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RELATION THEORETIC OPERATIONS ON BOX AND TOTALIZATION RELATIONS

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Abstract. In this paper, we shall only study the most simple relation theoretic operations on box and totalization relations

$$\Gamma_{(A,\,B)} = A \times B \qquad \text{and} \qquad \tilde{F} = F \cup \Gamma_{(D_F^c,\,Y)},$$

where $A \subset X$, $B \subset Y$ and F is a relation on X to Y with domain D_F. The intersection convolutions, and several algebraic and topological properties of these relations, will be studied elsewhere.

This line of investigations is mainly motivated by the fact that the relations

$$\tilde{\Gamma}_{(A,\,B)} = \widetilde{\Gamma_{(A,\,B)}} \qquad \text{and} \qquad \tilde{\Gamma}_A = \tilde{\Gamma}_{(A,\,A)}$$

play an important role in the uniformization of various topological structures such as proximities, closures and topologies, for instance. Moreover, the relations $\tilde{F}$ can be used to prove a useful reduction theorem for the intersection convolution of relations. The latter operation allows of a natural treatment of the Hahn-Banach type extension theorems.

1. Set theoretic operations on relations

A subset F of a product set $X{\times}Y$ is called a relation on X to Y. If in particular F is a relation on X to itself, then we may simply say that F is a relation on X. Thus, a relation F on X to Y is also a relation on $X \cup Y$.

If F is a relation on X to Y, then for any $x \in X$ and $A \subset X$ the sets $F(x) = \{\, y \in Y : \ (x,\,y) \in F\}$ and $F[\,A\,] = \bigcup_{a\in A} F(a)$ are called the images of x and A under F, respectively.

Moreover, the sets $D_F = \{\, x \in X : \ F(x) \neq \emptyset\,\}$ and $R_F = F[\,X\,]$ are called the domain and range of F, respectively. If in particular $X = D_F$, then we say that F is a relation of X to Y, or that F is a total relation on X to Y. While, if $Y = R_F$, then we say that F is a relation on X onto Y.

If F is a relation on X to Y and $U \subset D_F$, then the relation $F\,|\,U = F \cap (U{\times}Y)$ is called the restriction of F to U. Moreover, if F and G are relations on X to Y such that $D_F \subset D_G$ and $F = G\,|\,D_F$, then G is called an extension of F.

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In particular, a relation f on X to Y is called a function if for each $x \in D_f$ there exists $y \in Y$ such that $f(x) = \{y\}$. In this case, by identifying singletons with their elements, we may simply write $f(x) = y$ instead of $f(x) = \{y\}$.

Concerning relations, we shall only quote here the following basic theorems from [17].

Theorem 1.1. *If F is a relation on X to Y, then*

$$F = \bigcup_{x \in X} \{x\} \times F(x) = \bigcup_{x \in D_F} \{x\} \times F(x).$$

Remark 1.2. By this theorem, a relation F on X to Y can be naturally defined by specifying $F(x)$ for all $x \in X$, or by specifying D_F and $F(x)$ for all $x \in D_F$.

Corollary 1.3. *If F and G are relations on X to Y, then the following assertions are equivalent:*

(1) $F \subset G$;

(2) $F(x) \subset G(x)$ *for all* $x \in X$; *(3)* $F(x) \subset G(x)$ *for all* $x \in D_F$.

Corollary 1.4. *If F and G are relations on X to Y, then the following assertions are equivalent:*

(1) $F = G$;

(2) $F(x) = G(x)$ *for all* $x \in X$;

(3) $D_F = D_G$ *and* $F(x) = G(x)$ *for all* $x \in D_F$.

Theorem 1.5. *If F is a relation on X to Y, then for any $A, B \subset X$ we have*

(1) $F[A \cap B] \subset F[A] \cap F[B]$; *(2)* $F[A \cup B] = F[A] \cup F[B]$.

Theorem 1.6. *If F is a relation on X to Y, then for any $A, B \subset X$ we have*

$$F[A] \setminus F[B] \subset F[A \setminus B].$$

Corollary 1.7. *If F is a relation on X onto Y, then for any $A \subset X$ we have*

$$F[A]^c \subset F[A^c].$$

Remark 1.8. If in particular the inverse F^{-1} of F, defined in the next section, is a function, then the equality also holds in Theorems 1.5 and 1.6 and Corollary 1.7.

Theorem 1.9. *If F and G are relations on X to Y, then for any $x \in X$ we have*

(1) $\big(F \cap G\big)(x) = F(x) \cap G(x)$; *(2)* $\big(F \cup G\big)(x) = F(x) \cup G(x)$.

Theorem 1.10. *If F and G are relations on X to Y, then for any $A \subset X$ we have*

(1) $\big(F \cap G\big)[A] \subset F[A] \cap G[A]$; *(2)* $\big(F \cup G\big)[A] = F[A] \cup G[A]$.

Remark 1.11. Theorems 1.5, 1.9 and 1.10 can be naturally extended to arbitrary families of sets and relations.

Theorem 1.12. *If F and G are relations on X to Y, then for any $x \in X$ and $A \subset X$ we have*

(1) $(F \setminus G)(x) = F(x) \setminus G(x)$; *(2)* $F[A] \setminus G[A] \subset (F \setminus G)[A]$.

Corollary 1.13. *If F is a relation on X to Y, then for any $x \in X$ and $A \subset X$, with $A \neq \emptyset$, we have*

(1) $F^c(x) = F(x)^c$; *(2)* $F[A]^c \subset F^c[A]$.

Theorem 1.14. *If F is a relation on X to Y, then for any $A \subset X$ we have*

$$F^c[A]^c = \bigcap_{x \in A} F(x).$$

2. Relation theoretic operations on relations

If F is a relation on X to Y, then the relation

$$F^{-1} = \big\{ (y, x) : \quad (x, y) \in F \big\}$$

is called the inverse of F.

Moreover, if G is a relation on Y to Z, then the relation

$$G \circ F = \big\{ (x, z) : \quad \exists\ y \in Y : \quad (x, y) \in F, \quad (y, z) \in G \big\}$$

is called the composition of G and F.

Concerning inverse and composite relations, we shall only quote here the following basic theorems.

Theorem 2.1. *If F is a relation on X to Y, then for any $x \in X$ and $y \in Y$ the following assertions are equivalent:*

(1) $x \in F^{-1}(y)$; *(2)* $y \in F(x)$.

Theorem 2.2. *If F is a relation on X to Y, then for any $x \in X$ and $B \subset Y$ the following assertions are equivalent:*

(1) $x \in F^{-1}[B]$; *(2)* $F(x) \cap B \neq \emptyset$.

Theorem 2.3. *If F is a relation on X to Y, then for any $A \subset X$ and $B \subset Y$ the following assertions are equivalent:*

(1) $A \cap F^{-1}[B] \neq \emptyset$; *(2)* $F[A] \cap B \neq \emptyset$.

Theorem 2.4. *If F and G are relations on X to Y, then*

(1) $\big(F \cap G\big)^{-1} = F^{-1} \cap G^{-1}$; *(2)* $\big(F \cup G\big)^{-1} = F^{-1} \cup G^{-1}$.

Theorem 2.5. *If F and G are relations on X to Y, then*

$$\big(F \setminus G\big)^{-1} = F^{-1} \setminus G^{-1}.$$

Corollary 2.6. *If F is a relation on X to Y, then*

$$\big(F^c\big)^{-1} = \big(F^{-1}\big)^c.$$

Theorem 2.7. *If F is a relation on X to Y and G is a relation on Y to Z, then for any $x \in X$ and $A \subset X$ we have*

(1) $\big(G \circ F\big)(x) = G\big[F(x)\big]$; *(2)* $\big(G \circ F\big)[A] = G\big[F[A]\big]$.

