambda + \mu\,)\,U \subset \lambda\,U + \mu\,U$ for all $\lambda, \mu \in K$.

The corresponding elementwise operations with subsets of various algebraic structures allow of some more concise treatments of several basic theorems on substructures of these structures.

Remark 2.7. For instance, a subset U of a groupoid X is called a *subgroupoid* of X if U is itself a groupoid with respect to the restriction of the addition on X to $U \times U$.

Thus, U is a subgroupoid of X if and only if U is *superadditive* in the sense $U + U \subset U$. Moreover, if U is a subgroupoid of X, then U is in particular $\mathbb{N}$–*superhomogeneous* in the sense that $\mathbb{N}\,U \subset U$.

Concerning subgroups, we can prove some more interesting theorems.

Theorem 2.8. *If X is a group, then for a nonvoid subset U of X the following assertions are equivalent:*

(1) U is a subgroup of X, (2) $-U \subset U$ and $U + U \subset U$, (3) $U - U \subset U$.

Remark 2.9. Note that if U is a subset of a group X such that $-U \subset U$, then U is already *symmetric* in the sense that $-U = U$.

While, if U is a subset of a groupoid X with zero such that $U + U \subset U$ and $0 \in U$, then U is already *idempotent* in the sense that $U + U = U$.

Therefore, as an immediate consequence of Theorem 2.8, we can also state

Corollary 2.10. *A nonvoid subset U of a group X is a subgroup of X if and only if it is symmetric and idempotent.*

In addition to Theorem 2.8, we can also easily prove the following

Theorem 2.11. *If X is a group, then for any two symmetric subsets U and V of X the following assertions are equivalent:*

(1) $U+V=V+U$, *(2)* $U+V$ *is symmetric.*

Proof. If (1) holds, then $-(U+V)=-V+(-U)=V+U=U+V$, and thus (2) also holds.

While, if (2) holds, then $U+V=-(U+V)=-V+(-U)=V+U$, and thus (1) also holds.

Remark 2.12. If U and V are idempotent subsets of a semigroup X such that (1) holds, then

$$U+V+U+V=U+V+V+U=U+V+U=U+U+V=U+V,$$

and thus $U+V$ is also an idempotent subset of X.

Therefore, as an immediate consequence of Theorem 2.11 and Corollary 2.10, we can also state

Theorem 2.13. *If X is a group, then for any two subgroups U and V of X the following assertions are equivalent:*

(1) $U+V=V+U$, *(2)* $U+V$ *is a subgroup of* X.

Hence, it is clear that in particular we also have the following

Corollary 2.14. *If U and V are commuting subgroups of a group X, then $U+V$ is the smallest subgroup of X containing both U and V.*

Remark 2.15. In the standard textbooks, Theorem 2.13, or its corollary, is usually proved directly without using Theorems 2.8 and 2.11. (See, for instance, Sott [13, p. 18] and Burton [4, p. 118].)

3. Direct sums of subsets of groupoids

Analogously to Fuchs [6, p. 3.15], we may naturally introduce the following

Definition 3.1. If U, V and W are subsets of a groupoid X such that for every $x\in W$ there exists a unique pair $(u_x, v_x)\in U\times V$ such that

$$x=u_x+v_x\,,$$

then we say that W is the *direct sum* of U and V, and we write $W=U\oplus V$.

Remark 3.2. Thus, in particular we have $W=U+V$. Hence, if in addition X has a zero such that $0\in V$, we can infer that $U\subset W$.

Moreover, in this particular case for any $x\in U$ we have $x=x+0$. Hence, by using the unicity of u_x and v_x, we can infer that $u_x=x$ and $v_x=0$.

Remark 3.3. Therefore, if $W=U\oplus V$ and in particular X has a zero such that $0\in U\cap V$, then in addition to $W=U+V$ we can also state that $U\cup V\subset W$ and $U\cap V=\{0\}$.

Namely, by Remark 3.2 and its dual, we have $U \subset W$ and $V \subset W$, and thus $U \cup V \subset W$. Moreover, if $x \in U \cap V$, i. e., $x \in U$ and $x \in V$, then we have $v_x = 0$ and $u_x = 0$, and thus $x = u_x + v_x = 0$.

In this respect, we can also easily prove the following

Theorem 3.4. *If U and V are subgroups of a monoid X, with $0 \in U \cap V$, then the following assertions are equivalent:*

(1) $X = U \oplus V$; *(2)* $X = U + V$ *and* $U \cap V = \{0\}$.

Proof. If $x \in X$ such that $x = u_1 + v_1$ and $x = u_2 + v_2$ for some $u_1, u_2 \in U$ and $v_1, v_2 \in V$, then $u_1 + v_1 = u_2 + v_2$, and thus $-u_2 + u_1 = v_2 - v_1$. Moreover, we also have $-u_2 + u_1 \in U$ and $v_2 - v_1 \in V$. Hence, if the second part of (2) holds, we can infer that $-u_2 + u_1 = 0$ and $v_2 - v_1 = 0$. Therefore, $u_1 = u_2$, and $v_1 = v_2$ also hold.

Remark 3.5. Note that if U and V are subgroups of a monoid X, with $0 \in U \cap V$, such that $X = U + V$, then for any $x \in X$ there exist $u \in U$ and $v \in V$ such that $x = u + v$. Hence, by taking $y = -v - u$, we can see that $x + y = 0$ and $y + x = 0$. Therefore, $-x = y$, and thus X is also a group.

Remark 3.6. Note that if G is a group, then the Descartes product $X = G \times G$, with the coordinatewise addition, is also a group. Moreover,

$$U = \{ (x, 0) : \quad x \in G \} \qquad \text{and} \qquad V = \{ (0, y) : \quad y \in G \}$$

are subgroups of X such that $X = U + V$ and $U \cap V = \{ (0, 0) \}$. Therefore, by Theorem 3.4, we also have $X = U \oplus V$.

Furthermore, it is also worth noticing that the sets U and V are elementwise commuting in the sense that $u + v = v + u$ for all $u \in U$ and $v \in V$.

The importance of elementwise commuting sets is apparent from the following

Theorem 3.7. *If U and V are elementwise commuting subgroupoids of a semigroup X such that $X = U \oplus V$, then the mappings*

$$x \mapsto u_x \qquad \text{and} \qquad x \mapsto v_x,$$

where $x \in X$, are additive. Thus, in particular, they are $\mathbb{N}$–homogeneous.

Proof. If $x, y \in X$, then by the assumed associativity and commutativity properties of the addition in X we have

$$x + y = (u_x + v_x) + (u_y + v_y) = (u_x + u_y) + (v_x + v_y).$$

Therefore, since $u_x + u_y \in U$ and $v_x + v_y \in V$, the equalities

$$u_{x+y} = u_x + u_y \qquad \text{and} \qquad v_{x+y} = v_x + v_y$$

are also true.

Moreover, by induction, it can be easily seen that if f is an additive function of one groupoid X to another Y, then $f(n x) = n f(x)$ for all $n \in \mathbb{N}$ and $x \in X$.

