Tech. Rep., Inst. Math., Univ. Debrecen 2011/1, 14 pp.

SETS AND POSETS WITH INVERSIONS

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Abstract. In this paper, we investigate unary operations $\vee$, $\wedge$ and $\diamond$ on a set X satisfying

$$x = x^{\vee\vee} = x^{\wedge\wedge} \qquad \text{and} \qquad x^{\diamond} = x^{\vee\wedge} = x^{\wedge\vee}$$

for all $x \in X$.

Moreover, if in particular X is a meet-semilattice, then we also investigate the operations defined by

$$x_{\blacktriangledown} = x \wedge x^{\vee}, \qquad x_{\blacktriangle} = x \wedge x^{\wedge}, \qquad x_{\blacklozenge} = x \wedge x^{\diamond};$$

$$x_{\bullet} = x^{\vee} \wedge x^{\wedge}, \qquad x_{\clubsuit} = x^{\vee} \wedge x^{\diamond}, \qquad x_{\spadesuit} = x^{\wedge} \wedge x^{\diamond};$$

and $x_{\bigstar} = x \wedge x^{\vee} \wedge x^{\wedge} \wedge x^{\diamond}$ for all $x \in X$.

Our prime example for this is the set-lattice $\mathcal{P}(U, V)$ of all relations on one group U to another V equipped with the operations defined such that

$$F^{\vee}(u) = F(-u), \qquad F^{\wedge}(u) = -F(u) \qquad \text{and} \qquad F^{\diamond}(u) = -F(-u)$$

for all $F \subset X \times Y$ and $u \in U$.

1. A few basic facts on relations and functions

A subset F of a product set $X \times Y$ is called a relation on X to Y. If in particular $F \subset X^2$, then we may simply say that F is a relation on X. In particular, $\Delta_X = \{(x, x) : x \in X\}$ is called the identity relation on X.

If F is a relation on X to Y, then for any $x \in X$ and $A \subset X$ the sets $F(x) = \{ y \in Y : (x, y) \in F \}$ and $F[A] = \bigcup_{a \in A} F(a)$ are called the images of x and A under F, respectively.

Instead of $y \in F(x)$ sometimes we shall also write $x F y$. Moreover, the sets $D_F = \{ x \in X : F(x) \neq \emptyset \}$ and $R_F = F[X] = F[D_F]$ will be called the domain and range of F, respectively.

If in particular $D_F = X$, then we say that F is a relation of X to Y, or that F is a total relation on X to Y. While, if $R_F = Y$, then we say that F is a relation on X onto Y.

1991 *Mathematics Subject Classification.* Primary 06A05, 06A11; Secondary 06A12, 20M15.

Key words and phrases. Sets and posets, involutions and interiors, fixed and coincidence points.

The work of the author has been supported by the Hungarian Scientific Research Fund (OTKA) Grant NK-81402.

Typeset by $\mathcal{AMS}$-TEX

If F is a relation on X to Y, then $F = \bigcup_{x \in X} \{x\} \times F(x) = \bigcup_{x \in D_F} \{x\} \times F(x)$. Therefore, a relation F on X to Y can be naturally defined by by specifying $F(x)$ for all $x \in X$, or by specifying D_F and $F(x)$ for all $x \in D_F$.

For instance, if F is a relation on X to Y, then the inverse relation F^{-1} of F can be naturally defined such that $F^{-1}(y) = \{ x \in X : \ y \in F(x) \}$ for all $y \in Y$. Thus, we also have $F^{-1} = \{ (y, x) : \ (x, y) \in F \}$.

Moreover, if in addition G is a relation on Y to Z, then the composition relation $G \circ F$ of G and F can be naturally defined such that $(G \circ F)(x) = G[F(x)]$ for all $x \in X$. Thus, we also have $(G \circ F)[A] = G[F[A]]$ for all $A \subset X$.

Now, a relation F on X may be called reflexive, transitive and antisymmetric if $\Delta_X \subset F$, $F \circ F \subset F$ and $F \cap F^{-1} \subset \Delta_X$, respectively. Moreover, a relation having all these properties may be called a partial order relation.

In particular, a relation f on X to Y is called a function if for each $x \in D_f$ there exists $y \in Y$ such that $f(x) = \{y\}$. In this case, by identifying singletons with their elements, we may simply write $f(x) = y$ in place of $f(x) = \{y\}$.

The inverse f^{-1} of a function f is a function if and only if f is injective in the sense that $f(x) \neq f(y)$ for all $x, y \in D_f$ with $x \neq y$. Moreover, for a function g, we have $g = f^{-1}$ if and only if $g \circ f = \Delta_{D_f}$ and $f \circ g = \Delta_{D_g}$.

A function $\star$ of a set X to itself is called a unary operation on X. Moreover, a function $*$ of X^2 to X is called a binary operation in X. For any $x, y \in X$, we usually write $x^\star$ or $x_\star$ and $x * y$ in place of $\star(x)$ and $*\big((x, y)\big)$, respectively.

A unary operation $\star$ on a set X is called an involution if $x = x^{\star\star}$ for all $x \in X$. Hence, it is clear that $\star$ is injective and onto X. Moreover, we can also note that a unary operation $\star$ is an involution if and only if $\star = \star^{-1}$.

If X is a group, then for any $A, B \subset X$, we may also naturally define $A + B = \{x + y : \ x \in A, \ y \in B\}$ and $-A = \{-x : \ x \in A\}$. However, thus the family $\mathcal{P}(X)$ of all subsets of X is only a monoid with involution.

Finally, we note that a relation F on one group X to another Y will be called here odd, even and symmetric-valued if $F(-x) = -F(x)$, $F(-x) = F(x)$ and $-F(x) = F(x)$ hold for all $x \in X$, respectively.

2. A few basic facts on partially ordered sets

According to Birkhoff [1], a set X, equipped with a partially order relation $\leq$, is called a poset (partially ordered set). In this case, for any $x, y \in X$, we write $x < y$ if $x \leq y$ and $x \neq y$.

A poset X is called a chain (totally ordered set) if for any $x, y \in X$ we have either $x \leq y$ or $y \leq x$. Thus, X is a chain if and only if at least (exactly) one of the alternatives $x < y$, $x = y$ and $y < x$ holds.

If X is a poset with the relation $\leq$, then X is also a poset with the inverse relation $\geq$ of $\leq$. This poset is denoted by X^* and called the dual of X. Thus, if in particular X is a chain, then X^* is also a chain.

If X and Y are posets, then for any (x, y), $(z, w) \in X \times Y$ we may naturally write $(x, y) \leq (z, w)$ if $x \leq z$ and $y \leq w$. Thus, $X \times Y$ is also a poset. However, if X and Y are chains, then $X \times Y$ need not be a chain.