Theorem 2.8. *If F is a relation on X to Y and G is a relation on Y to Z, then*

$$(G \circ F)^{-1} = F^{-1} \circ G^{-1}.$$

Theorem 2.9. *If F and G are relations on X to Y and H is a relation on Y to Z, then*

(1) $H \circ \big(F \cap G\big) \subset \big(H \circ F\big) \cap \big(H \circ G\big)$; *(2)* $H \circ \big(F \cup G\big) = \big(H \circ F\big) \cup \big(H \circ G\big)$.

Theorem 2.10. *If F is a relation on X to Y and G and H are relations on Y to Z, then*

(1) $\big(G \cap H\big) \circ F \subset \big(G \circ F\big) \cap \big(H \circ F\big)$; *(2)* $\big(G \cup H\big) \circ F = \big(G \circ F\big) \cup \big(H \circ F\big)$.

Hint. The latter two theorems can be most easily proved with the help of Theorems 2.7 and 1.10 and Corollary 1.4.

Remark 2.11. Theorems 2.4, 2.9 and 2.10 can be naturally to arbitrary families of relations.

Theorem 2.12. *If F and G are relations on X to Y and H is a relation on Y to Z, then*

$$\big(H \circ F\big) \setminus \big(H \circ G\big) \subset H \circ \big(F \setminus G\big).$$

Theorem 2.13. *If F is a relation on X to Y and G and H are relations on Y to Z, then*

$$\big(G \circ F\big) \setminus \big(H \circ F\big) \subset \big(G \setminus H\big) \circ F.$$

Proof. To check this, note that by Theorems 1.12 and 2.7, we have

$$\begin{aligned} \big(\big(G \circ F\big) \setminus \big(H \circ F\big)\big)(x) &= \big(G \circ F\big)(x) \setminus \big(H \circ F\big)(x) \\ &= G\big[F(x)\big] \setminus H\big[F(x)\big] \subset \big(G \setminus H\big)\big[F(x)\big] = \big(\big(G \setminus H\big) \circ F\big)(x) \end{aligned}$$

for all $x \in X$. Therefore, by Corollary 1.4, the required equality is also true.

Corollary 2.14. *If F is a relation on X to Y and G is a relation on Y to Z, then*

$$\text{(1)}\quad \big(G\circ F\big)^c \subset G^c\circ F \quad \text{if} \quad X=D_F; \qquad \text{(2)}\quad \big(G\circ F\big)^c \subset G\circ F^c \quad \text{if} \quad Z=R_G.$$

Proof. To check (2), note that if $Z=R_G$, then by Theorem 2.7

$$\big(G\circ\big(X\times Y\big)\big)(x)=G\big[\big(X\times Y\big)(x)\big]=G[Y]=R_G=Z=\big(X\times Z\big)(x)$$

for all $x\in X$. Therefore, by Corollary 1.4, $G\circ\big(X\times Y\big)=X\times Z$. And thus, by Theorem 2.12,

$$\begin{aligned}\big(G\circ F\big)^c=\big(X\times Z\big)\setminus\big(G\circ F\big)=\big(G\circ\big(X\times Y\big)\big)\setminus\big(G\circ F\big)\\ \subset G\circ\big(\big(X\times Y\big)\setminus F\big)=G\circ F^c.\end{aligned}$$

Theorem 2.15. *If F is a relation on X to Y and G is a relation on Y to Z, then for any $x\in X$ we have*

$$\big(G^c\circ F\big)^c(x)=\bigcap_{y\in F(x)} G(y).$$

Proof. By Corollary 1.13 and Theorems 2.7 and 1.14, we have

$$\big(G^c\circ F\big)^c(x)=\big(G^c\circ F\big)(x)^c=G^c[F(x)]^c=\bigcap_{y\in F(x)} G(y).$$

3. Box and totalization relations

Definition 3.1. For any $A\subset X$ and $B\subset Y$, we define

$$\Gamma_{(A,\,B)}=A\times B.$$

Remark 3.2. In particular, we shall also write

$$\Gamma_A=\Gamma_{(A,\,A)} \qquad \text{and} \qquad \Gamma_{(a,\,B)}=\Gamma_{(\{a\},\ B)}$$

for any $a\in X$.

Concerning box relations, the following theorems have been proved in [17].

Theorem 3.3. *If $A\subset X$ and $B\subset Y$, then for any $x\in X$ we have*

$$\Gamma_{(A,\,B)}(x)=\begin{cases} B & \text{if} \quad x\in A,\\ \emptyset & \text{if} \quad x\notin A.\end{cases}$$

Remark 3.4. Thus, in particular if $A \subset X$, then for any $x \in X$ we have

$$\Gamma_A(x) = \begin{cases} A & \text{if} \quad x \in A\,, \\ \emptyset & \text{if} \quad x \notin A\,. \end{cases}$$

Theorem 3.5. *If $A \subset X$ and $B \subset Y$, then for any $U \subset X$ we have*

$$\Gamma_{(A,\,B)}\,[\,U\,] = \begin{cases} \emptyset & \textit{if} \quad U \subset A^c, \\ B & \textit{if} \quad U \not\subset A^c. \end{cases}$$

Remark 3.6. Thus, in particular if $A \subset X$, then for any $U \subset X$ we have

$$\Gamma_A\,[\,U\,] = \begin{cases} \emptyset & \text{if} \quad U \subset A^c, \\ A & \text{if} \quad U \not\subset A^c. \end{cases}$$

Definition 3.7. For any relation F on one set X to another Y, we define

$$\tilde{F} = F \cup \Gamma_{(\,D_F^c\,,\,Y\,)}\,.$$

Remark 3.8. If $Y \neq \emptyset$, then the relation $\tilde{F}$ may be called the natural totalization of F. Its usefulness will be cleared up by the forthcoming results.

In particular, for any $A\,,\,B \subset X$, the totalizations

$$\tilde{\Gamma}_A = \widetilde{\Gamma_A} \qquad\qquad \text{and} \qquad\qquad \tilde{\Gamma}_{(A,\,B)} = \widetilde{\Gamma_{(A,\,B)}}$$

may be called the Davis–Pervin and the Hunsaker–Lindgren relations on X, respectively.

The latter relations play an important role in the generalized uniformization of various topological structures such as proximities, closures, topologies, and filters, for instance. (See [2], [10], [15] and [1, pp. 42 , 193], [5], [11].)

While, the relations $\tilde{F}$ can be used to prove a useful reduction theorem for the intersection convolution of relations [16]. The latter operation allows of a natural treatment of the Hahn-Banach type extension theorems. (See [12] and [4].)

Concerning totalization relations, the following theorems have been proved in [17].

Theorem 3.9. *If F is a relation on X to Y, then for any $U \subset X$ we have*

$$\tilde{F}\,[\,U\,] = \begin{cases} F\,[\,U\,] & \textit{if} \quad U \subset D_F, \\ Y & \textit{if} \quad U \not\subset D_F. \end{cases}$$

Corollary 3.10. *If F is a relation on X to Y, then for some $U \subset X$ we have $F[\,U\,] = \tilde{F}\,[\,U\,]$ if and only if either $U \subset D_F$ or $F\,[\,U\,] = Y$.*

Corollary 3.11. *If F is a relation on X to Y, then for some $x \in X$ we have $F(x) = \tilde{F}(x)$ if and only if either $x \in D_F$ or $F(x) = Y$.*

Theorem 3.12. *If F is a relation on X to Y, then for any $x \in X$ we have*

$$\tilde{F}(x) = \begin{cases} F(x) & \text{if} \quad x \in D_F, \\ Y & \text{if} \quad x \notin D_F. \end{cases}$$

Corollary 3.13. *If F is a relation on X to Y, then $\tilde{F}$ is an extension of F such that $F = \tilde{F}$ if and only if $F(x) = Y$ for all $x \in D_F^c$.*

Corollary 3.14. *If F is a relation on X to Y and $Y \neq \emptyset$, then $F = \tilde{F}$ if and only if F is total.*

Theorem 3.15. *If $A \subset X$ and $B \subset Y$, then*

$$\tilde{\Gamma}_{(A,\,B)} = \begin{cases} \Gamma_{(X,\,Y)} & \text{if} \quad B = \emptyset, \\ \Gamma_{(A,\,B)} \cup \Gamma_{(A^c,\,Y)} & \text{if} \quad B \neq \emptyset. \end{cases}$$

Remark 3.16. Thus, in particular if A is a nonvoid subset of X, then

$$\tilde{\Gamma}_A = \Gamma_A \cup \Gamma_{(A^c,\,X)}.$$

Moreover, we can at once see that the latter equality is also true for $A = \emptyset$.