Remark 3.8. Note that if in particular X has a zero such that $0 \in V$, then by Remark 3.2 the mapping $x \mapsto u_x$, where $x \in X$, is idempotent. Moreover, if $0 \in U$ also holds, then $u_0 = 0$. Thus, the above mapping is also zero-homogeneous.

Remark 3.9. In this respect, it is also worth noticing that if in particular X is a monoid, and U and V are subgroups of X, with $0 \in U \cap V$, then by Remark 3.5 X is also a group, and thus the mappings considered in Theorem 3.7 are actually $\mathbb{Z}$–homogeneous.

Remark 3.10. If in particular X is a vector space, then by using Zorn's lemma [14, p. 38] it can be shown that for each subspace U of X there exists a subspace V of X such that $X = U \oplus V$.

In the standard textbooks, this fundamental decomposition theorem is usually proved with the help of Hamel bases. (See, for instance, Cotlar and Cignoli [5, p. 15] and Taylor and Lay [14, p. 43].)

4. Some further results on elementwise commuting sets

The importance of elementwise commuting sets is also apparent from the following

Theorem 4.1. *If U and V are elementwise commuting, comutative subsets of a semigroup X, then $U+V$ is also commutative.*

Proof. Namely, if $x, y \in U+V$, then there exist $u, \omega \in U$ and $v, w \in V$ such that $x = u+v$ and $y = \omega + w$. Hence, we can already see that

$$x+y = u+v+\omega+w = u+\omega+v+w = \omega+u+w+v = \omega+w+u+v = y+x\,.$$

Therefore, the required assertion is also true.

Remark 4.2. Conversely, we can also easily note that if U and V are subsets of a groupoid X such that $U+V$ is commutative and $U \cup V \subset U+V$, then U and V are commutative and elementwise commuting.

Therefore, as an immediate consequence of Theorem 4.1, we can also state

Corollary 4.3. *If U and V are subsets of monoid X such that $0 \in U \cap V$, then the following assertions are equivalent:*

(1) $U+V$ is commutative,

(2) U and V are commutative and elementwise commuting.

Remark 4.4. Note that if U and V are elementwise commuting subsets of a groupoid X, then we have not only $U+V = V+U$, but also $u+V = V+u$ and $U+v = v+U$ for all $u \in U$ and $v \in V$.

Therefore, it is of some interest to note that we also have the following

Theorem 4.5. *If U and V are subsets of a groupoid X such that $U+V = U \oplus V$, then the following assertions are equivalent:*

(1) U and V are elementwise commuting,

(2) $u+V = V+u$ and $v+U = U+v$ for all $u \in U$ and $v \in V$,

(3) $u+V \subset V+u$ and $v+U \subset U+v$ for all $u \in U$ and $v \in V$,

(4) $V+u \subset u+V$ and $U+v \subset v+U$ for all $u \in U$ and $v \in V$.

Proof. Namely, if for instance (3) holds, then for any $u \in U$ and $v \in V$ we have $u+v \in u+V \subset V+u$. Therefore, there exists $w \in V$ such that $u+v = w+u$. Moreover, again by (3), we can see that $w+u \in w+U \subset U+w$. Therefore, there exists $\omega \in U$ such that $w+u = \omega + w$. Thus, we also have $u+v = \omega + w$. Hence, by using that $U+V = U \oplus V$, we can infer that $u = \omega$ and $v = w$. Therefore, $u+v = v+u$, and thus (1) is also true.

Remark 4.6. In this respect, it is also worth noticing that if U is a subset and V is a subgroup of a monoid X, then the following assertions are also equivalent:

(1) $U+v = v+U$ for all $v \in V$,

(2) $U+v \subset v+U$ for all $v \in V$, (3) $v+U \subset U+v$ for all $v \in V$.

Namely, if for instance (2) holds, then we have

$$v+U = v+U+0 = v+U+(-v)+v \subset v+(-v)+U+v = 0+U+v = U+v$$

for all $v \in V$, and thus (1) also holds.

Concerning elementwise commuting sets, by Theorems 1.5 and 1.9, we can at once state the following two theorems.

Theorem 4.7. *If U and V are elementwise commuting sets of a semigroup X, then the sets $\mathbb{N}U$ and $\mathbb{N}V$ are also also elementwise commuting.*

Theorem 4.8. *If U and V are elementwise commuting subsets of a group X, then the sets $\mathbb{Z}U$ and $\mathbb{Z}V$ are also also elementwise commuting.*

Moreover, concerning elementwise commuting sets, we can also easily prove

Theorem 4.9. *If U and V are elementwise commuting subsets of a semigroup X such that U is commutative, then U and $U+V$ are also elementwise commuting.*

Proof. Suppose that $x \in U$ and $y \in U+V$. Then, there exist $u \in U$ and $v \in V$ such that $y = u+v$. Moreover, by the assumed commutativity properties of U and V, we have

$$x+y = x+u+v = u+x+v = u+v+x = y+x\,.$$

Therefore, the required assertion is also true.

Remark 4.10. The importance of elementwise commuting subsets will also be well shown by the forthcoming theorems of Section 10.

5. Divisible and cancellable subsets of groupoids

Analogously to Hall [10, p. 197], Fuchs [6, p. 58] and Scott [13, p. 95], we may naturally introduce the following

Definition 5.1. A subset U of a groupoid X is called *n–divisible*, for some $n \in \mathbb{N}$, if $U \subset nU$.

Now, the subset U may also be naturally called *A–divisible*, for some $A \subset \mathbb{N}$, if it is n–divisible for all $n \in A$.

Remark 5.2. Thus, U is n-divisible if and only if it is *n–subhomogeneous.* That is, for each $x \in U$ there exists $y \in U$ such that $x = n\,y$.

Therefore, the set U may be naturally called *uniquely n–divisible* if for each $x \in U$ there exists a unique $y \in U$ such that $x = n\,y$.

Moreover, the subset U may also be naturally called *uniquely A–divisible* if it is uniquely n–divisible for all $n \in A$.

Now, in addition to Definition 5.1, we may also naturally introduce the following definition which has also been utilized in [8] .

Definition 5.3. A subset U of a groupoid X is called *n–cancellable*, for some $n \in \mathbb{N}$, if $n\,x = n\,y$ implies $x = y$ for all $x\,,\, y \in U$.

Now, the set U may also be naturally called *A–cancellable*, for some $A \subset \mathbb{N}$, if it is n–cancellable for all $n \in A$.

Remark 5.4. Thus, if U is both n–divisible and n–cancellable, then U is already uniquely n–divisible.

Namely, if $x \in U$ such that $x = n\,y_1$ and $x = n\,y_2$ for some $y_1\,,\, y_2 \in U$, then we also have $n\,y_1 = n\,y_2$, and hence $y_1 = y_2$.

Remark 5.5. Moreover, by using some obvious analogues of Definitions 5.1 and 5.3, we can also see that if U is a both k–divisible and k–cancellable subset of a group X, for some $k \in \mathbb{Z}$, then U is already uniquely k–divisible.