Therefore, under the above assumptions, it is frequently more convenient to write $(x, y) \leq (z, w)$ if either $x < z$ or $x = z$ and $y \leq w$. Thus, $X \times Y$ is also a poset. Moreover, if X and Y are chains, then $X \times Y$ is also a chain.

A function f of one poset X to another Y is called increasing if $f(x) \leq f(y)$ for all $x, y \in X$ with $x \leq y$. Moreover, f is called strictly increasing if $f(x) < f(y)$ for all $x, y \in X$ with $x < y$.

Decreasing functions can be defined quite similarly. Note that a function f of one poset X to another Y is decreasing if and only if it is an increasing function of X^* to Y, or equivalently of X to Y^*.

If X is a poset, then for any $\alpha \in X$ and $A \subset X$ we write $\alpha \in \mathrm{lb}(A)$ if $\alpha \leq x$ for all $x \in A$. Moreover, we write $\min(A) = A \cap \mathrm{lb}(A)$. Note that, by the antisymmetry of the inequality, $\min(A)$ is at most a singleton.

The expressions $\mathrm{ub}(A)$ and $\max(A)$ can be defined quite similarly. Moreover, we may write $\inf(A) = \max\big(\mathrm{lb}(A)\big)$ and $\sup(A) = \min\big(\mathrm{ub}(A)\big)$. Thus, for instance, $\inf(A) = \max\big(\mathrm{lb}(A)\big) = \mathrm{lb}(A) \cap \mathrm{ub}\big(\mathrm{lb}(A)\big)$.

Therefore, by identifying singletons with their elements, we have $\alpha = \inf(A)$ if and only if $\alpha \in \mathrm{lb}(A)$ and $\alpha \in \mathrm{ub}\big(\mathrm{lb}(A)\big)$. That is, $\alpha \leq x$ for all $x \in A$ and $\beta \leq \alpha$ for all $\beta \in \mathrm{lb}(A)$, i.e., for all $\beta \in X$ with $\beta \leq x$ for all $x \in A$.

A poset X is called a meet-semilattice (join-semilattice) if $x \wedge y = \inf\{x, y\}$ ($x \vee y = \sup\{x, y\}$) exists for $x, y \in X$. Thus, X is a join-semilattice if and only if X^* is a meet-semilattice.

In particular, a poset X is called a lattice if it is both a meet-semilattice and a join-semilattice. Note that every chain X is a lattice. Namely, if $x, y \in X$ such that $x \leq y$, then we evidently have $x = x \wedge y$ and $y = x \vee y$.

Concerning increasing functions, we can easily prove the following theorems.

Theorem 2.1. *For any function f of one poset X to another Y, the following assertions hold:*

(1) If f is strictly increasing, then f is increasing;

(2) If f is injective and increasing, then f is strictly increasing.

Proof. To prove (2), assume that the conditions of (2) hold and $x, y \in X$ such that $x < y$. Then, by the definition of $<$, we have $x \leq y$ and $x \neq y$. Hence, since f is increasing and injective, we can infer that $f(x) \leq f(y)$ and $f(x) \neq f(y)$. Therefore, $f(x) < f(y)$ also holds. This shows that f is strictly increasing.

Theorem 2.2. *For any function f of a chain X to a poset Y, the following assertions hold:*

(1) If f is strictly increasing, then f is injective;

(2) If f is injective and increasing, then f^{-1} is strictly increasing.

Proof. To prove (2), assume that the conditions of (2) holds and $z, w \in f[X]$ such that $z < w$. Then, by the definition of $f[X]$, there exist $x\ y \in X$ such that $z = f(x)$ and $w = f(y)$. Hence, since $z < w$, and thus $z \neq w$, we can see that $x \neq y$. Thus, since X is totally ordered, we have either $x < y$ or $y < x$.

However, if $y < x$ were true, then by Theorem 2.1 we would have $f(y) < f(x)$, and hence $w < z$. This contradicts the assumption that $z < w$. Namely, if both $z < w$ and $w < z$ were true, then $z \leq w$ and $w \leq z$, and thus $z = w$ would also be true.

Therefore, we can only have $x < y$. However, since $z = f(x)$ and $w = f(y)$, we also have $x = f^{-1}(z)$ and $y = f^{-1}(w)$ by the injectivity of f. Therefore, $f^{-1}(z) < f^{-1}(w)$ also holds. This shows that f^{-1} is strictly increasing.

Theorem 2.3. *If f is an injective increasing function of one poset X onto another Y such that f^{-1} is also increasing, then for any $A \subset X$ we have*

(1) $f\big(\inf(A)\big) = \inf\big(f[A]\big)$ *if at least one of these infima exists;*

(2) $f\big(\sup(A)\big) = \sup\big(f[A]\big)$ *if at least one of these suprema exists.*

Proof. To prove the first part of (1), suppose that $A \subset X$ such that $\alpha = \inf(A)$ exists. Then, for any $x \in A$, we have $\alpha \leq x$, and hence $f(\alpha) \leq f(x)$. Therefore, $f(\alpha) \in \operatorname{lb}\big(f[A]\big)$.

On the other hand, if $\beta \in \operatorname{lb}\big(f[A]\big)$, then for any $x \in A$, we have $\beta \leq f(x)$, and hence $f^{-1}(\beta) \leq x$. Therefore, $f^{-1}(\beta) \in \operatorname{lb}(A)$, and thus $f^{-1}(\beta) \leq \alpha$. Hence, we can infer that $\beta \leq f(\alpha)$, and thus $f(\alpha) \in \operatorname{ub}\big(\operatorname{lb}\big(f[A]\big)\big)$ also holds.

The above arguments show that $f(\alpha) = \max\big(\operatorname{lb}\big(f[A]\big)\big) = \inf\big(f[A]\big)$, and thus the required equality is also true.

The second part of (1) can be proved quite similarly. Moreover, it can also easily derived from the first part of (1) by taking f^{-1} and $f[A]$ in place of f and A, respectively.

Corollary 2.4. *If f is an injective increasing function of one poset X onto another Y such that f^{-1} is also increasing, then for any $x, y \in X$ we have*

(1) $f(x \wedge y) = f(x) \wedge f(y)$ *if X and Y are meet-semilattices;*

(2) $f(x \vee y) = f(x) \vee f(y)$ *if X and Y are join-semilattices.*

Remark 2.5. Note that if for instance f is a function of one poset X to another Y such that $f\,(x \wedge y) = f(x) \wedge f(y)$ whenever $x, y \in X$ such that $x \wedge y$ exits, then f is necessarily increasing.

3. Sets with inversions

Definition 3.1. Let X be a set, and assume that $\vee$, $\wedge$ and $\diamond$ are unary operations on X such that

$$x = x^{\vee\vee} = x^{\wedge\wedge} \qquad \text{and} \qquad x^{\diamond} = x^{\vee\wedge} = x^{\wedge\vee}$$

for all $x \in X$. Then, we say that X is a set with inversions $\vee$, $\wedge$ and $\diamond$.