Theorem 3.17. *If $A \subset X$ and $B \subset Y$ such that $B \neq \emptyset$, then for any $U \subset X$, with $U \neq \emptyset$, we have*

$$\tilde{\Gamma}_{(A,\,B)}[U] = \begin{cases} B & \text{if} \quad U \subset A, \\ Y & \text{if} \quad U \not\subset A. \end{cases}$$

Remark 3.18. Thus, in particular if A and U are nonvoid subsets of X, then

$$\tilde{\Gamma}_A[U] = \begin{cases} A & \text{if} \quad U \subset A, \\ X & \text{if} \quad U \not\subset A. \end{cases}$$

Moreover, we can easily see that the latter equality is also true for $A = \emptyset$.

Corollary 3.19. *If $A \subset X$ and $B \subset Y$ such that $B \neq \emptyset$, then for any $x \in X$ we have*

$$\tilde{\Gamma}_{(A,\,B)}(x) = \begin{cases} B & \text{if} \quad x \in A, \\ Y & \text{if} \quad x \notin A. \end{cases}$$

Remark 3.20. Thus, in particular if A is a nonvoid subset of X and $x \in X$, then

$$\tilde{\Gamma}_A(x) = \begin{cases} A & \text{if} \quad x \in A, \\ X & \text{if} \quad x \notin A. \end{cases}$$

Moreover, we can easily see that the latter equality is also true for $A = \emptyset$.

Remark 3.21. Note that if $A \subset X$ and $\emptyset$ is considered as a subset of Y, then by Theorems 3.15 and 3.3 we have $\tilde{\Gamma}_{(A,\,\emptyset)}(x) = \Gamma_{(X,\,Y)}(x) = Y$ for all $x \in X$. Therefore, the assumption $B \neq \emptyset$ is indispensable in Corollary 3.19 and Theorem 3.17.

4. Inverses of box and totalization relations

Theorem 4.1. *If* $A \subset X$ *and* $B \subset Y$, *then*

$$\Gamma_{(A,\,B)}^{-1} = \Gamma_{(B\,,\,A)}\,.$$

Proof. By the corresponding definitions, for any $x \in X$ and $y \in Y$, we have

$$\begin{aligned}(y,\,x) \in \Gamma_{(A,\,B)}^{-1} &\iff (x,\,y) \in \Gamma_{(A,\,B)} \iff (x,\,y) \in A \times B \\ &\iff x \in A\,, \quad y \in B \iff (y,\,x) \in B \times A \iff (y,\,x) \in \Gamma_{(B,\,A)}\,.\end{aligned}$$

Therefore, the required equality is also true.

Remark 4.2. Thus, in particular if $A \subset X$, then

$$\Gamma_A^{-1} = \Gamma_A\,.$$

Corollary 4.3. *If* $A \subset X$ *and* $B \subset Y$, *then for any* $V \subset Y$ *we have*

$$\Gamma_{(A,\,B)}^{-1}[\,V\,] = \begin{cases} \emptyset & \textit{if} \quad V \subset B^c\,, \\ A & \textit{if} \quad V \not\subset B^c\,. \end{cases}$$

Proof. By Theorems 4.1 and 3.5, we have

$$\Gamma_{(A,\,B)}^{-1}[\,V\,] = \Gamma_{(B,\,A)}[\,V\,] = \begin{cases} \emptyset & \text{if} \quad V \subset B^c\,, \\ A & \text{if} \quad V \not\subset B^c\,. \end{cases}$$

Remark 4.4. Thus, in particular if $A \subset X$, then for any $V \subset X$ we have

$$\Gamma_A^{-1}[\,V\,] = \begin{cases} \emptyset & \text{if} \quad V \subset A^c\,, \\ A & \text{if} \quad V \not\subset A^c\,. \end{cases}$$

Corollary 4.5. *If* $A \subset X$ *and* $B \subset Y$, *then for any* $y \in Y$ *we have*

$$\Gamma_{(A,\,B)}^{-1}(y) = \begin{cases} A & \textit{if} \quad y \in B\,, \\ \emptyset & \textit{if} \quad y \notin B\,. \end{cases}$$

Remark 4.6. Thus, in particular if $A \subset X$, then for any $y \in X$ we have

$$\Gamma_A^{-1}(y) = \begin{cases} A & \text{if} \quad y \in A\,, \\ \emptyset & \text{if} \quad y \notin A\,. \end{cases}$$

Theorem 4.7. *If* F *is a relation on* X *to* Y, *then*

$$\tilde{F}^{-1} = F^{-1} \cup \Gamma_{(Y,\ D_F^c)}$$

Proof. By Definition 3.7 and Theorems 2.4 and 4.1, we have

$$\tilde{F}^{-1} = \big(F \cup \Gamma_{(D_F^c,\ Y)}\big)^{-1} = F^{-1} \cup \Gamma_{(D_F^c,\ Y)}^{-1} = F^{-1} \cup \Gamma_{(Y,\ D_F^c)}\,.$$

Corollary 4.8. *If F is a relation on X to Y, then for any nonvoid subset V of Y, we have*

$$\tilde{F}^{-1}[V] = D_F^c \cup F^{-1}[V].$$

Proof. By Theorems 4.7, 1.10 and 3.5, we have

$$\tilde{F}^{-1}[V] = \big(F^{-1} \cup \Gamma_{(Y,\, D_F^c)}\big)[V] = F^{-1}[V] \cup \Gamma_{(Y,\, D_F^c)}[V] = F^{-1}[V] \cup D_F^c.$$

Therefore, the required equality is also true.

Corollary 4.9. *If F is a relation on X to Y, then for any $y \in Y$ we have*

$$\tilde{F}^{-1}(y) = D_F^c \cup F^{-1}(y).$$

Theorem 4.10. *If $A \subset X$ and $B \subset Y$ such that $A \neq X$ and $B \neq \emptyset$, then*

$$\tilde{\Gamma}^{-1}_{(A,\, B)} = \tilde{\Gamma}_{(B^c,\, A^c)}.$$

Proof. Because of $B \neq \emptyset$ and Theorem 2.3, we have $A = D_{\Gamma_{(A,\, B)}}$. Therefore, if $y \in Y$, then by Corollaries 4.9 and 4.5 we have

$$\tilde{\Gamma}^{-1}_{(A,\, B)}(y) = A^c \cup \Gamma^{-1}_{(A,\, B)}(y) = A^c \cup \begin{cases} A & \text{if} \quad y \in B, \\ \emptyset & \text{if} \quad y \notin B, \end{cases}$$

and hence

$$\tilde{\Gamma}^{-1}_{(A,\, B)}(y) = \begin{cases} A^c & \text{if} \quad y \in B^c, \\ X & \text{if} \quad y \notin B^c. \end{cases}$$

Moreover, because of $A \neq X$ and Corollary 3.19, we also have

$$\tilde{\Gamma}_{(B^c,\, A^c)}(y) = \begin{cases} A^c & \text{if} \quad y \in B^c, \\ X & \text{if} \quad y \notin B^c. \end{cases}$$

Hence, by Corollary 1.4, it is clear that the required equality is also true.