In this respect, it is worth noticing that the following two theorems are also true.

Theorem 5.6. *If U is an n–superhomogeneous subset of a groupoid X, for some $n \in \mathbb{N}$, then the following assertions are equivalent :*

(1) U is uniquely n–divisible , *(2) U is both n–divisible and n–cancellable.*

Proof. Namely, if (1) holds and $x\,,\, y \in U$ such that $n\,x = n\,y$, then because of $n\,x \in U$ and (1) we also have $x = y$. Therefore, U is n–cancellable, and thus (2) also holds. The converse implication (2) $\Longrightarrow$ (1) has been proved in Remark 5.4.

Theorem 5.7. *If U is a k–superhomogeneous subset of a group X, for some $k \in \mathbb{Z}$, then following assertions are equivalent :*

(1) U is uniquely k–divisible , *(2) U is both k–divisible and k–cancellable.*

By using the corresponding definitions and Theorems 1.4 and 1.8, we can easily prove the following two theorems.

Theorem 5.8. *If U is an n–divisible subset of a semigroup X, for some $n \in N$, and $p\,,\, q \in \mathbb{N}$ such that $n = p\,q$ and U is q–superhomogeneous, then U is also p–divisible.*

Proof. If $x \in U$, then by the n–divisibility of U there exists $y \in U$ such that $x = ny$. Now, by using Theorem 1.4, we can see that $x = n\,y = (p\,q\,)y = p\,(\,q\,y\,)$. Hence, because of $q\,y \in U$, it is clear that U is also p–divisible.

Theorem 5.9. *If U is an k–divisible subset of a semigroup X, for some $k \in Z$, and $p\,,\, q \in \mathbb{Z}$ such that $k = p\,q$ and U is q–superhomogeneous, then U is also p–divisible.*

In addition to the latter two theorems, it is also worth proving the following

Theorem 5.10. *For a subset U of a monoid X, the following assertions are equivalent:*

(1) $U \subset \{0\}$, *(2)* U *is* 0*-divisible,* *(3)* U *is* $\mathbb{N}_0$*–divisible.*

By using the corresponding definitions and Theorems 1.4 and 1.8, we can also easily prove the following counterparts of Theorems 5.8, 5.9 and 5.10.

Theorem 5.11. *If U is an m–superhomogeneous, both n– and m–cancellable subset of a semigroup X, for some $m, n \in \mathbb{N}$, then U is also nm–cancellable.*

Proof. If $x, y \in U$ such that $(nm)x = (nm)y$, then by Theorem 1.4 we also have $n(mx) = n(my)$. Hence, by using the n–cancelability of U, and the fact that $mx, my \in U$, we can infer that $mx = my$. Now, by the m–cancelability of U, we can see that $x = y$. Therefore, U is also nm–cancellable.

Theorem 5.12. *If U is an l–superhomogeneous, both k– and l–cancellable subset of a group X, for some $k, l \in \mathbb{N}$, then U is also kl–cancellable.*

Theorem 5.13. *For a subset U of a monoid X, the following assertions are equivalent:*

(1) $\operatorname{card}(U) \leq 1$, *(2)* U *is* 0*-cancellable,* *(3)* U *is* $\mathbb{N}_0$*–cancellable.*

In addition to Theorems 5.8 and 5.9, we can also prove the following two theorems.

Theorem 5.14. *If U is a uniquely n–divisible, n–superhomogeneous subset of a semigroup X for some $n \in N$, and $p, q \in \mathbb{N}$ such that $n = pq$ and U is q–superhomogeneous, then U is also uniquely p–divisible.*

Proof. By Theorem 5.8 and Remark 5.4, we need only show that now U is also p–cancellable.

For this, note that if $x, y \in U$ such that $px = py$, then by Theorem 1.4 we also have $nx = (qp)x = q(px) = q(py) = (qp)x = ny$. Moreover, by Theorem 5.6, U is now n–cancellable. Therefore, we necessarily have $x = y$.

Theorem 5.15. *If U is a uniquely k–divisible, k–superhomogeneous subset of a group X, for some $k \in Z$, and $p, q \in \mathbb{Z}$ such that $n = pq$ and U is q–superhomogeneous, then U is also uniquely p–divisible.*

Remark 5.16. Note that in assertion (3) of Theorem 5.10 we may also write "uniquely $\mathbb{N}_0$–divisible" instead of "$\mathbb{N}_0$–divisible".

6. Some further results on divisible and cancellable sets

Theorem 6.1. *If U is a k–divisible, symmetric subset of a group X, for some $k \in \mathbb{Z}$, then U is also $-k$–divisible.*

Proof. If $x \in U$, then by the k–divisibility of U there exists $y \in U$ such that $x = ky$. Now, by using Theorem 1.8, we can see that

$$x = ky = \big((-k)(-1)\big)y = (-k)\big((-1)y\big) = (-k)(-y).$$

Hence, since now we also have $-y \in -U = U$, it is clear that U is also $-k$–divisible.

From this theorem, it is clear that in particular we also have

Corollary 6.2. *If U is an $\mathbb{N}$–divisible, symmetric subset of a group X, then U is $\mathbb{Z}_0$–divisible.*

Analogously to Theorem 6.1, we can also easily prove the following

Theorem 6.3. *If U is a k–cancellable subset of a group X, for some $k \in \mathbb{Z}$, then U is also $-k$–cancellable.*

Proof. If $x, y \in U$ such that $(-k)\,x = (-k)\,y$, then by Theorem 1.8 we also have

$$k\,x = \big((-1)(-k)\big)\,x = (-1)\big((-k)\,x\big) = (-1)\big((-k)\,y\big) = \big((-1)(-k)\big)\,y = k\,y\,.$$

Hence, by the assumption, it follows that $x = y$, and thus the required assertion is also true.

From this theorem, it is clear that in particular we also have

Corollary 6.4. *If U is an $\mathbb{N}$–cancellable subset of a group X, then U is also $\mathbb{Z}_0$–cancellable.*

Now, as an immediate consequence of Theorems 6.1 and 6.3 and Remark 5.5, we can also state

Theorem 6.5. *If U is a uniquely k–divisible, symmetric subset of a group X, for some $k \in \mathbb{Z}$, then U is also uniquely $-k$–divisible.*

Hence, it is clear that in particular we also have

Corollary 6.6. *If U is a uniquely $\mathbb{N}$–divisible, symmetric subset of a group X, then U is also uniquely $\mathbb{Z}_0$–divisible.*

Remark 6.7. By using some obvious analogues of Definition 5.1 and Remark 5.2, we can also easily see that a subset U of a vector space X over K is k–divisible (uniquely k–divisible), for some $k \in K_0$, if and only if $k^{-1}x \in U$ for all $x \in U$. That is, $k^{-1}U \subset U$.

Remark 6.8. If U is an n–cancellable subset of a groupoid X with zero, for some $n \in \mathbb{N}$, such that $0 \in U$, then $n\,x = 0$ implies $x = 0$ for all $x \in U$.

Namely, if $x \in U$ such that $n\,x = 0$, then by the corresponding definitions we also have $n\,x = n\,0$, and hence $x = 0$.