The introduction of the above definition has been suggested by the following obvious examples.

Example 3.2. Let $\mathbb{R}$ be the set of all real numbers, and for any $x \in \mathbb{R}$ define

$$x^{\vee} = -x\,, \qquad x^{\wedge} = \begin{cases} 0 & \text{if} \quad x = 0\,, \\ x^{-1} & \text{if} \quad x \neq 0\,, \end{cases} \qquad \text{and} \qquad x^{\diamond} = \begin{cases} 0 & \text{if} \quad x = 0\,, \\ -x^{-1} & \text{if} \quad x \neq 0\,. \end{cases}$$

Then, $\mathbb{R}$ is a set with inversions $\vee$, $\wedge$ and $\diamond$.

Example 3.3. Let $\mathbb{C} = \mathbb{R}^2$, and for any $z = (u, v) \in \mathbb{C}$ define

$$z^{\vee} = (-u, v)\,, \qquad z^{\wedge} = (u, -v) \qquad \text{and} \qquad z^{\diamond} = (-u, -v)\,.$$

Then, $\mathbb{C}$ is a set with inversions $\vee$, $\wedge$ and $\diamond$.

Remark 3.4. Note that $z^{\wedge}$ and $z^{\diamond}$ are just the complex conjugate and the ordinary negative of z, respectively.

Example 3.5. Let U and V be groups, and for any function f of U to V and $u \in U$ define

$$f^{\vee}(u) = f(-u), \qquad f^{\wedge}(u) = -f(u) \qquad \text{and} \qquad f^{\diamond}(u) = -f(-u).$$

Then, the family V^U of all functions f of U to V is a set with inversions $\vee$, $\wedge$ and $\diamond$.

Remark 3.6. Note that, for any $u \in U$ and $v \in V$, we have

$$\begin{aligned}(u, v) \in f^{\diamond} &\iff v = f^{\diamond}(u) \iff v = -f(-x) \\ &\iff -v = f(-u) \iff (-u, -v) \in f \iff -(u, v) \in f.\end{aligned}$$

Therefore, $f^{\diamond}$ is just the global negative of f.

Remark 3.7. The global negative $f^{\diamond}$ has to be carefully distinguished from the pointwise one $f^{\wedge}$ despite that both can be naturally denoted by $-f$.

Namely, for instance, if $\Delta = \Delta_U$ is the identity function of U, then $\Delta^{\diamond} = \Delta$. But, $\Delta^{\wedge} = \Delta$ if and only if $-u = u$, or equivalently $2u = 0$ for all $u \in U$.

Example 3.8. Let U and V be groups, and for any relation F on X to Y and $u \in U$ define

$$F^{\vee}(u) = F(-u), \qquad F^{\wedge}(u) = -F(u) \qquad \text{and} \qquad F^{\diamond}(u) = -F(-u).$$

Then, the family $\mathcal{P}(U \times V)$ of all relations F on U to V is a set with inversions $\vee$, $\wedge$ and $\diamond$.

Remark 3.9. It can be easily seen that

$$F^{\vee} = \big\{ (-u, v) : \quad (u, v) \in F \big\}, \qquad F^{\wedge} = \big\{ (u, -v) : \quad (u, v) \in F \big\},$$

and

$$F^{\diamond} = \big\{ (-u, -v) : \quad (u, v) \in F \big\}.$$

Therefore, Example 3.8 is a generalization of not only Example 3.5, but also Example 3.3 too.

Example 3.10. Let U be a group, and for any relation F on U define

$$F^{\#} = \big\{ (-v, -u) : \quad (u, v) \in F \big\}.$$

Then, the family $\mathcal{P}(U^2)$ of all relations F on U is a set with inversions ${}^{-1}$, $\diamond$ and $\#$.

Remark 3.11. It can be easily seen that

$$\big(F^{-1}\big)^{\vee} = \big(F^{\wedge}\big)^{-1} \qquad \text{and} \qquad \big(F^{\vee}\big)^{-1} = \big(F^{-1}\big)^{\wedge}.$$

Moreover, we can also note that above relations are, in general, quite different. Therefore, we cannot write $\vee$ or $\wedge$ in place of $\diamond$ in the above example.

However, from the above examples we can immediately get several further examples with the help of the following

Theorem 3.12. *If* X *is a set with inversions* $\vee$, $\wedge$ *and* $\diamond$, *then*

(1) X *is set with inversions* $\wedge$, $\vee$ *and* $\diamond$;

(2) X *is set with inversions* $\vee$, $\diamond$ *and* $\wedge$.

Proof. To check (2), note that if $x \in X$, then we have

$$x^{\vee\diamond} = x^{\vee\vee\wedge} = x^{\wedge} \qquad \text{and} \qquad x^{\diamond\vee} = x^{\wedge\vee\vee} = x^{\wedge}\,.$$

Hence, we can see that $x^{\diamond\diamond} = x^{\diamond\vee\wedge} = x^{\wedge\wedge} = x$. Therefore, (2) is also true.

Remark 3.13. By the above theorem, for instance, we can also state that X is a set with inversions $\diamond$, $\vee$ and $\wedge$.

Moreover, as an immediate consequence of Definition 3.1 and Theorem 3.12, we can also state

Theorem 3.14. *If* X *is a set with inversions* $\vee$, $\wedge$ *and* $\diamond$, *then the operation* $\square = \vee$, $\wedge$ *or* $\diamond$ *is injective and onto* X. *Moreover, we have* $\square = \square^{-1}$.

4. Fixed points of the operations $\vee$, $\wedge$ and $\diamond$

Definition 4.1. If X is a set and $\square$ is an unary operation on X, then we write

$$X_{\square} = \left\{ x \in X : \quad x = x^{\square} \right\},$$

Example 4.2. Thus, according to Example 3.2, we have

$$\mathbb{R}_{\vee} = \{0\}\,, \qquad \mathbb{R}_{\wedge} = \{-1,\,0,\,1\} \qquad \text{and} \qquad \mathbb{R}_{\diamond} = \{0\}\,.$$

Example 4.3. Moreover, according to Example 3.3, we have

$$\mathbb{C}_{\vee} = \{0\} \times \mathbb{R}\,, \qquad \mathbb{C}_{\wedge} = \mathbb{R} \times \{0\} \qquad \text{and} \qquad \mathbb{C}_{\diamond} = \{(0,\,0)\}\,.$$

Example 4.4. Furthermore, according to Example 3.8, for any relation F on U to V we have

(1) $F \in \mathcal{P}(U, V)_{\diamond} \iff F$ is odd;

(2) $F \in \mathcal{P}(U, V)_{\vee} \iff F$ is even;

(3) $F \in \mathcal{P}(U, V)_{\wedge} \iff F$ is symmetric-valued.