Remark 4.11. Thus, in particular if A is a proper, nonvoid subset of X, then

$$\tilde{\Gamma}^{-1}_{A} = \tilde{\Gamma}_{A^c}.$$

Moreover, by Remark 3.16, we can see that $\tilde{\Gamma}_{\emptyset} = X^2$ and $\tilde{\Gamma}_X = X^2$. Therefore, the above equality is also true for $A = \emptyset$ and $A = X$.

Remark 4.12. However, if X is a set and B is a nonvoid subset of Y, then by Theorems 3.15 and 4.1 we can see that

$$\tilde{\Gamma}^{-1}_{(X,\, B)} = \Gamma^{-1}_{(X,\, B)} = \Gamma_{(B,\, X)} \qquad \text{and} \qquad \tilde{\Gamma}_{(B^c,\, X^c)} = \tilde{\Gamma}_{(B^c,\, \emptyset)} = \Gamma_{(Y,\, X)}.$$

Thus, in particular the assumption $A \neq X$ is indispensable in Theorem 4.10.

Corollary 4.13. *If $A \subset X$ and $B \subset Y$ such that $B \neq \emptyset$, then for any nonvoid subset V of Y, we have*

$$\tilde{\Gamma}^{-1}_{(A,\,B)}[\,V\,] = \begin{cases} A^c & \text{if} \quad V \subset B^c, \\ X & \text{if} \quad V \not\subset B^c. \end{cases}$$

Proof. If in addition $A \neq X$, then by Theorems 4.10 and 3.17 we have

$$\tilde{\Gamma}^{-1}_{(A,\,B)}[\,V\,] = \tilde{\Gamma}_{(B^c,\,A^c)}[\,V\,] = \begin{cases} A^c & \text{if} \quad V \subset B^c, \\ X & \text{if} \quad V \not\subset B^c. \end{cases}$$

Moreover, by Remark 4.12 and Theorem 3.5, we have

$$\tilde{\Gamma}^{-1}_{(X,\,B)}[\,V\,] = \Gamma_{(B,\,X)}[\,V\,] = \begin{cases} \emptyset & \text{if} \quad V \subset B^c, \\ X & \text{if} \quad V \not\subset B^c. \end{cases}$$

Thus, since $\emptyset = X^c$, the required equality is again true.

Remark 4.14. Thus, in particular if A is a nonvoid subset of X, then for any nonvoid subset V of X we have

$$\tilde{\Gamma}^{-1}_{A}[\,V\,] = \begin{cases} A^c & \text{if} \quad V \subset A^c, \\ X & \text{if} \quad V \not\subset A^c. \end{cases}$$

Moreover, we can easily see the above equality is also true for $A = \emptyset$.

Corollary 4.15. *If $A \subset X$ and $B \subset Y$ such that $B \neq \emptyset$, then for any $y \in Y$ we have*

$$\tilde{\Gamma}^{-1}_{(A,\,B)}(y) = \begin{cases} X & \text{if} \quad y \in B, \\ A^c & \text{if} \quad y \notin B. \end{cases}$$

Remark 4.16. Thus, in particular if A is a nonvoid subset of X, then for any $y \in X$ we have

$$\tilde{\Gamma}^{-1}_{A}(y) = \begin{cases} X & \text{if} \quad y \in A, \\ A^c & \text{if} \quad y \notin A. \end{cases}$$

Moreover, we can easily see the above equality is also true for $A = \emptyset$.

Remark 4.17. Note that if A is a proper subset of X and $\emptyset$ is considered as a subset of Y, then by Theorems 3.15 and 4.1 we have

$$\tilde{\Gamma}^{-1}_{(A,\,\emptyset)}(y) = \Gamma^{-1}_{(X,\,Y)}(y) = \Gamma_{(Y,\,X)}(y) = X$$

and

$$\tilde{\Gamma}_{(\emptyset^c,\,A^c)}(y) = \tilde{\Gamma}_{(Y,\,A^c)}(y) = \Gamma_{(Y,\,A^c)}(y) = A^c$$

for all $y \in Y$. Therefore, the assumption $B \neq \emptyset$ is indispensable in Corollaries 4.13 and 4.15 and Theorem 4.10.

5. Compositions of box relations with arbitrary ones

Theorem 5.1. *If* $A \subset X$ *and* $B \subset Y$, *and* G *is a relation on* Y *to* Z, *then*

$$G \circ \Gamma_{(A,\,B)} = \Gamma_{(A,\; G[\,B\,])}\,.$$

Proof. By Theorems 2.7 and 3.3, for any $x \in X$, we have

$$\big(G \circ \Gamma_{(A,\,B)}\big)(\,x) = G\left[\,\Gamma_{(A,\,B)}(\,x)\right] = \begin{cases} G\,[\,B\,] & \text{if} \quad x \in A\,, \\ G\,[\,\emptyset\,] & \text{if} \quad x \notin A\,. \end{cases}$$

Hence, since $G\,[\,\emptyset\,] = \emptyset$, by Theorem 3.3 we can see that

$$\big(G \circ \Gamma_{(A,\,B)}\big)(\,x) = \Gamma_{(A,\; G[\,B\,])}(\,x)$$

for all $x \in X$. Therefore, by Corollary 1.4, the required equality is also true.

Remark 5.2. Thus, in particular if $A \subset X$ and G is a relation on X to Y, then

$$G \circ \Gamma_A = \Gamma_{(A,\; G[\,A\,])}\,.$$

Theorem 5.3. *If* F *is a relation on* X *to* Y, *and* $C \subset Y$ *and* $D \subset Z$, *then*

$$\Gamma_{(C,\,D)} \circ F = \Gamma_{(F^{-1}[\,C\,],\; D)}\,.$$

Proof. By Theorems 2.7 and 3.5, for any $x \in X$, we have

$$\big(\Gamma_{(C,\,D)} \circ F\big)(\,x) = \Gamma_{(C,\,D)}\left[\,F(\,x)\right] = \begin{cases} \emptyset & \text{if} \quad F(x) \subset C^c, \\ D & \text{if} \quad F(x) \not\subset C^c. \end{cases}$$

Moreover, by Theorem 2.2, we have

$$F(x) \not\subset C^c \iff F(x) \cap C \neq \emptyset \iff x \in F^{-1}[\,C\,]\,.$$

Therefore,

$$\big(\Gamma_{(C,\,D)} \circ F\big)(x) = \begin{cases} D & \text{if} \quad x \in F^{-1}\,[\,C\,]\,, \\ \emptyset & \text{if} \quad x \notin F^{-1}\,[\,C\,]\,. \end{cases}$$

Hence, by Theorem 3.3, we can see that

$$\big(\Gamma_{(C,\,D)} \circ F\big)(x) = \Gamma_{(\,F^{-1}[\,C\,],\; D)}(x)$$

for all $x \in X$. Thus, by Corollary 1.4, the required equality is also true.

Remark 5.4. Thus, in particular if F is a relation on X to Y, and $B \subset Y$, then

$$\Gamma_B \circ F = \Gamma_{(\,F^{-1}[\,B\,],\; B)}\,.$$

Theorem 5.5. *If* $A \subset X$, $B, C \subset Y$ *and* $D \subset Z$ *, then*

$$\Gamma_{(C,\,D)} \circ \Gamma_{(A,\,B)} = \begin{cases} \emptyset & \textit{if} \quad B \subset C^c, \\ \Gamma_{(A,\,D)} & \textit{if} \quad B \not\subset C^c. \end{cases}$$

Proof. By Theorem 5.1, we have

$$\Gamma_{(C,\,D)} \circ \Gamma_{(A,\,B)} = \Gamma_{(A,\ \Gamma_{(C,\,D)}[\,B\,])} \,.$$

Moreover, by Theorem 3.5, we have

$$\Gamma_{(C,\,D)}\,[\,B\,] = \begin{cases} \emptyset & \text{if} \quad B \subset C^c, \\ D & \text{if} \quad B \not\subset C^c. \end{cases}$$

Thus, since $\Gamma_{(A,\,\emptyset)} = \emptyset$, the required equality is also true.