Remark 6.9. Quite similarly, we can also see that if U is a k–cancellable subset of a group X, for some $k \in \mathbb{Z}$, such that $0 \in U$, then $k\,x = 0$ implies $x = 0$ for all $x \in U$.

Now, by using the letter observation and Corollary 6.4, we can also easily prove

Theorem 6.10. *If U is an $\mathbb{N}$–cancellable subset of a group X such that $0 \in U$, then $k\,x = l\,x$ implies $k = l$ for all $k, l \in \mathbb{Z}$ and $x \in U_0$.*

Proof. Assume on the contrary that there exist $k, l \in \mathbb{Z}$ and $x \in U_0$ such that $k\,x = l\,x$, but $k \neq l$. Then, by using Theorem 1.8, we can see that

$$(k - l)\,x = \big(k + (-l)\big)\,x = k\,x + (-l)\,x = l\,x + (-l)\,x = \big(l + (-l)\big)\,x = 0\,x = 0\,.$$

Hence, by using Corollary 6.4 and Remark 6.9, we can infer that $x = 0$. This contradiction proves the theorem.

From the above theorem, by taking $l = 0$, we can immediately derive

Corollary 6.11. *If U is an $\mathbb{N}$–cancellable subset of group X such that $0 \in U$, then $k\,x = 0$ implies $k = 0$ for all $k \in \mathbb{Z}$ and $x \in U_0$.*

In addition to Remark 6.9, we can also easily prove the following

Theorem 6.12. *If X is a commutative group, then for each $k \in \mathbb{Z}$ the following assertions are equivalent:*

(1) X is k–cancellable; *(2) $k\,x = 0$ implies $x = 0$ for all $x \in X$.*

Proof. From Remark 6.9, we can see that (1) $\Longrightarrow$ (2) even if the group X is not assumed to be commutative.

Moreover, if $x,\, y \in X$ such that $k\,x = k\,y$, then by using Theorem 1.9 we can see that

$$k\,(\,x - y\,) = k\,(\,x + (-y\,)\,) = k\,x + k\,(-y) = k\,y + k\,(-y) = k\,(\,y + (-y\,)\,) = k\,0 = 0\,.$$

Hence, if (2) holds, then we can already infer that $x - y = 0$, and thus $x = y$. Therefore, (1) also holds.

From this theorem, by using Corollary 6.4, we can immediately derive

Corollary 6.13. *If X is a commutative group such that $n\,x = 0$ implies $x = 0$ for all $n \in \mathbb{N}$ and $x \in X$, then X is $\mathbb{Z}_0$–cancellable.*

Remark 6.14. By using an obvious analogue of Definition 5.3, we can also easily see that every subset U of a vector space X over K is K_0–cancellable. Moreover, $k\,x = l\,x$ implies $k = l$ for all $k, l \in K$ and $x \in X_0$.

7. Characterizations of divisible and cancellable sets

Definition 7.1. If X is a groupoid, then for any $x \in X$ and $n \in \mathbb{N}$ we define

$$n^{-1}\,x = \big\{\, y \in X : \quad x = n\,y \,\big\}\,.$$

Remark 7.2. Now, having in mind the definition of the image of a set under a relation, for any $U \subset X$, we may also naturally define $n^{-1}\,U = \bigcup_{x \in U} n^{-1}\,x$.

Thus, we can easily see that $n^{-1}\,U = \big\{\, y \in X : \quad n\,y \in U \,\big\}$. Namely, if for instance, $y \in n^{-1}\,U$, then by the above definition there exists $x \in U$ such that $y \in n^{-1}\,x$. Hence, by Definition 7.1, it already follows that $ny = x \in U$.

By using Definition 7.1, we can also easily prove the following

Theorem 7.3. *If X is a groupoid, then for any $x \in X$ and $n \in \mathbb{N}$ we have*

(1) $n\,\big(\,n^{-1}x\,\big) \subset \{x\}$, *(2) $\{x\} \subset n^{-1}\big(\,n\,x\,\big)$.*

Proof. Since $n\,x = n\,x$, it is clear that $x \in n^{-1}\,\big(\,n\,x\,\big)$. Therefore, (2) is true.

Moreover, if $z \in n\,\big(\,n^{-1}x\,\big)$ then there exists $y \in n^{-1}x$ such that $z = n\,y$. Hence, since $y \in n^{-1}x$ implies $ny = x$, we can infer that $z = x$. Therefore, (1) is also true.

Remark 7.4. Now, by using this theorem, for any $U \subset X$, we can also easily prove that $n\,\big(\,n^{-1}U\,\big) \subset U \subset n^{-1}\big(\,n\,U\,\big)$.

For instance, by using Theorem 7.3 and Remark 7.2, we can easily see that

$$U = \bigcup_{x \in U} \{x\} \subset \bigcup_{x \in U} n^{-1}(\,n\,x\,) = n^{-1}\left(\,\bigcup_{x \in U} \{n\,x\}\right) = n^{-1}\,(\,n\,U\,)\,.$$

By using an obvious analogue of Definition 7.1, we can also easily prove the following

Theorem 7.5. *If X is a group, then for any $x \in X$ and $k \in \mathbb{Z}$ we have*

(1) $k\left(k^{-1}x\right) \subset \{x\}$, *(2)* $\{x\} \subset k^{-1}\left(kx\right)$.

Remark 7.6. Now, by using this theorem, for any $U \subset X$, we can also easily prove that $k\left(k^{-1}U\right) \subset U \subset k^{-1}\left(kU\right)$.

However, it is now more important to note that, by using the corresponding definitions, we can also easily prove the following three theorems.

Theorem 7.7. *If X is a groupoid, then for any $U \subset X$ and $n \in \mathbb{N}$ the following assertions are equivalent:*

(1) *U is n–divisible,* *(2)* $U \cap n^{-1}x \neq \emptyset$ *for all* $x \in U$.

Theorem 7.8. *If X is a groupoid, then for any $U \subset X$ and $n \in \mathbb{N}$ the following assertions are equivalent:*

(1) *U is uniquely n–divisible,* *(2)* $\operatorname{card}\left(U \cap n^{-1}x\right) = 1$ *for all* $x \in U$.

Theorem 7.9. *If X is a groupoid, then for any $U \subset X$ and $n \in \mathbb{N}$ the following assertions are equivalent:*

(1) *U is n–cancellable,* *(2)* $\operatorname{card}\left(U \cap n^{-1}(nx)\right) \leq 1$ *for all* $x \in U$.

Proof. If $x \in X$ and $y_1, y_2 \in U \cap n^{-1}(nx)$, then $y_1, y_2 \in U$ and $y_1, y_2 \in n^{-1}(nx)$, and thus $ny_1 = nx = ny_2$. Hence, if (1) holds, we can infer that $y_1 = y_2$, and thus (2) also holds.