Concerning the set $X_{\vee}$, we can easily prove the following

Theorem 4.5. *If X is a set with inversions $\vee$, $\wedge$ and $\diamond$, then for any $x \in X$, the following assertions are equivalent:*

(1) $x \in X_{\vee}$; *(2)* $x^{\wedge} = x^{\diamond}$;

(3) $x^{\vee} \in X_{\vee}$; *(4)* $x^{\wedge} \in X_{\vee}$; *(5)* $x^{\diamond} \in X_{\vee}$.

Proof. By the corresponding definitions and Theorem 3.14, it is clear that

$$x^{\vee} \in X_{\vee} \iff x^{\vee} = x^{\vee\vee} \iff x^{\vee} = x \iff x \in X_{\vee},$$

$$x^{\wedge} = x^{\diamond} \iff x^{\wedge} = x^{\vee\wedge} \iff x = x^{\vee} \iff x \in X_{\vee}.$$

Therefore, (3) and (2) are equivalent to (1).

Moreover, by the corresponding definitions and Theorem 3.12, it is clear that

$$x^{\wedge} \in X_{\vee} \iff x^{\wedge} = x^{\wedge\vee} \iff x^{\wedge} = x^{\diamond},$$

$$x^{\diamond} \in X_{\vee} \iff x^{\diamond} = x^{\diamond\vee} \iff x^{\diamond} = x^{\wedge}.$$

Therefore, (4) and (5) are equivalent to (2), and thus also to (1).

Analogously to the above theorem, we can also easily prove the following theorems which can also be derived from Theorem 4.5 by using Theorem 3.12.

Theorem 4.6. *If X is a set with inversions $\vee$, $\wedge$ and $\diamond$, then for any $x \in X$, the following assertions are equivalent:*

(1) $x \in X_{\wedge}$; *(2)* $x^{\vee} = x^{\diamond}$;

(3) $x^{\vee} \in X_{\wedge}$; *(4)* $x^{\wedge} \in X_{\wedge}$; *(5)* $x^{\diamond} \in X_{\wedge}$.

Theorem 4.7. *If X is a set with inversions $\vee$, $\wedge$ and $\diamond$, then for any $x \in X$, the following assertions are equivalent:*

(1) $x \in X_{\diamond}$; *(2)* $x^{\vee} = x^{\wedge}$;

(3) $x^{\vee} \in X_{\diamond}$; *(4)* $x^{\wedge} \in X_{\diamond}$; *(5)* $x^{\diamond} \in X_{\diamond}$.

Now, as an immediate consequence of the latter theorem and Example 4.4, we can also state

Theorem 4.8. *For any relation F on one group U to another V, the following assertions are equivalent:*

(1) F *is odd;* *(2)* $F = F^{\diamond}$; *(3)* $F^{\vee} = F^{\wedge}$;

(3) $F^{\vee}$ *is odd;* *(4)* $F^{\wedge}$ *is odd;* *(5)* $F^{\diamond}$ *is odd.*

Hence, it is clear that in particular, we also have

Corollary 4.9. *If f is an additive function of one group X to another Y, then* $f = f^{\diamond}$.

Remark 4.10. The latter statements, under different notation, have already been established in our former paper [3].

5. Posets with inversions

Definitin 5.1. A poset X, with inversions $\vee$, $\wedge$ and $\diamond$, is said to be of

(1) $\uparrow\uparrow$–type if both $\vee$ and $\wedge$ are increasing;

(2) $\uparrow\downarrow$–type if $\vee$ is increasing and $\wedge$ is decreasing.

Some further similar types of posets with inversions are to be defined analogously.

Example 5.2. If U and V are groups, then the family $\mathcal{P}(U, V)$ of all relations on U to V, equipped with the ordinary set inclusion and the operations $\vee$, $\wedge$ and $\diamond$ defined in Example 3.8, is an $\uparrow\uparrow$–type poset with inversions.

Example 5.3. If U is a group, then the family $\mathcal{P}(U^2)$ of all relations on U, equipped with the ordinary set inclusion and the operations -1, $\diamond$ and $\#$ considered in Example 3.10, is also an $\uparrow\uparrow$–type poset with inversions.

Example 5.4. If U is a group and V is a partially ordered group, then the family V^U of all functions of U to V, equipped with the pointwise inequality and the operations $\vee$, $\wedge$ and $\diamond$ defined in Example 3.5, is an $\uparrow\downarrow$–type poset with inversions.

Example 5.5. The family $\mathbb{R}$ of all real numbers, equipped with the usual inequality and the operations $\vee$, $\wedge$ and $\diamond$ defined in Example 3.2, is a $\downarrow\downarrow$–type poset with inversions.

Example 5.6. If the family $\mathbb{C}$ of all complex numbers is equipped with either the coordinate-wise inequality or the lexicographic order considered in Section 2, then the operations $\vee$ and $\wedge$ defined in Example 3.3 are not monotonic. However, the operation $\diamond$ defined there is decreasing.

Namely, if for instance $z = (0, 0)$ and $w = (1, 0)$, then $z < w$, $w^{\vee} = (-1, 0) < (0, 0) = z^{\vee}$ and $z^{\wedge} = (0, 0) < (1, 0) = w^{\wedge}$. Thus, $\vee$ is not increasing and $\wedge$ is not decreasing.

Moreover, if for instance $\omega = (0, 1)$, then $z < \omega$, $z^{\vee} = (0, 0) < (0, 1) = \omega^{\vee}$ and $\omega^{\wedge} = (0, -1) < (0, 0) = z^{\wedge}$. Thus, $\vee$ is not decreasing and $\wedge$ is not increasing.

Now, as some immediate consequences of Definition 5.1 and Theorems 3.12, we can also state the following two theorems.

Theorem 5.7. *If X is an $\uparrow\uparrow$–type poset with inversions $\vee$, $\wedge$ and $\diamond$, then*

(1) X is an $\uparrow\uparrow$–type poset with inversions $\wedge$, $\vee$ and $\diamond$;

(2) X is an $\uparrow\uparrow$–type poset with inversions $\vee$, $\diamond$ and $\wedge$.

Theorem 5.8. *If X is an $\uparrow\downarrow$–type poset with inversions $\vee$, $\wedge$ and $\diamond$, then*

(1) X is a $\downarrow\uparrow$–type poset with inversions $\wedge$, $\vee$ and $\diamond$;

(2) X is also an $\uparrow\downarrow$–type poset with inversions $\vee$, $\diamond$ and $\wedge$.