Remark 5.6. Thus, in particular if A , $B \subset X$, then

$$\Gamma_B \circ \Gamma_A = \begin{cases} \emptyset & \text{if} \quad A \subset B^c, \\ \Gamma_{(A,\,B)} & \text{if} \quad A \not\subset B^c. \end{cases}$$

Hence, since $A \subset A^c \iff A = \emptyset$, and $\Gamma_\emptyset = \emptyset$, we can also see that $\Gamma_A^2 = \Gamma_A$.

6. Composition of box relations with totalization ones

Theorem 6.1. *If* $A \subset X$ *and* $B \subset Y$ *and* G *is a relation on* Y *to* Z *, then*

$$\tilde{G} \circ \Gamma_{(A,\,B)} = \begin{cases} \Gamma_{(A,\ G[\,B\,])} & \textit{if} \quad B \subset D_G, \\ \Gamma_{(A,\,Z\,)} & \textit{if} \quad B \not\subset D_G. \end{cases}$$

Proof. By Theorem 5.1, we have

$$\tilde{G} \circ \Gamma_{(A,\,B)} = \Gamma_{(A,\ \tilde{G}[\,B\,])} \,.$$

Moreover, by Theorem 3.9, we have

$$\tilde{G}\,[\,B\,] = \begin{cases} G\,[\,B\,] & \text{if} \quad B \subset D_G, \\ Z & \text{if} \quad B \not\subset D_G. \end{cases}$$

Therefore, the required equality is also true.

Remark 6.2. Thus, in particular if $A \subset X$ and G is a relation on X to Y, then

$$\tilde{G} \circ \Gamma_A = \begin{cases} \Gamma_{(A,\ G[\,A\,])} & \text{if} \quad A \subset D_G, \\ \Gamma_{(A,\,Y\,)} & \text{if} \quad A \not\subset D_G. \end{cases}$$

Theorem 6.3. *If F is a relation on X to Y and $C \subset Y$ and $D \subset Z$ such that $C \neq \emptyset$, then*

$$\Gamma_{(C,D)} \circ \tilde{F} = \Gamma_{(D_F^c,\, D)} \cup \Gamma_{(F^{-1}[C],\, D)}.$$

Proof. By Theorem 5.3, we have

$$\Gamma_{(C,D)} \circ \tilde{F} = \Gamma_{(\tilde{F}^{-1}[C],\, D)}.$$

Moreover, since $C \neq \emptyset$, by Corollary 4.8 we have

$$\tilde{F}^{-1}[C] = D_F^c \cup F^{-1}[C].$$

Therefore,

$$\Gamma_{(C,D)} \circ \tilde{F} = \Gamma_{(D_F^c \cup F^{-1}[C],\, D)},$$

and thus by [, Theorem 3.3] the required equality is also true.

Remark 6.4. Thus, in particular if F is a relation on X to Y and B is a nonvoid subset of Y, then

$$\Gamma_B \circ \tilde{F} = \Gamma_{(D_F^c,\, B)} \cup \Gamma_{(F^{-1}[B],\, B)}.$$

Moreover, we can at once see that the above equality is also true for $B = \emptyset$.

Remark 6.5. However, we can note that

$$\Gamma_{(\emptyset,D)} \circ \tilde{F} = \emptyset \circ \tilde{F} = \emptyset \qquad \text{and} \qquad \Gamma_{(D_F^c,\, D)} \cup \Gamma_{(F^{-1}[\emptyset],\, D)} = \Gamma_{(D_F^c,\, D)}.$$

Therefore, the condition $C \neq \emptyset$ is indispensable in Theorem 6.3.

Theorem 6.6. *If $A \subset X$, $B, C \subset Y$ and $D \subset Z$ such that $B \neq \emptyset$ and $D \neq \emptyset$, then*

$$\tilde{\Gamma}_{(C,D)} \circ \Gamma_{(A,B)} = \begin{cases} \Gamma_{(A,D)} & \text{if} \quad B \subset C, \\ \Gamma_{(A,Z)} & \text{if} \quad B \not\subset C. \end{cases}$$

Proof. Because of $D \neq \emptyset$ and Theorem 3.3, we have $C = D_{\Gamma_{(C,D)}}$. Therefore, by Theorem 6.1, we have

$$\tilde{\Gamma}_{(C,D)} \circ \Gamma_{(A,B)} = \begin{cases} \Gamma_{(A,\, \Gamma_{(C,D)}[B])} & \text{if} \quad B \subset C, \\ \Gamma_{(A,Z)} & \text{if} \quad B \not\subset C. \end{cases}$$

Moreover, by Theorem 3.4, we have

$$\Gamma_{(C,D)}[B] = \begin{cases} \emptyset & \text{if} \quad B \subset C^c, \\ D & \text{if} \quad B \not\subset C^c. \end{cases}$$

Hence, since $\emptyset \neq B \subset C$ implies that $B \not\subset C^c$, it is clear that the required equality is also true.

Remark 6.7. Thus, in particular if A and B are nonvoid subsets of X, then

$$\tilde{\Gamma}_B \circ \Gamma_A = \begin{cases} \Gamma_{(A,\,B)} & \text{if} \quad A \subset B, \\ \Gamma_{(A,\,X)} & \text{if} \quad A \not\subset B. \end{cases}$$

Moreover, we can easily see that the above equality is also true whenever $A = \emptyset$ or $B = \emptyset$. Namely, for instance by Remarks 3.16 and 5.6 we have

$$\tilde{\Gamma}_\emptyset \circ \Gamma_A = \Gamma_X \circ \Gamma_A = \begin{cases} \emptyset & \text{if} \quad A = \emptyset, \\ \Gamma_{(A,\,X)} & \text{if} \quad A \neq \emptyset. \end{cases}$$

Remark 6.8. However, it is clear that the condition $B \neq \emptyset$ cannot be omitted from Theorem 6.6. Moreover, by using Theorems 3.15 and 5.5, we can see that

$$\tilde{\Gamma}_{(C,\,\emptyset)} \circ \Gamma_{(A,B)} = \Gamma_{(Y,\,Z)} \circ \Gamma_{(A,B)} = \begin{cases} \emptyset & \text{if} \quad B = \emptyset, \\ \Gamma_{(A,\,Z)} & \text{if} \quad B \neq \emptyset. \end{cases}$$

Therefore, the condition $D \neq \emptyset$ is also indispensable in Theorem 6.6.

Theorem 6.9. *If* $A \subset X$, $B, C \subset Y$ *and* $D \subset Z$ *such that* $B \neq \emptyset$ *and* $C \neq \emptyset$, *then*

$$\Gamma_{(C,\,D)} \circ \tilde{\Gamma}_{(A,\,B)} = \begin{cases} \Gamma_{(A^c,\,D)} & \textit{if} \quad B \subset C^c, \\ \Gamma_{(X,\,D)} & \textit{if} \quad B \not\subset C^c. \end{cases}$$

Proof. Because of $B \neq \emptyset$ and Theorem 3.3, we have $A = D_{\Gamma_{(A,\,B)}}$. Therefore, by Theorem 6.3, we have

$$\Gamma_{(C,\,D)} \circ \tilde{\Gamma}_{(A,B)} = \Gamma_{(A^c \cup \Gamma^{-1}_{(A,\,B)}[\,C\,],\; D)}\,.$$

Moreover, by Corollary 4.3, we have

$$\Gamma^{-1}_{(A,\,B)}[\,C\,] = \begin{cases} \emptyset & \text{if} \quad C \subset B^c, \\ A & \text{if} \quad C \not\subset B^c. \end{cases}$$

Hence, since $C \subset B^c \iff B \subset C^c$, it is clear that the required equality is also true.