Conversely, if $x, y \in U$ such that $nx = ny$, then by Definition 7.1 we have $y \in n^{-1}(nx)$. Moreover, by Theorem 7.3, we also have $x \in n^{-1}(nx)$. Therefore, $x, y \in U \cap n^{-1}(nx)$. Hence, if (2) holds, we can infer that $x = y$. Therefore, (1) also holds.

Analogously to the latter three theorems, we can also easily prove the following three theorems.

Theorem 7.10. *If X is a group, then for any $U \subset X$ and $k \in \mathbb{Z}$ the following assertions are equivalent:*

(1) *U is k–divisible,* *(2)* $U \cap k^{-1}x \neq \emptyset$ *for all* $x \in U$.

Theorem 7.11. *If X is a group, then for any $U \subset X$ and $k \in \mathbb{Z}$ the following assertions are equivalent:*

(1) *U is uniquely k–divisible,* *(2)* $\operatorname{card}\left(U \cap k^{-1}x\right) = 1$ *for all* $x \in U$.

Theorem 7.12. *If X is a group, then for any $U \subset X$ and $k \in \mathbb{Z}$ the following assertions are equivalent:*

(1) *U is k–cancellable,* *(2)* $\operatorname{card}\left(U \cap k^{-1}(kx)\right) \leq 1$ *for all* $x \in X$.

Moreover, as a simple reformulation of Theorem 6.12, we can also state

Theorem 7.13. *A commutative group X, then for any $k \in \mathbb{Z}$ the following assertions are equivalent:*

(1) *X is k–cancellable,* *(2)* $k^{-1}0 \subset \{0\}$, *(3)* $k^{-1}0 = \{0\}$.

Remark 7.14. Quite similarly, by Remark 6.8, we can also state that if U is an n–cancellable subset of groupoid X with zero, for some $n \in \mathbb{N}$, such that $0 \in U$, then $U \cap n^{-1} 0 = \{ 0 \}$.

Remark 7.15. Moreover, by Remark 6.9, we can also state that if U is a k–cancellable subset of group X, for some $k \in \mathbb{Z}$, such that $0 \in U$, then $U \cap k^{-1} 0 = \{ 0 \}$.

In addition to Theorem 7.13 and Remarks 7.14 and 7.15, it is also worth proving

Theorem 7.16. *The following assertions hold:*

(1) If X is a commutative group, then $k^{-1} 0$ is a subgroup of X for all $k \in \mathbb{Z}$.

(2) If X is a commutative monoid, then $n^{-1} 0$ is a submonoid of X for all $n \in \mathbb{N}_0$.

However, it is now more important to note that in addition to Theorems 7.7, 7.10, 7.9 and 7.12, we can also easily prove the following four theorems.

Theorem 7.17. *If X is a groupoid, then for any $n \in \mathbb{N}$ the following assertions are equivalent:*

(1) X is n–divisible,

(2) $\{ x \} \subset n \left(n^{-1} x \right)$ for all $x \in X$,

(3) $\{ x \} = n \left(n^{-1} x \right)$ for all $x \in X$.

Proof. If (1) holds, then by Theorem 7.7, for every $x \in X$, we have $n^{-1} x \neq \emptyset$, and thus $n \left(n^{-1} x \right) \neq \emptyset$. Moreover, by Theorem 7.3, we also have $n \left(n^{-1} x \right) \subset \{ x \}$. Therefore, (3) also holds. The implication $(2) \Longrightarrow (1)$ is even more obvious by Theorem 7.7.

Theorem 7.18. *If X is a group, then for any $k \in \mathbb{Z}$ the following assertions are equivalent:*

(1) X is k–divisible,

(2) $\{ x \} \subset k \left(k^{-1} x \right)$ for all $x \in X$,

(3) $\{ x \} = k \left(k^{-1} x \right)$ for all $x \in X$.

Theorem 7.19. *If X is a groupoid, then for any $n \in \mathbb{N}$ the following assertions are equivalent:*

(1) X is n–cancellable,

(2) $n^{-1} \left(n x \right) \subset \{ x \}$ for all $x \in X$,

(3) $n^{-1} \left(n x \right) = \{ x \}$ for all $x \in X$.

Proof. If (1) holds, then by Theorem 7.9, for every $x \in X$, we have $\operatorname{card} \left(n^{-1} (n x) \right) \leq 1$. Moreover, by Theorem 7.3, we also have $\{ x \} \subset n^{-1} (n x)$. Therefore, (3) also holds. The implication $(2) \Longrightarrow (1)$ is even more obvious by Theorem 7.9.

Theorem 7.20. *If X is a group, then for any $k \in \mathbb{Z}$ the following assertions are equivalent:*

(1) X is k–cancellable,

(2) $k^{-1} \left(k x \right) \subset \{ x \}$ for all $x \in X$,

(3) $k^{-1} \left(k x \right) = \{ x \}$ for all $x \in X$.

Now, as some immediate consequences of the latter four theorems, and Theorems 5.6 and 5.7, we can also state the following two theorems.

Theorem 7.21. *If X is a groupoid, then for any $n \in \mathbb{N}$ the following assertions are equivalent:*

(1) X *is uniquely* n*–divisible,*

(2) $n^{-1}\big(n\,x\big) \subset \{x\} \subset n\,\big(n^{-1}x\big)$ *for all* $x \in X$,

(3) $n^{-1}\big(n\,x\big) = \{x\} = n\,\big(n^{-1}x\big)$ *for all* $x \in X$.

Theorem 7.22. *If X is a group, then for any $k \in \mathbb{Z}$ the following assertions are equivalent:*

(1) X *is uniquely* k*–divisible,*

(2) $k^{-1}\big(k\,x\big) \subset \{x\} \subset k\,\big(k^{-1}x\big)$ *for all* $x \in X$,

(2) $k^{-1}\big(k\,x\big) = \{x\} = k\,\big(k^{-1}x\big)$ *for all* $x \in X$.

8. Some further results on the sets $n^{-1}x$ and $k^{-1}x$

In addition to Theorem 7.3, we can also prove the following

Theorem 8.1. *If X is a semigroup, then for any $x \in X$ and $m, n \in \mathbb{N}$ we have:*

(1) $m\,\big(n^{-1}x\big) \subset n^{-1}\big(m\,x\big)$, *(2)* $m^{-1}\big(n^{-1}x\big) \subset (m\,n)^{-1}x$,

(3) $m\,\big((m\,n)^{-1}x\big) \subset n^{-1}x$, *(4)* $n^{-1}x \subset (m\,n)^{-1}(m\,x)$.

Proof. If $y \in n^{-1}x$, then by Definition 7.1 we have $x = n\,y$. Hence, by using Theorem 1.4, we can infer that

$$m\,x = m\,(n\,y) = (m\,n)\,y = (n\,m)\,y = n\,(m\,y).$$

Thus, by Definition 7.1, we also have

$$y \in (m\,n)^{-1}(m\,x) \qquad \text{and} \qquad m\,y \in n^{-1}(m\,x).$$

Hence, we can already see that (4) and (1) are true.