Moreover, an immediate consequence Theorem 3.14 and Corollary 2.4 , we can also at once state the following

Theorem 5.9. *If* X *is an* $\uparrow\uparrow$*–type poset with inversions* $^\vee$, $^\wedge$ *and* $^\diamond$, *then for any* $x, y \in X$ *and* $\square \in \{ \vee, \wedge, \diamond \}$, *we have*

(1) $(x \wedge y)^\square = x^\square \wedge y^\square$ *if* X *is a meet-semilattice;*

(2) $(x \vee y)^\square = x^\square \vee y^\square$ *if* X *is a join-semilattice.*

Furthermore, by using Theorems 3.14 and a dual of Theorem 2.3, we can also easily establish the following

Theorem 5.10. *If* X *is an* $\uparrow\downarrow$*–type lattice with inversions* $^\vee$, $^\wedge$ *and* $^\diamond$, *then for any* $x, y \in X$ *and* $\square \in \{ \wedge, \diamond \}$, *we have*

(1) $(x \wedge y)^\vee = x^\vee \wedge y^\vee$; (2) $(x \vee y)^\vee = x^\vee \vee y^\vee$;

(3) $(x \wedge y)^\square = x^\square \vee y^\square$; (4) $(x \vee y)^\square = x^\square \wedge y^\square$.

6. Compound operations on meet-semilattices with inversions

Definition 6.1. If X is a meet-semilattice with inversions $^\vee$, $^\wedge$ and $^\diamond$, then for any $x \in X$ we write

$$x_\blacktriangledown = x \wedge x^\vee, \qquad x_\blacktriangle = x \wedge x^\wedge, \qquad x_\blacklozenge = x \wedge x^\diamond;$$
$$x_\bullet = x^\vee \wedge x^\wedge, \qquad x_\clubsuit = x^\vee \wedge x^\diamond, \qquad x_\spadesuit = x^\wedge \wedge x^\diamond.$$

Remark 6.2. If X is a join-semilattice with inversions $^\vee$, $^\wedge$ and $^\diamond$, then for any $x \in X$ we may naturally write $x^\blacktriangledown = x \vee x^\vee$.

Namely, if in particular X is a lattice with inversions $^\vee$, $^\wedge$ and $^\diamond$, then for any $x \in X$ we have $x_\blacktriangledown \leq x \leq x^\blacktriangledown$.

Concerning the above operations, we can easily prove the following

Theorem 6.3. *If* X *is a meet-semilattice with inversions* $^\vee$, $^\wedge$ *and* $^\diamond$ *such that* $^\vee$ *is increasing, then for any* $x \in X$ *we have*

(1) $x_\blacktriangledown = (x^\vee)_\blacktriangledown = (x_\blacktriangledown)^\vee$; (2) $x_\clubsuit = (x^\vee)_\blacktriangle = (x_\blacktriangle)^\vee$;

(3) $x_\bullet = (x^\vee)_\blacklozenge = (x_\blacklozenge)^\vee$; (4) $x_\blacklozenge = (x^\vee)_\bullet = (x_\bullet)^\vee$;

(5) $x_\blacktriangle = (x^\vee)_\clubsuit = (x_\clubsuit)^\vee$; (6) $x_\spadesuit = (x^\vee)_\spadesuit = (x_\spadesuit)^\vee$.

Proof. By the corresponding definitions and Theorem 3.12, we have

$$(x^\vee)_\blacktriangledown = x^\vee \wedge x^{\vee\vee} = x^\vee \wedge x = x_\blacktriangledown; \qquad (x^\vee)_\blacktriangle = x^\vee \wedge x^{\vee\wedge} = x^\vee \wedge x^\diamond = x_\clubsuit;$$
$$(x^\vee)_\blacklozenge = x^\vee \wedge x^{\vee\diamond} = x^\vee \wedge x^\wedge = x_\bullet; \qquad (x^\vee)_\spadesuit = x^{\vee\wedge} \wedge x^{\vee\diamond} = x^\diamond \wedge x^\wedge = x_\spadesuit.$$

Moreover, by Theorem 3.14 and Corollary 2.4, we also have

$$(x_\blacktriangledown)^\vee = (x \wedge x^\vee)^\vee = x^\vee \wedge x^{\vee\vee} = x^\vee \wedge x = x_\blacktriangledown;$$
$$(x_\blacktriangle)^\vee = (x \wedge x^\wedge)^\vee = x^\vee \wedge x^{\wedge\vee} = x^\vee \wedge x^\diamond = x_\clubsuit;$$
$$(x_\blacklozenge)^\vee = (x \wedge x^\diamond)^\vee = x^\vee \wedge x^{\diamond\vee} = x^\vee \wedge x^\wedge = x_\bullet;$$
$$(x_\spadesuit)^\vee = (x^\wedge \wedge x^\diamond)^\vee = x^{\wedge\vee} \wedge x^{\diamond\vee} = x^\diamond \wedge x^\wedge = x_\spadesuit.$$

Therefore, assertions (1)–(3) and (6) are true.

Moreover, from (2) and (3), we can immediately infer that

$$(x_{\clubsuit})^{\vee} = (x_{\blacktriangle})^{\vee\vee} = x_{\blacktriangle}\,; \qquad (x_{\bullet})^{\vee} = (x_{\blacklozenge})^{\vee\vee} = x_{\blacklozenge}\,;$$

$$(x^{\vee})_{\clubsuit} = (x^{\vee\vee})_{\blacktriangle} = x_{\blacktriangle}\,; \qquad (x^{\vee})_{\bullet} = (x^{\vee\vee})_{\blacklozenge} = x_{\blacklozenge}\,.$$

Therefore, assertions (5) and (4) are also true.

Analogously to the above theorem, we can also easily prove the following theorems which can also be derived from Theorem 6.3 by using Theorem 3.12.

Theorem 6.4. *If X is a meet-semilattice with inversions $\vee$, $\wedge$ and $\diamond$ such that $\wedge$ is increasing, then*

(1) $x_{\spadesuit} = (x^{\wedge})_{\blacktriangledown} = (x_{\blacktriangledown})^{\wedge}$; *(2)* $x_{\blacktriangle} = (x^{\wedge})_{\blacktriangle} = (x_{\blacktriangle})^{\wedge}$;

(3) $x_{\bullet} = (x^{\wedge})_{\blacklozenge} = (x_{\blacklozenge})^{\wedge}$; *(4)* $x_{\blacklozenge} = (x^{\wedge})_{\bullet} = (x_{\bullet})^{\wedge}$;

(5) $x_{\clubsuit} = (x^{\wedge})_{\clubsuit} = (x_{\clubsuit})^{\wedge}$; *(6)* $x_{\blacktriangledown} = (x^{\wedge})_{\spadesuit} = (x_{\spadesuit})^{\wedge}$.