Remark 6.10. Thus, in particular if A and B are nonvoid subsets of X, then

$$\Gamma_B \circ \tilde{\Gamma}_A = \begin{cases} \Gamma_{(A^c,\,B)} & \text{if} \quad A \subset B^c, \\ \Gamma_{(X,\,B)} & \text{if} \quad A \not\subset B^c. \end{cases}$$

Moreover, we can easily see that the above equality is also true whenever $A = \emptyset$ or $B = \emptyset$. Namely, for instance, by Remarks 3.16 and 5.6, we have

$$\Gamma_B \circ \tilde{\Gamma}_\emptyset = \Gamma_B \circ \Gamma_X = \begin{cases} \emptyset & \text{if} \quad B = \emptyset, \\ \Gamma_{(X,\,B)} & \text{if} \quad B \neq \emptyset. \end{cases}$$

Remark 6.11. However, it is clear that the condition $C \neq \emptyset$ cannot be omitted from Theorem 6.9. Moreover, by using Theorems 3.15 and 5.5, we can see that

$$\Gamma_{(C,\,D)} \circ \tilde{\Gamma}_{(A,\,\emptyset)} = \Gamma_{(C,\,D)} \circ \Gamma_{(X,Y)} = \begin{cases} \emptyset & \text{if} \quad C = \emptyset, \\ \Gamma_{(X,\,D)} & \text{if} \quad C \neq \emptyset. \end{cases}$$

Therefore, the condition $B \neq \emptyset$ is also indispensable in Theorem 6.9.

7. Compositions of totalization relations with arbitrary ones

Theorem 7.1. *If F is a relation on X to Y and G is a relation on Y to Z, then*

$$G \circ \tilde{F} = G \circ F \, \cup \, \Gamma_{(D_F^c , \; G[Y])} \, .$$

Proof. By Definition 3.7 and Theorems 2.9 and 5.1, we have

$$G \circ \tilde{F} = G \circ \big(F \cup \Gamma_{(D_F^c , \; Y)} \big) = G \circ F \, \cup \, G \circ \Gamma_{(D_F^c , \; Y)} = G \circ F \, \cup \, \Gamma_{(D_F^c , \; G[Y])} \, .$$

Theorem 7.2. *If F is a relation on X to Y and G is a relation on Y to Z, then*

$$\tilde{G} \circ F = G \circ F \cup \Gamma_{(F^{-1}[D_G^c] , \; Z)} \, .$$

Proof. By Definition 3.7 and Theorems 2.10 and 5.3, we have

$$\tilde{G} \circ F = \big(G \cup \Gamma_{(D_G^c , \; Z)} \big) \circ F = G \circ F \, \cup \, \Gamma_{(D_G^c , \; Z)} \circ F = G \circ F \cup \Gamma_{(F^{-1}[D_G^c] , \; Z)} \, .$$

Theorem 7.3. *If F is an arbitrary relation on X to Y and G is a non-total relation of Y to Z, then*

$$\tilde{G} \circ \tilde{F} = G \circ F \, \cup \, \Gamma_{(D_F^c , \; Z)} \, \cup \, \Gamma_{(F^{-1} [D_G^c] , \; Z)} \, .$$

Proof. By Theorem 7.2 and 3.9, we have

$$\tilde{G} \circ \tilde{F} = \tilde{G} \circ F \, \cup \, \Gamma_{(D_F^c , \; \tilde{G} [Y])} = G \circ F \, \cup \, \Gamma_{(F^{-1} [D_G^c] , \; Z)} \, \cup \, \Gamma_{(D_F^c , \; Z)} \, .$$

Namely, by the hypothesis $D_G \neq Y$, we have $Y \not\subset D_G$, and thus $\tilde{G}[Y] = Z$.

Remark 7.4. Note that if F is a relation on X to Y and G is a relation of Y to Z, then by Definition 3.7 and Theorem 7.1 we only have

$$\tilde{G} \circ \tilde{F} = G \circ \tilde{F} = G \circ F \, \cup \, \Gamma_{(D_F^c , \; G[Y])} \, .$$

Thus, since now $\Gamma_{(F^{-1} [D_G^c] , \; Z)} = \Gamma_{(F^{-1} [\emptyset] , \; Z)} = \Gamma_{(\emptyset , Z)} = \emptyset$, the extra condition $D_G \neq Y$ is not dispensable in Theorem 7.3.

Theorem 7.5. *If $A \subset X$ and $B \subset Y$ such that $B \neq \emptyset$, and G is a relation on Y to Z, then*

$$G \circ \tilde{\Gamma}_{(A , B)} = \Gamma_{(A , G [B])} \, \cup \, \Gamma_{(A^c , \; G [Y])} \, .$$

Proof. Because of $B \neq \emptyset$ and Theorem 3.3, we have $A = D_{\Gamma_{(A , \, B)}}$. Therefore, by Theorems 7.1 and 5.1, we have

$$G \circ \tilde{\Gamma}_{(A , B)} = G \circ \Gamma_{(A , B)} \, \cup \, \Gamma_{(A^c , \; G [Y])} = \Gamma_{(A , G [B])} \, \cup \, \Gamma_{(A^c , \; G [Y])} \, .$$

Remark 7.6. Thus in particular if A is a nonvoid subset of X and G is a relation on X to Y, then

$$G \circ \tilde{\Gamma}_A = \Gamma_{(A,\; G[A])} \cup \Gamma_{(A^c,\; G[X])}\,.$$

Moreover, by Remarks 3.16 and 5.2 and the corresponding definitions, we can see that

$$G \circ \tilde{\Gamma}_{\emptyset} = G \circ \Gamma_X = \Gamma_{(X,\; G[X])} = \Gamma_{(\emptyset, G[\emptyset])} \cup \Gamma_{(\emptyset^c,\; G[X])}$$

is also true.

Remark 7.7. However, by Theorems 3.15 and 5.1 and the corresponding definitions, we can see that

$$G \circ \tilde{\Gamma}_{(A,\emptyset)} = G \circ \Gamma_{(X,Y)} = \Gamma_{(X,\; G[Y])}\,.$$

and

$$\Gamma_{(A, G[\emptyset])} \cup \Gamma_{(A^c,\; G[Y])} = \Gamma_{(A^c,\; G[Y])}\,.$$

Therefore, the assumption $B \neq \emptyset$ is indispensable in Theorem 7.5.

Theorem 7.8. *If F is a relation on X to Y, and moreover $C \subset Y$ and $D \subset Z$ such that $D \neq \emptyset$, then*

$$\tilde{\Gamma}_{(C,D)} \circ F = \Gamma_{(F^{-1}[C],\; D)} \cup \Gamma_{(F^{-1}[C^c],\; Z)}\,.$$

Proof. Because of $D \neq \emptyset$ and Theorem 3.3, we have $C = D_{\Gamma_{(C,D)}}$. Therefore, by Theorems 7.2 and 5.3, we have

$$\tilde{\Gamma}_{(C,D)} \circ F = \Gamma_{(C,D)} \circ F \cup \Gamma_{(F^{-1}[C^c],\; Z)} = \Gamma_{(F^{-1}[C],\; D)} \cup \Gamma_{(F^{-1}[C^c],\; Z)}\,.$$

Remark 7.9. Thus, in particular if F is a relation on X to Y and B is a nonvoid subset of Y, then

$$\tilde{\Gamma}_B \circ F = \Gamma_{(F^{-1}[B],\; B)} \cup \Gamma_{(F^{-1}[B^c],\; Y)}\,.$$

Moreover, by Remarks 3.16 and 5.4 and the corresponding definitions, we can see that

$$\tilde{\Gamma}_{\emptyset} \circ F = \Gamma_Y \circ F = \Gamma_{(F^{-1}[Y],\; Y)} = \Gamma_{(F^{-1}[\emptyset],\; \emptyset)} \cup \Gamma_{(F^{-1}[\emptyset^c],\; Y)}$$

is also true.