On the other hand, if $y \in (m\,n)^{-1}x$, then by Definition 7.1 and Theorem 1.4 we have

$$x = (m\,n)\,y = (n\,m)\,y = n\,(m\,y).$$

Thus, by Definition 7.1, we also have $m\,y \in n^{-1}x$. Hence, we can already see that (3) is also true.

Finally, if $y \in m^{-1}\big(n^{-1}x\big)$, then by Remark 7.2, we have $m\,y \in n^{-1}x$. Hence, by using Definition 7.1 and Theorem 1.4, we can infer that

$$x = n\,(m\,y) = (n\,m)\,y = (m\,n)\,y.$$

Thus, by Definition 7.1, we also have $y = (m\,n)^{-1}x$. Hence, we can already see that (2) is also true.

From this theorem, by Theorem 7.8, it is clear that in particular we also have

Corollary 8.2. *If X is a uniquely $\mathbb{N}$–divisible semigroup, then for any $x \in X$ and $m, n \in \mathbb{N}$ we have:*

(1) $m\,\big(n^{-1}x\big) = n^{-1}\big(m\,x\big)$, *(2)* $m^{-1}\big(n^{-1}x\big) = (m\,n)^{-1}x$,

(3) $m\,\big((m\,n)^{-1}x\big) = n^{-1}x$, *(4)* $n^{-1}x = (m\,n)^{-1}\big(m\,x\big)$.

Analogously to Theorem 8.1, we can also prove the following

Theorem 8.3. *If X is a group, then for any $x \in X$ and $k, l \in \mathbb{Z}$ we have:*

(1) $k\big(l^{-1}x\big) \subset l^{-1}\big(kx\big)$, *(2)* $k^{-1}\big(l^{-1}x\big) \subset (kl)^{-1}x$,

(3) $k\big((kl)^{-1}x\big) \subset l^{-1}x$, *(4)* $l^{-1}x \subset (kl)^{-1}\big(kx\big)$.

Hence, by Corollary 6.6 and Theorem 7.11, it is clear that in particular we have

Corollary 8.4. *If X is a uniquely $\mathbb{N}$–divisible group, then for any $x \in X$ and $k, l \in \mathbb{Z}_0$ we have:*

(1) $k\big(l^{-1}x\big) = l^{-1}\big(kx\big)$, *(2)* $k^{-1}\big(l^{-1}x\big) = (kl)^{-1}x$,

(3) $k\big((kl)^{-1}x\big) = l^{-1}x$, *(4)* $l^{-1}x = (kl)^{-1}\big(kx\big)$.

In addition to Theorem 8.1, we can also prove the following

Theorem 8.5. *If X is a commutative semigroup, then for any $x, y \in X$ and $n \in \mathbb{N}$ we have*

$$n^{-1}x + n^{-1}y \subset n^{-1}(x+y).$$

Proof. If $z \in n^{-1}x$ and $w \in n^{-1}y$, then by using Definition 7.1 and Theorem 1.5, we can see that

$$x + y = nz + nw = n(z+w).$$

Therefore, by Definition 7.1, we also have $z + w \in n^{-1}(x+y)$. Hence, we can already see that the required inclusion is also true.

From this theorem, by Theorem 7.8, it is clear that in particular we also have

Corollary 8.6. *If X is a uniquely $\mathbb{N}$–divisible commutative semigroup, then for any $x, y \in X$ and $n \in \mathbb{N}$ we have*

$$n^{-1}(x+y) = n^{-1}x + n^{-1}y.$$

Analogously to Theorem 8.5, we can also prove the following

Theorem 8.7. *If X is a commutative group, then for any $k \in \mathbb{Z}$ and $x, y \in X$ we have*

$$k^{-1}x + k^{-1}y \subset k^{-1}(x+y).$$

Hence, by Corollary 6.6 and Theorem 5.11, it is clear that in particular we also have

Corollary 8.8. *If X is a uniquely $\mathbb{N}$–divisible commutative semigroup, then for any $k \in \mathbb{Z}_0$ and $x, y \in X$ we have*

$$k^{-1}(x+y) = k^{-1}x + k^{-1}y.$$

Remark 8.9. In the latter two theorems and their corollaries, the commutativity assumptions on X can be weakened.

For instance, in Theorem 8.5 it would be enough to assume only that the sets $n^{-1}x$ and $n^{-1}y$ are elementwise commuting.

9. Uniquely ℕ–divisible semigroups

In addition to Corollary 8.2, we can also easily prove the following

Lemma 9.1. *If X is a uniquely ℕ–divisible semigroup and $m, n, p, q \in \mathbb{N}$ such that $m/n = p/q$, then for every $x \in X$ we have*

$$m\left(n^{-1}x\right) = p\left(q^{-1}x\right).$$

Proof. By Theorem 7.21, we have

$$n\left(n^{-1}x\right) = \{x\} = q\left(q^{-1}x\right).$$

Hence, by using that $mq = pn$, we can infer that

$$(mq)\left(n\left(n^{-1}x\right)\right) = (pn)\left(q\left(q^{-1}x\right)\right).$$

Now, by using Theorem 1.4, we can also see that

$$(nq)\left(m\left(n^{-1}x\right)\right) = (nq)\left(p\left(q^{-1}x\right)\right).$$

Hence, by using Theorem 5.6 and 5.11, we can see that the required equality is also true.

Analogously to this lemma, we can also prove the following

Lemma 9.2. *If X is a uniquely ℕ–divisible group and $n, q \in \mathbb{N}$ and $m, p \in \mathbb{Z}$ such that $m/n = p/q$, then for every $x \in X$ we have*

$$m\left(n^{-1}x\right) = p\left(q^{-1}x\right).$$

Because of the above lemmas, we may naturally introduce the following two definitions.

Definition 9.3. If X is a uniquely ℕ–divisible semigroup, then for any $x \in X$ and $m, n \in \mathbb{N}$ we define

$$\left(m/n\right)x = m\left(n^{-1}x\right).$$

Definition 9.4. If X is a uniquely ℕ–divisible group, then for any $x \in X$, $n \in \mathbb{N}$ and $m \in \mathbb{Z}$ we define

$$\left(m/n\right)x = m\left(n^{-1}x\right).$$

By using Definition 9.3 and Corollary 8.2, we can easily prove the following

Theorem 9.5. *If X is a uniquely ℕ–divisible semigroup, then for any $x \in X$ and $r, s \in \mathbb{Q}$, with $r, s > 0$, we have*

(1) $(r+s)x = rx + sx$, (2) $(rs)x = r(sx)$.

Proof. By the definition of $\mathbb{Q}$, there exists $m, n, p, q \in \mathbb{N}$ such that $r = m/n$ and $s = p/q$.