Theorem 6.5. *If X is a meet-semilattice with inversions $\vee$, $\wedge$ and $\diamond$ such that $\diamond$ is increasing, then*

(1) $x_{\spadesuit} = (x^{\diamond})_{\blacktriangledown} = (x_{\blacktriangledown})^{\diamond}$; *(2)* $x_{\clubsuit} = (x^{\diamond})_{\blacktriangle} = (x_{\blacktriangle})^{\diamond}$;

(3) $x_{\blacklozenge} = (x^{\diamond})_{\blacklozenge} = (x_{\blacklozenge})^{\diamond}$; *(4)* $x_{\bullet} = (x^{\diamond})_{\bullet} = (x_{\bullet})^{\diamond}$;

(5) $x_{\blacktriangle} = (x^{\diamond})_{\clubsuit} = (x_{\clubsuit})^{\diamond}$; *(6)* $x_{\blacktriangledown} = (x^{\diamond})_{\spadesuit} = (x_{\spadesuit})^{\diamond}$.

7. A further important operation on meet-semilattices with inversions

Definition 7.1. If X is a meet-semilattice with inversions $\vee$, $\wedge$ and $\diamond$, then for any $x \in X$ we write

$$x_{\bigstar} = x \wedge x^{\vee} \wedge x^{\wedge} \wedge x^{\diamond}\,.$$

Remark 7.2. If X is a join-semilattice with inversions $\vee$, $\wedge$ and $\diamond$, then for any $x \in X$ we may also naturally write $x^{\bigstar} = x \vee x^{\vee} \vee x^{\wedge} \vee x^{\diamond}$.

A simple computation gives the following

Theorem 7.3. *If X is a meet-semilattice with inversions $\vee$, $\wedge$ and $\diamond$, then for any $x \in X$ we have*

$$x_{\bigstar} = x_{\blacktriangledown} \wedge x_{\spadesuit} = x_{\blacktriangle} \wedge x_{\clubsuit} = x_{\bullet} \wedge x_{\blacklozenge}\,.$$

Proof. By the corresponding definitions and the commutativity and associativity of the operation $\wedge$, we have

$$x_{\bigstar} = x \wedge x^{\vee} \wedge x^{\wedge} \wedge x^{\diamond} = (x^{\vee} \wedge x^{\wedge}) \wedge (x \wedge x^{\diamond}) = x_{\bullet} \wedge x_{\blacklozenge}\,.$$

The proof of the other equalities are even more obvious.

Now, in addition to Theorems 6.3–6.5, we can easily prove the following

Theorem 7.4. *If X is an $\uparrow\uparrow$–type meet-semilattice with inversions $\vee$, $\wedge$ and $\diamond$, then for any $x \in X$ we have*

(1) $x_{\blacktriangledown} = x_{\blacktriangledown\blacktriangledown}$; (2) $x_{\spadesuit} = x_{\blacktriangledown\spadesuit} = x_{\spadesuit\blacktriangledown}$;

(3) $x_{\bigstar} = x_{\blacktriangledown\blacksquare} = x_{\blacksquare\blacktriangledown}$ *with* $\blacksquare \in \{ \blacktriangle , \blacklozenge , \bullet , \clubsuit \}$.

Proof. By the corresponding definitions and Theorems 6.3–6.5 and 7.3, we have

$x_{\blacktriangledown\blacktriangledown} = x_{\blacktriangledown} \wedge (x_{\blacktriangledown})^{\vee} = x_{\blacktriangledown} \wedge x_{\blacktriangledown} = x_{\blacktriangledown}$; $x_{\blacktriangledown\blacktriangle} = x_{\blacktriangledown} \wedge (x_{\blacktriangledown})^{\wedge} = x_{\blacktriangledown} \wedge x_{\spadesuit} = x_{\bigstar}$;

$x_{\blacktriangledown\blacklozenge} = x_{\blacktriangledown} \wedge (x_{\blacktriangledown})^{\diamond} = x_{\blacktriangledown} \wedge x_{\spadesuit} = x_{\bigstar}$; $x_{\blacktriangledown\bullet} = (x_{\blacktriangledown})^{\vee} \wedge (x_{\blacktriangledown})^{\wedge} = x_{\blacktriangledown} \wedge x_{\spadesuit} = x_{\bigstar}$;

$x_{\blacktriangledown\clubsuit} = (x_{\blacktriangledown})^{\vee} \wedge (x_{\blacktriangledown})^{\diamond} = x_{\blacktriangledown} \wedge x_{\spadesuit} = x_{\bigstar}$; $x_{\blacktriangledown\spadesuit} = (x_{\blacktriangledown})^{\wedge} \wedge (x_{\blacktriangledown})^{\diamond} = x_{\spadesuit} \wedge x_{\spadesuit} = x_{\spadesuit}$;

and quite similarly

$x_{\blacktriangle\blacktriangledown} = x_{\blacktriangle} \wedge (x_{\blacktriangle})^{\vee} = x_{\blacktriangle} \wedge x_{\clubsuit} = x_{\bigstar}$;

$x_{\blacklozenge\blacktriangledown} = x_{\blacklozenge} \wedge (x_{\blacklozenge})^{\vee} = x_{\blacklozenge} \wedge x_{\bullet} = x_{\bigstar}$; $x_{\bullet\blacktriangledown} = x_{\bullet} \wedge (x_{\bullet})^{\vee} = x_{\bullet} \wedge x_{\blacklozenge} = x_{\bigstar}$;

$x_{\clubsuit\blacktriangledown} = x_{\clubsuit} \wedge (x_{\clubsuit})^{\vee} = x_{\clubsuit} \wedge x_{\blacktriangle} = x_{\bigstar}$; $x_{\spadesuit\blacktriangledown} = x_{\spadesuit} \wedge (x_{\spadesuit})^{\vee} = x_{\spadesuit} \wedge x_{\spadesuit} = x_{\spadesuit}$.

Analogously to Theorem 7.3, we can also easily prove the following theorems.

Theorem 7.5. *If X is an $\uparrow\uparrow$–type meet-semilattice with inversions $\vee$, $\wedge$ and $\diamond$, then for any $x \in X$ we have*

(1) $x_{\blacktriangle} = x_{\blacktriangle\blacktriangle}$; (2) $x_{\clubsuit} = x_{\blacktriangle\clubsuit} = x_{\clubsuit\blacktriangle}$;

(3) $x_{\bigstar} = x_{\blacktriangle\blacksquare} = x_{\blacksquare\blacktriangle}$ *with* $\blacksquare \in \{ \blacklozenge , \bullet , \spadesuit \}$.

Theorem 7.6. *If X is an $\uparrow\uparrow$–type meet-semilattice with inversions $\vee$, $\wedge$ and $\diamond$, then for any $x \in X$ we have*

(1) $x_{\blacklozenge} = x_{\blacklozenge\blacklozenge}$; (2) $x_{\bullet} = x_{\blacklozenge\bullet} = x_{\bullet\blacklozenge}$;

(3) $x_{\bigstar} = x_{\blacklozenge\blacksquare} = x_{\blacksquare\blacklozenge}$ *with* $\blacksquare \in \{ \clubsuit , \spadesuit \}$.