Remark 7.10. However, by Theorems 3.15 and 5.3 and the corresponding definitions, we can see that

$$\tilde{\Gamma}_{(C,\emptyset)} \circ F = \Gamma_{(Y,Z)} \circ F = \Gamma_{(F^{-1}[Y],\; Z)} = \Gamma_{(D_F,\; Z)}$$

and

$$\Gamma_{(F^{-1}[C],\; \emptyset)} \cup \Gamma_{(F^{-1}[C^c],\; Z)} = \Gamma_{(F^{-1}[C^c],\; Z)}\,.$$

Therefore, the assumption $D \neq \emptyset$ is indispensable in Theorem 7.8.

8. Compositions of totalization relations with totalizations of box relations

Theorem 8.1. *If $A \subset X$ and $B \subset Y$, and G is a non-total relation on Y to Z, then*

$$\tilde{G} \circ \tilde{\Gamma}_{(A,B)} = \begin{cases} \tilde{\Gamma}_{(A,\, G[B])} & \text{if} \quad B \subset D_G, \\ \Gamma_{(X,Z)} & \text{if} \quad B \not\subset D_G. \end{cases}$$

Proof. If $B \neq \emptyset$, then by Theorem 3.3 we have $A = D_{\Gamma_{(A,B)}}$. Moreover, by Theorems 7.3, 5.1 and 4.3, we have

$$\tilde{G} \circ \tilde{\Gamma}_{(A,B)} = G \circ \Gamma_{(A,B)} \cup \Gamma_{(A^c,Z)} \cup \Gamma_{(\Gamma^{-1}_{(A,B)}[D_G^c],\, Z)}$$
$$= \Gamma_{(A,\, G[B])} \cup \Gamma_{(A^c,Z)} \cup \begin{cases} \Gamma_{(\emptyset, Z)} & \text{if} \quad B \subset D_G, \\ \Gamma_{(A,Z)} & \text{if} \quad B \not\subset D_G. \end{cases}$$

Furthermore, if $B \subset D_G$, then because of $B \neq \emptyset$, we can note that $G[B] \neq \emptyset$. Therefore, by Theorem 3.3, have $A = D_{\Gamma_{(A,\, G[B])}}$. Hence, by Definition 3.7, we can see that

$$\Gamma_{(A,\, G[B])} \cup \Gamma_{(A^c,Z)} = \tilde{\Gamma}_{(A,\, G[B])}.$$

Moreover, by the corresponding definitions, we can see that $\Gamma_{(\emptyset,Z)} = \emptyset$,

$$\Gamma_{(A,Z)} \cup \Gamma_{(A^c,Z)} = \Gamma_{(X,Z)} \qquad \text{and} \qquad \Gamma_{(A,\, G[B])} \subset \Gamma_{(X,Z)}.$$

Hence, it is clear that the required equality is also true whenever $B \neq \emptyset$.

Moreover, by Theorems 3.15 and 6.1, we can see that

$$\tilde{G} \circ \tilde{\Gamma}_{(A,\emptyset)} = \tilde{G} \circ \Gamma_{(X,Y)} = \Gamma_{(X,Z)} = \tilde{\Gamma}_{(A,\emptyset)} = \tilde{\Gamma}_{(A,\, G[\emptyset])}.$$

Therefore, the required equality is also true for $B = \emptyset$.

Remark 8.2. Thus, in particular if $A \subset X$ and G is a non-total relation on X to Y, then

$$\tilde{G} \circ \tilde{\Gamma}_A = \begin{cases} \tilde{\Gamma}_{(A,\, G[A])} & \text{if} \quad A \subset D_G, \\ \Gamma_{(X,Y)} & \text{if} \quad A \not\subset D_G. \end{cases}$$

Remark 8.3. Note that if $A \subset X$ and G is a total relation on Y to Z, then by Remarks 2.16 and 7.6 we have

$$\tilde{G} \circ \tilde{\Gamma}_A = G \circ \tilde{\Gamma}_A = \Gamma_{(A,G[A])} \cup \Gamma_{(A^c,\, G[X])}$$

and

$$\tilde{\Gamma}_{(A,\, G[A])} = \Gamma_{(A,G[A])} \cup \Gamma_{(A^c,Y} = \Gamma_{(A,G[A])} \cup \Gamma_{(A^c,\, G[A])} \cap \Gamma_{(A^c,\, G[X]^c)},$$

whenever $\emptyset \neq A \subset D_G$. Therefore, the condition that G is non-total is indispensable in Remark 8.2 and Theorem 8.1.

Theorem 8.4. *If* F *is a relation on* X *to* Y, *and* $C \subset Y$ *and* $D \subset Z$ *such that* $D \neq \emptyset$ *and* $C \neq Y$, *then*

$$\tilde{\Gamma}_{(C,D)} \circ \tilde{F} = \Gamma_{(F^{-1}[C],\, D)} \cup \Gamma_{(D_F^c, Z)} \cup \Gamma_{(F^{-1}[C^c],\, Z)}.$$

Proof. Because of $D \neq \emptyset$ and Theorem 3.3, we have $C = D_{\Gamma_{(C,D)}}$. Thus, since $C \neq Y$, the relation $\Gamma_{(C,D)}$ is non-total. Now, by Theorems 7.3 and 5.3, we can see that

$$\begin{aligned}\tilde{\Gamma}_{(C,D)} \circ \tilde{F} &= \Gamma_{(C,D)} \circ F \cup \Gamma_{(D_F^c,\, Z)} \cup \Gamma_{(F^{-1}[C^c],\, Z)} \\ &= \Gamma_{(F^{-1}[C],\, D)} \cup \Gamma_{(D_F^c, Z)} \cup \Gamma_{(F^{-1}[C^c],\, Z)}.\end{aligned}$$

Remark 8.5. Thus, in particular if F is a relation on X to Y, and B is a proper, nonvoid subset of Y, then

$$\tilde{\Gamma}_B \circ \tilde{F} = \Gamma_{(F^{-1}[B],\, B)} \cup \Gamma_{(D_F^c,\, Y)} \cup \Gamma_{(F^{-1}[B^c],\, Y)}.$$

Moreover, by Remarks 3.16 and 6.4 and the corresponding definitions, we can see that

$$\tilde{\Gamma}_\emptyset \circ \tilde{F} = \Gamma_Y \circ \tilde{F} = \Gamma_{(D_F^c,\, Y)} \cup \Gamma_{(F^{-1}[Y],\, Y)} = \Gamma_{(F^{-1}[\emptyset],\, \emptyset)} \cup \Gamma_{(D_F^c,\, Y)} \cup \Gamma_{(F^{-1}[\emptyset^c],\, Y)}$$

and

$$\tilde{\Gamma}_Y \circ \tilde{F} = \Gamma_Y \circ \tilde{F} = \Gamma_{(D_F^c, Y)} \cup \Gamma_{(F^{-1}[Y], Y)} = \Gamma_{(D_F^c, Y)} \cup \Gamma_{(F^{-1}[Y], Y)} \cup \Gamma_{(F^{-1}[Y^c], Y)}.$$

Therefore, the above equality is also true for $B = \emptyset$ and $B = Y$.