Now, by using Theorems 7.8 and 1.4 and Corollary 8.2, we can see that

$$\begin{aligned}(r+s)x &= \left((m/n)+(p/q)\right)x = \left((mq+pn)/(nq)\right)x \\ &= (mq+pn)\left((nq)^{-1}x\right) = (mq)\left((nq)^{-1}x\right) + (pn)\left((nq)^{-1}x\right) \\ &= m\left(q\left((nq)^{-1}x\right)\right) + p\left(n\left((nq)^{-1}x\right)\right) = m\left(n^{-1}x\right) + p\left(q^{-1}x\right) \\ &= (m/n)x + (p/q)x = rx + sx\end{aligned}$$

and

$$\begin{aligned}(r\,s)\,x &= \big((m/n)\,(p/q)\big)\,x = \big((m\,p)/(n\,q)\big)\,x = (m\,p)\,\big((n\,q)^{-1}\,x\big)\\ &= m\,\big(p\,\big((n\,q)^{-1}\,x\big)\big) = m\,\big(p\,\big(n^{-1}\,\big(q^{-1}x\big)\big)\big) = m\,\big(n^{-1}\,\big(p\,\big(q^{-1}x\big)\big)\big)\\ &= m\,\big(n^{-1}\,\big((p/q)\,x\big)\big) = (m/n)\,\big((p/q)\,x\big) = r\,(s\,x)\,.\end{aligned}$$

Analogously to this theorem, we can also prove the following

Theorem 9.6. *If X is a uniquely $\mathbb{N}$–divisible group, then for any $x \in X$ and $r\,,\, s \in \mathbb{Q}$ we have*

(1) $(r+s)\,x = r\,x + s\,x\,,$ *(2)* $(r\,s)\,x = r\,(s\,x)\,.$

By using Definition 9.3 and Corollary 8.6, we can also easily prove the following

Theorem 9.7. *If X is a uniquely $\mathbb{N}$–divisible commutative semigroup, then for any $x\,,\, y \in X$ and $r \in \mathbb{Q}\,,$ with $r > 0\,,$ we have*

$$r\,(x+y) = r\,x + r\,y\,.$$

Proof. By the definition of $\mathbb{Q}\,,$ there exist $m\,,\, n \in \mathbb{N}$ such that $r = m/n\,.$

Now, by using Corollary 8.6 and Theorem 1.5, we can see that

$$\begin{aligned}r\,(x+y) &= (m/n)\,(x+y) = m\,\big(n^{-1}\,(x+\,y)\big) = m\,\big(n^{-1}\,x + n^{-1}\,y)\big)\\ = m\,\big(n^{-1}\,x\big)+m\,\big(n^{-1}\,y\big) &= m\,\big(n^{-1}\,x\big)+m\,\big(n^{-1}\,y\big) = (m/n)\,x+(m/n)\,y = r\,x+r\,y\end{aligned}$$

Analogously to this theorem, we can also prove the following

Theorem 9.8. *If X is a uniquely $\mathbb{N}$–divisible commutative group, then for any $x\,,\, y \in X$ and $r \in \mathbb{Q}\,,$ we have*

$$r\,(x+y) = r\,x + r\,y\,.$$

Now, as an immediate consequence of Theorems 9.6 and 9.7, we can also state

Corollary 9.9. *If X is a uniquely $\mathbb{N}$–divisible commutative group, then $X\,,$ with the multiplication given in Definition 9.4, is a vector space over $\mathbb{Q}\,.$*

Remark 9.10. Note that, by Remark 6.7, every vector space X over $\mathbb{Q}$ is uniquely $\mathbb{Q}_0$–divisible.

Now, by using Corollary 9.9, from the basic decomposition theorem of vector spaces, mentioned in Remark 3.10, we can immediately derive the following

Theorem 9.11. *If X is a uniquely $\mathbb{N}$–divisible commutative group, then for each $\mathbb{N}$–divisible subgroup U of X there exists an $\mathbb{N}$–divisible subgroup V of X such that $X = U \oplus V$.*

Remark 9.12. Note that now, by Theorem 5.6, X is $\mathbb{N}$–cancellable, and thus actually both U and V are also uniquely $\mathbb{N}$–divisible. Moreover, by Corollary 6.6, U, V and X are uniquely $\mathbb{Z}_0$–divisible.

Remark 9.13. To see that the $\mathbb{N}$–divisibility of U is an essential condition in the above theorem, we can note that $\mathbb{Z}$ is an additive subgroup of the field $\mathbb{Q}$ such that, for any $\mathbb{N}$–superhomogeneous subset V of $\mathbb{Q}$ with $\mathbb{Z} \cap V \subset \{0\}$, we have $V \subset \{0\}$, and thus $\mathbb{Z} + V \subset \mathbb{Z}$.

Namely, if $x \in V$, then since $V \subset \mathbb{Q}$ there exist $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ such that $x = m/n$. Moreover, since V is $\mathbb{N}$–superhomogeneous, we have

$$m = n\,(\,m/n\,) = n\,x \in V\,.$$

Hence, since $m \in \mathbb{Z}$ and $\mathbb{Z} \cap V \subset \{0\}$ also hold, we can infer that $m = 0$, and thus $x = 0$. Therefore, $V \subset \{0\}$, and thus $\mathbb{Z} + V \subset \mathbb{Z} + \{0\} = \mathbb{Z}$.

In addition to Remark 9.13, it is also worth proving the following

Theorem 9.14. *If X is an $\mathbb{N}$–cancellable group and $a \in X$, then $U = \mathbb{Z}\,a$ is a commutative subgroup of X such that, for every $\mathbb{N}$–divisible symmetric subset V of $X \setminus \{a\}$, we have $U \cap V \subset \{0\}$.*

Proof. By Theorems 1.8, 1.9 and 2.8, it is clear that U is a commutative subgroup of X. Therefore, we need only prove that $U \cap V \subset \{0\}$.

For this, assume on the contrary that there exists $x \in U \cap V$ such that $x \neq 0$. Then, by the definition of U, there exists $k \in Z$ such that $x = k\,a$. Hence, since $x \neq 0$, we can infer that $k \neq 0$. Therefore, by Corollary 6.2, there exists $v \in V$ such that $x = k\,v$. Thus, we have $k\,a = k\,v$. Hence, by using Corollary 6.4, we can infer that $a = v$, and thus $a \in V$. This contradiction proves the required inclusion.

From this theorem, by using Theorem 3.4, we can immediately derive

Corollary 9.15. *If X and U are as in Theorem 9.14, then for every $\mathbb{N}$–divisible subgroup V of X with $a \notin V$ and $X = U + V$ we have $X = U \oplus V$.*

Remark 9.16. Concerning Theorem 9.11, it is also worth mentioning that Baer [1] in 1936 already proved that if U is an $\mathbb{N}$–divisible subgroup of a commutative group X, then there exists a subgroup V of X such that $X = U \oplus V$.

Moreover, Kertész [11] in 1951 proved that if X is a commutative group such that the order of each element of X is a square-free number, then for every subgroup U of X there exists a subgroup V of X such that $X = U \oplus V$.

Surprisingly, the above two results were already considered to be well-known by Baer in [1, p.1] and [3, p. 504]. Moreover, it is also worth mentioning that Hall [9], analogously to Kertész [11], also proved an "if and only if result".