Theorem 7.7. *If X is an $\uparrow\uparrow$–type meet-semilattice with inversions $\vee$, $\wedge$ and $\diamond$, then for any $x \in X$ we have*

(1) $x_{\blacklozenge} = x_{\bullet\bullet}$; (2) $x_{\bigstar} = x_{\bullet\blacksquare} = x_{\blacksquare\bullet}$ *with* $\blacksquare \in \{ \clubsuit , \spadesuit \}$.

Theorem 7.8. *If X is an $\uparrow\uparrow$–type meet-semilattice with inversions $\vee$, $\wedge$ and $\diamond$, then for any $x \in X$ we have*

(1) $x_{\blacktriangle} = x_{\clubsuit\clubsuit}$; (2) $x_{\bigstar} = x_{\clubsuit\spadesuit} = x_{\spadesuit\clubsuit}$; (3) $x_{\blacktriangledown} = x_{\spadesuit\spadesuit}$.

Finally, we note that the following theorem is also true.

Theorem 7.9. *If X is an $\uparrow\uparrow$–type meet-semilattice with inversions $\vee$, $\wedge$ and $\diamond$, then for any $x \in X$ we have*

(1) $x_{\bigstar} = \left(x^{\square}\right)_{\bigstar} = \left(x_{\bigstar}\right)^{\square}$ *with* $\square \in \{ \vee , \wedge , \diamond \}$;

(2) $x_{\bigstar} = x_{\blacksquare\bigstar} = x_{\bigstar\blacksquare}$ *with* $\blacksquare \in \{ \blacktriangledown , \blacktriangle , \blacklozenge , \bullet , \clubsuit , \spadesuit , \bigstar \}$.

8. Maximality properties of the compound operations

Theorem 8.1. *If X is a meet-semilattice with inversions $\vee$, $\wedge$ and $\diamond$, such that $\vee$ is increasing, then for any $x \in X$ we have*

$$x_{\blacktriangledown} = \max \{ y \in X_{\vee} : \quad y \leq x \}.$$

Proof. Define $A = \{ y \in X_{\vee} : \quad y \leq x \}$. Then, we can easily see that $x_{\blacktriangledown} \in A$. Namely, by the corresponding definitions, we have $x_{\blacktriangledown} = x \wedge x^{\vee} \leq x$. Moreover, by Theorem 6.3, we also have $x_{\blacktriangledown} = (x_{\blacktriangledown})^{\vee}$, and thus $x_{\blacktriangledown} \in X_{\vee}$.

On the other hand, if $y \in A$, then we have $y \in X_{\vee}$ and $y \leq x$. Hence, by using the corresponding definitions, we can infer that $y = y^{\vee} \leq x^{\vee}$. Therefore, $y \leq x \wedge x^{\vee} = x_{\blacktriangledown}$, and thus $x_{\blacktriangledown} \in \mathrm{ub}(A)$. This shows that $x_{\blacktriangledown} = \max(A)$, and thus the required equality is also true.

From the above theorem, by using Theorems 6.3–6.5 and 7.4 and 7.9, we can immediately derive the following corollaries.

Corollary 8.2. *If X is a meet-semilattice with inversions $\vee$, $\wedge$ and $\diamond$, such that $\vee$ is increasing, then for any $x \in X$ we have*

(1) $x_{\blacktriangledown} = \max \{ y \in X_{\vee} : \quad y \leq x^{\vee} \}$ *;*

(2) $x_{\spadesuit} = \max \{ y \in X_{\vee} : \quad y \leq x^{\square} \}$ *with* $\square \in \{ \wedge, \diamond \}$.

Corollary 8.3. *If X is an $\uparrow\uparrow$–type meet-semilattice with inversions $\vee$, $\wedge$ and $\diamond$, then for any $x \in X$ we have*

(1) $x_{\blacktriangledown} = \max \{ y \in X_{\vee} : \quad y \leq x_{\blacktriangledown} \}$ *;*

(2) $x_{\bigstar} = \max \{ y \in X_{\vee} : \quad y \leq x_{\blacksquare} \}$ *with* $\blacksquare \in \{ \blacktriangle, \blacklozenge, \bullet, \clubsuit, \bigstar \}$.

Analogously to the above theorem, we can also easily prove the following theorems which can also be derived from Theorem 8.1 by using Theorem 3.12.

Theorem 8.4. *If X is a meet-semilattice with inversions $\vee$, $\wedge$ and $\diamond$, such that $\wedge$ is increasing, then for any $x \in X$ we have*

$$x_{\blacktriangle} = \max \{ y \in X_{\wedge} : \quad y \leq x \}.$$

Hence, by Theorems 6.4, 6.3 and 7.5 and 7.9, it is clear that we also have the following corollaries.

Corollary 8.5. *If X is a meet-semilattice with inversions $\vee$, $\wedge$ and $\diamond$, such that $\wedge$ is increasing, then for any $x \in X$ we have*

(1) $x_{\blacktriangle} = \max \{ y \in X_{\wedge} : \quad y \leq x^{\wedge} \}$ *;*

(2) $x_{\clubsuit} = \max \{ y \in X_{\wedge} : \quad y \leq x^{\square} \}$ *with* $\square \in \{ \vee, \diamond \}$.

Corollary 8.6. *If X is an $\uparrow\uparrow$–type meet-semilattice with inversions $\vee$, $\wedge$ and $\Diamond$, then for any $x \in X$ we have*

(1) $x_{\blacktriangle} = \max\left\{ y \in X_{\wedge} : \quad y \le x_{\blacktriangle} \right\}$;

(2) $x_{\bigstar} = \max\left\{ y \in X_{\wedge} : \quad y \le x_{\blacksquare} \right\}$ *with* $\blacksquare \in \{ \blacklozenge, \bullet, \spadesuit, \bigstar \}$.

Theorem 8.7. *If X is a meet-semilattice with inversions $\vee$, $\wedge$ and $\Diamond$, such that $\Diamond$ is increasing, then for any $x \in X$ we have*

$$x_{\blacklozenge} = \max\left\{ y \in X_{\Diamond} : \quad y \le x \right\}.$$

Hence, by Theorems 6.5, 6.3, 6.4 and 7.6, it is clear that we also have the following corollaries.

Corollary 8.8. *If X is a meet-semilattice with inversions $\vee$, $\wedge$ and $\Diamond$, such that $\Diamond$ is increasing, then for any $x \in X$ we have*

(1) $x_{\blacklozenge} = \max\left\{ y \in X_{\Diamond} : \quad y \le x^{\Diamond} \right\}$;

(2) $x_{\bullet} = \max\left\{ y \in X_{\Diamond} : \quad y \le x^{\Box} \right\}$ *with* $\Box \in \{ \vee, \wedge \}$.