10. Operations with divisible and cancellable sets

Theorem 10.1. *If U is an n–divisible subset of a semigroup X, for some $n \in \mathbb{N}$, then for every $m \in \mathbb{N}$ the set $m\,U$ is also n–divisible.*

Proof. If $x \in m\,U$, then by the definition of $m\,U$ there exists $u \in U$ such that $x = m\,u$. Moreover, by the n–divisibility of U, there exists $v \in U$ such that $u = n\,v$. Hence, by using Theorem 1.4, we can see that $x = m\,u = m\,(\,n\,v\,) = n\,(\,m\,v\,)$. Thus, since $m\,v \in m\,U$, the required assertion is also true.

Moreover, as a certain converse to this theorem, we can also prove

Theorem 10.2. *If U is an m–cancellable, n–superhomogeneous subset of a semigroup X, for some $m, n \in \mathbb{N}$, such that mU is n–divisible, then U is also n–divisible.*

Proof. If $x \in U$, then by the definition mU we also have $mx \in mU$. Therefore, by the n–divisibility of mU, there exists $v \in mU$ such that $mx = nv$. Moreover, by the definition of mU, there exists $y \in U$ such that $v = my$. Now, by using Theorem 1.4, we can see that $mx = nv = n(my) = m(ny)$. Hence, by using the m–cancellability of U and the fact that $ny \in U$, we can already infer that $x = ny$. Therefore, the required assertion is also true.

Quite similarly to Theorems 10.1 and 10.2, we can also prove the following two theorems.

Theorem 10.3. *If U is a k–divisible subset of a group X, for some $k \in \mathbb{Z}$, then for every $l \in \mathbb{Z}$ the set lU is also k–divisible.*

Theorem 10.4. *If U is an l–cancellable, k–superhomogeneous subset of a group X, for some $l, k \in \mathbb{N}$, such that lU is k–divisible, then U is also k–divisible.*

In addition to Theorem 10.1, we can also easily prove the following

Theorem 10.5. *If U and V are elementwise commuting, n–divisible subsets of a semigroup X, for some $n \in \mathbb{N}$, then $U + V$ is also n–divisible.*

Proof. If $x \in U+V$, then by the definition of $U+V$ there exist $u \in U$ and $v \in V$ such that $x = u + v$. Moreover, since U and V are n–divisible, there exist $\omega \in U$ and $w \in V$ such that $u = n\omega$ and $v = nw$. Hence, by using Theorem 1.5, we can see that $x = u + v = n\omega + nw = n(\omega + w)$. Thus, since $\omega + w \in U + V$, the required assertion is also true.

Moreover, as a certain converse to this theorem, we can also prove

Theorem 10.6. *If U and V are elementwise commuting, n–superhomogeneous subsets of a monoid X, for some $n \in \mathbb{N}$, such that $U + V$ is n–divisible, and $U + V = U \oplus V$ and $0 \in V$, then U is also n–divisible.*

Proof. If $x \in U$, then because of $0 \in V$ we also have $x \in U + V$. Thus, by the n–divisibility of $U + V$, there exists $y \in U + V$ such that $x = ny$. Moreover, by the definition of $U + V$, there exist $u \in U$ and $v \in V$ such that $y = u + v$. Now, by using Theorem 1.5, we can see that

$$x = ny = n(u + v) = nu + nv .$$

Moreover, we can also note that $x \in U + V$, $nu \in U$ and $nv \in V$. Hence, since $x = x + 0$ also holds with $x \in U$ and $0 \in V$, by using the assumption $U + V = U \oplus V$, we can already infer that $x = nu$. Therefore, U is also n–divisible.

Quite similarly to Theorems 10.5 and 10.6, we can also prove the following two theorems.

Theorem 10.7. *If U and V are elementwise commuting, k–divisible subsets of a semigroup X, for some $k \in \mathbb{Z}$, then $U + V$ is also k–divisible.*

Theorem 10.8. *If U and V are elementwise commuting, k–superhomogeneous subsets of a group X, for some $k \in \mathbb{Z}$, such that $U+V$ is k–divisible, and $U+V = U \oplus V$ and $0 \in V$, then U is also k–divisible.*

Hence, by Theorem 3.4, it is clear that in particular we also have

Corollary 10.9. *If U and V are elementwise commuting subgroups of a group X such that $U+V$ is k–divisible, for some $k \in \mathbb{Z}$ such that $U \cap V = \{0\}$, then U and V are also n–divisible.*

In addition to Theorem 10.5, we can also prove the following

Theorem 10.10. *If U and V are elementwise commuting, n–superhomogeneous subsets of a semigroup X, for some $n \in \mathbb{N}$ such that U and V are n–cancellable and $U+V = U \oplus V$, then $U+V$ is also n–cancellable.*

Proof. For this, assume that $x, y \in U+V$ such $n\,x = n\,y$. Then, by the definition of $U+V$, there exist $u, \omega \in U$ and $v, w \in V$ such that $x = u+v$ and $y = \omega + w$. Hence, by using Theorem 1.5, we can see that

$$n\,u + n\,v = n\,(u+v) = n\,x = n\,y = n\,(\omega + w) = n\,\omega + n\,w\,.$$

Moreover, we can also note that $n\,u, n\omega \in U$ and $n\,v, n\,w \in V$, and thus $n\,u + n\,v$, $n\omega + n\,w \in U+V$. Now, by using that $U+V = U \oplus V$, we can see that $n\,u = n\,\omega$ and $n\,v = n\,w$. Hence, by using the n–cancellability of U and V, we can already infer that $u = \omega$ and $v = w$. Therefore, $x = u+v = \omega + w = y$, and thus the required assertion is also true.

Remark 10.11. Now, as a trivial converse to this theorem, we can also state that if U and V subsets of a monoid X such that $U+V$ is n–cancellable, for some $n \in \mathbb{Z}$, and $0 \in U \cap V$, then U and V are also n–cancellable.

Quite similarly to Theorem 10.10, we can also prove the following

Theorem 10.12. *If U and V are elementwise commuting, k–superhomogeneous subsets of a group X, for some $k \in \mathbb{Z}$ such that U and V are k–cancellable and $U+V = U \oplus V$, then $U+V$ is also k–cancellable.*

Hence, by Theorem 3.4, it is clear that in particular we also have

Corollary 10.13. *If U and V are elementwise commuting subgroups of a group X such that U and V are k–cancellable for some $k \in \mathbb{Z}$, and $U \cap V = \{0\}$, then $U+V$ is also k–cancellable.*

Remark 10.14. In an immediate continuation of this paper, by using the notion of the order

$$n_a = \inf \{\, n \in \mathbb{N} : \quad n\,a = 0 \,\}$$

of an element a of a monoid (resp. group) X, we shall investigate the divisibility and cancellability properties of the set $\mathbb{N}_0\,a + V$ (resp. $\mathbb{Z}\,a + V$) for some substructures V of X.

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Tamás Glavosits, Department of Applied Mathematics, University of Miskolc, H–3515 Miskolc-Egyetemváros, Hungary

E-mail address: `matgt@uni-miskolc.hu`

Árpád Száz, Department of Mathematics, University of Debrecen, H–4010 Debrecen, P.O.B. 12, Hungary

E-mail address: `szaz@science.unideb.hu`