Corollary 8.9. *If X is an $\uparrow\uparrow$–type meet-semilattice with inversions $\vee$, $\wedge$ and $\Diamond$, then for any $x \in X$ we have*

(1) $x_{\blacklozenge} = \max\left\{ y \in X_{\Diamond} : \quad y \le x_{\blacklozenge} \right\}$;

(2) $x_{\bigstar} = \max\left\{ y \in X_{\Diamond} : \quad y \le x_{\blacksquare} \right\}$ *with* $\blacksquare \in \{ \clubsuit, \spadesuit, \bigstar \}$.

9. Some further results on the fixed points of the operations $\vee$, $\wedge$ and $\Diamond$

Theorem 9.1. *If $\Box$ is an increasing involution on a poset X, then*

$$X_{\Box} = \left\{ x \in X : \quad x \le x^{\Box} \right\} = \left\{ x \in X : \quad x^{\Box} \le x \right\}.$$

Proof. Define $A = \left\{ x \in X : \quad x \le x^{\Box} \right\}$. Then, by the definition of $X_{\Box}$ and the reflexivity of the inequality in X, it is clear that $X_{\Box} \subset A$.

Moreover, if $x \in A$, then we have $x \in X$ and $x \le x^{\Box}$. Hence, by using the corresponding properties of $\vee$, we can infer that $x^{\Box} \le x^{\Box\Box} = x$. Therefore, we actually have $x = x^{\Box}$, and thus $x \in X_{\Box}$. This shows that $A \subset X_{\Box}$.

Therefore, $X_{\Box} = A$, and thus the first part of the theorem is true. The second part of the theorem can be proved quite similarly.

Theorem 9.2. *If $\Box$ is an increasing involution on a meet-semilattice X and $x_{\blacksquare} = x \wedge x^{\Box}$ for all $x \in X$, then $X_{\Box} = X_{\blacksquare}$.*

Proof. If $x \in X_{\Box}$, then $x_{\blacksquare} = x \wedge x^{\Box} = x \wedge x = x$, and thus $x \in X_{\blacksquare}$. Therefore, $X_{\Box} \subset X_{\blacksquare}$.

While, if $x \in X_{\blacksquare}$, then $x = x_{\blacksquare} = x \wedge x^{\Box} \le x^{\Box}$. Hence, by Theorem 9.1, we can see that $x \in X_{\Box}$. Therefore, $X_{\blacksquare} \subset X_{\Box}$ is also true.

Remark 9.3. Note that if X and $\square$ are as in the above theorems, then X is already an $\uparrow\uparrow$–type poset with inversions either $\square$, Δ_X and $\square$ or $\square$, $\square$ and $\square$.

Therefore, the above theorems can be immediately derived from the corresponding results for the operation $\vee$.

Theorem 9.4. *If X is an $\uparrow\uparrow$–type meet-semilattice with inversions $\vee$, $\wedge$ and $\diamond$, then for any $x \in X$ the following assertions are equivalent:*

(1) $x \in X_{\vee}$; *(2)* $x^{\vee} = x_{\blacktriangledown}$; *(3)* $x^{\wedge} = x_{\spadesuit}$; *(4)* $x^{\diamond} = x_{\spadesuit}$.

Proof. By Theorems 4.5, 9.2 and 6.3–6.5, we can see that

$$x \in X_{\vee} \iff x^{\vee} \in X_{\vee} \iff x^{\vee} \in X_{\blacktriangledown} \iff x^{\vee} = (x^{\vee})_{\blacktriangledown} \iff x^{\vee} = x_{\blacktriangledown};$$
$$x \in X_{\vee} \iff x^{\wedge} \in X_{\vee} \iff x^{\wedge} \in X_{\blacktriangledown} \iff x^{\wedge} = (x^{\wedge})_{\blacktriangledown} \iff x^{\wedge} = x_{\spadesuit};$$
$$x \in X_{\vee} \iff x^{\diamond} \in X_{\vee} \iff x^{\diamond} \in X_{\blacktriangledown} \iff x^{\diamond} = (x^{\diamond})_{\blacktriangledown} \iff x^{\diamond} = x_{\spadesuit}.$$

Analogously to the above theorem, we can also easily prove the following theorems which can also be derived from Theorem 9.4 by using Theorem 3.12.

Theorem 9.5. *If X is an $\uparrow\uparrow$–type meet-semilattice with inversions $\vee$, $\wedge$ and $\diamond$, then for any $x \in X$ the following assertions are equivalent:*

(1) $x \in X_{\wedge}$; *(2)* $x^{\wedge} = x_{\blacktriangle}$; *(3)* $x^{\wedge} = x_{\clubsuit}$; *(4)* $x^{\diamond} = x_{\clubsuit}$.

Theorem 9.6. *If X is an $\uparrow\uparrow$–type meet-semilattice with inversions $\vee$, $\wedge$ and $\diamond$, then for any $x \in X$ the following assertions are equivalent:*

(1) $x \in X_{\diamond}$; *(2)* $x^{\diamond} = x_{\blacklozenge}$; *(3)* $x^{\wedge} = x_{\bullet}$; *(4)* $x^{\diamond} = x_{\bullet}$.

Finally, we note that, in addition to Theorem 9.2, the following theorem can also be easily proved.

Theorem 9.7. *If X is an $\uparrow\uparrow$–type meet-semilattice with inversions $\vee$, $\wedge$ and $\diamond$, then*

(1) $X_{\bullet} = X_{\vee} \cap X_{\wedge}$; *(2)* $X_{\clubsuit} = X_{\vee} \cap X_{\diamond}$;

(3) $X_{\spadesuit} = X_{\wedge} \cap X_{\diamond}$; *(4)* $X_{\star} = X_{\vee} \cap X_{\wedge} \cap X_{\diamond}$.

Proof. If $x \in X_{\vee} \cap X_{\wedge}$, then $x \in X_{\vee}$ and $x \in X_{\wedge}$. Therefore, $x_{\bullet} = x^{\vee} \cap x^{\wedge} = x \cap x = x$, and thus $x \in X_{\bullet}$. Consequently, $X_{\vee} \cap X_{\wedge} \subset X_{\bullet}$.

While, if $x \in X_{\bullet}$, then $x = x_{\bullet} = x^{\vee} \wedge x^{\wedge}$. Therefore, $x \leq x^{\vee}$ and $x \leq x^{\wedge}$. Hence, by using Theorem 9.1, we can infer that $x \in X_{\vee}$ and $x \in X_{\wedge}$, and thus $x \in X_{\vee} \cap X_{\wedge}$. Consequently, $X_{\bullet} \subset X_{\vee} \cap X_{\wedge}$ also holds.

Therefore, (1) is true. The proofs of (2)–(4) are quite similar.

Acknowledgement. The author is indebted to Gábor Horváth for several helpful discussions and suggested corrections.

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