Remark 8.6. However, if F is a relation on X to Y and C is a nonvoid subset of Y, then by Theorems 3.15 and 6.3 and the corresponding definitions we can see that

$$\tilde{\Gamma}_{(C,\emptyset)} \circ \tilde{F} = \Gamma_{(Y,Z)} \circ \tilde{F} = \Gamma_{(D_F^c,\, \emptyset)} \cup \Gamma_{(F^{-1}[C],\, \emptyset)} = \emptyset$$

and

$$\Gamma_{(F^{-1}[C],\, \emptyset)} \cup \Gamma_{(D_F^c, Z)} \cup \Gamma_{(F^{-1}[C^c],\, Z)} = \Gamma_{(D_F^c, Z)} \cup \Gamma_{(F^{-1}[C^c],\, Z)}.$$

While, if F is a relation on X to Y, $Y \neq \emptyset$ and $D \subset Z$, then by Theorems 3.15 and 6.3 we can see that

$$\tilde{\Gamma}_{(Y,D)} \circ \tilde{F} = \Gamma_{(Y,D)} \circ \tilde{F} = \Gamma_{(D_F^c,\, D)} \cup \Gamma_{(F^{-1}[Y],\, D)}$$

and

$$\Gamma_{(F^{-1}[Y],\, D)} \cup \Gamma_{(D_F^c, Z)} \cup \Gamma_{(F^{-1}[Y^c],\, Z)} = \Gamma_{(F^{-1}[Y],\, D)} \cup \Gamma_{(D_F^c, Z)}.$$

Therefore, the conditions $D \neq \emptyset$ and $C \neq Y$ are indispensable in Theorem 8.4.

Theorem 8.7. *If* $A \subset X$, $B, C \subset Y$ *and* $D \subset Z$ *such that* $B \neq \emptyset$, $D \neq \emptyset$ *and* $C \neq Y$, *then*

$$\tilde{\Gamma}_{(C,D)} \circ \tilde{\Gamma}_{(A,B)} = \begin{cases} \tilde{\Gamma}_{(A,D)} & \text{if} \quad B \subset C\,, \\ \Gamma_{(X,Z)} & \text{if} \quad B \not\subset C\,. \end{cases}$$

Proof. Because of $D \neq \emptyset$ and Theorem 3.3, we have $C = D_{\Gamma_{(C,D)}}$. Thus, since $C \neq Y$, the relation $\Gamma_{(C,D)}$ is non-total. Now, by Theorem 8.1, we can see that

$$\tilde{\Gamma}_{(C,D)} \circ \tilde{\Gamma}_{(A,B)} = \begin{cases} \tilde{\Gamma}_{(A,\,\Gamma_{(C,D)}[B])} & \text{if} \quad B \subset C\,, \\ \Gamma_{(X,Z)} & \text{if} \quad B \not\subset C\,. \end{cases}$$

Moreover, by Theorem 3.5, we can see that

$$\Gamma_{(C,D)}[B] = \begin{cases} \emptyset & \text{if} \quad B \subset C^c, \\ D & \text{if} \quad B \not\subset C^c. \end{cases}$$

Now, since $\emptyset \neq B \subset C$ implies $B \not\subset C^c$, it is clear that the required equality is also true.

Remark 8.8. Thus, in particular if A is a nonvoid and B is a proper, nonvoid subset of X, then

$$\tilde{\Gamma}_B \circ \tilde{\Gamma}_A = \begin{cases} \tilde{\Gamma}_{(A,B)} & \text{if} \quad A \subset B\,, \\ \Gamma_X & \text{if} \quad A \not\subset B\,. \end{cases}$$

Moreover, if A, $B \subset X$, then by Remarks 3.16, 6.7 and 6.10, we can see that

$$\tilde{\Gamma}_B \circ \tilde{\Gamma}_\emptyset = \tilde{\Gamma}_B \circ \Gamma_X = \begin{cases} \Gamma_{(X,B)} & \text{if} \quad X \subset B, \\ \Gamma_{(X,X)} & \text{if} \quad X \not\subset B, \end{cases}$$

and thus

$$\tilde{\Gamma}_B \circ \tilde{\Gamma}_\emptyset = \Gamma_X = \begin{cases} \tilde{\Gamma}_{(\emptyset,B)} & \text{if} \quad \emptyset \subset B\,, \\ \Gamma_X & \text{if} \quad \emptyset \not\subset B\,. \end{cases}$$

On the other hand, by Remarks 3.13 and 6.10, we can see that

$$\tilde{\Gamma}_\emptyset \circ \tilde{\Gamma}_A = \Gamma_X \circ \tilde{\Gamma}_A = \begin{cases} \Gamma_{(A^c,X)} & \text{if} \quad A \subset X^c, \\ \Gamma_{(X,X)} & \text{if} \quad A \not\subset X^c, \end{cases}$$

and thus

$$\tilde{\Gamma}_\emptyset \circ \tilde{\Gamma}_A = \Gamma_X = \begin{cases} \tilde{\Gamma}_{(A,\emptyset)} & \text{if} \quad A \subset \emptyset\,, \\ \Gamma_X & \text{if} \quad A \not\subset \emptyset\,. \end{cases}$$

Moreover,

$$\tilde{\Gamma}_X \circ \tilde{\Gamma}_A = \Gamma_X \circ \tilde{\Gamma}_A = \Gamma_X = \begin{cases} \tilde{\Gamma}_{(A,X)} & \text{if} \quad A \subset X\,, \\ \Gamma_X & \text{if} \quad A \not\subset X\,, \end{cases}$$

since $\tilde{\Gamma}_{(A,X)} = \Gamma_{(A,X)} \cup \tilde{\Gamma}_{(A^c,X)} = \Gamma_X$ even if $X = \emptyset$. Therefore, the required equality is also true if $A = \emptyset$ or $B = \emptyset$ or $B = X$.

Remark 8.9. Thus, in particular if $A \subset X$, then

$$\tilde{\Gamma}_A \circ \tilde{\Gamma}_A = \tilde{\Gamma}_A\,.$$

Therefore, $\tilde{\Gamma}_A$ is transitive. Moreover, by Remark 3.20, we can see that $\tilde{\Gamma}_A$ is reflexive on X. Thus, $\tilde{\Gamma}_A$ is actually a preorder relation on X.

In this respect, it is also worth noticing that, by Remark 6.11, we have

$$\tilde{\Gamma}_A^{-1} = \tilde{\Gamma}_{A^c}.$$

Therefore, $\tilde{\Gamma}_A$ is symmetric if and only if $\tilde{\Gamma}_A = \tilde{\Gamma}_{A^c}$. That is, either $A = \emptyset$ or $A = X$ by [17, Remark 5.30].

9. Some reduction theorems for compositions with totalization relations

Theorem 9.1. *If F is a relation on X to Y and G is a relation on Y to Z, then the following assertions are equivalent*

(1) $G \circ \tilde{F} = G \circ F$; *(2)* $X = D_F$ *or* $G = \emptyset$.

Proof. If $x \in X$, then by Theorems 7.1 and 1.9, we have

$$\big(G \circ \tilde{F} \big)(x) = \big(G \circ F \cup \Gamma_{(D_F^c,\ G[Y])} \big)(x) = \big(G \circ F \big)(x) \cup \Gamma_{(D_F^c,\ G[Y])}(x).$$

Moreover, by Theorem 3.3, we have

$$\Gamma_{(D_F^c,\ G[Y])}(x) = \begin{cases} \emptyset & \text{if} \quad x \in D_F, \\ G[Y] & \text{if} \quad x \notin D_F. \end{cases}$$

Thus, since $\big(G \circ F \big)(x) = G[F(x)] \subset G[Y]$, we have

$$\big(G \circ \tilde{F} \big)(x) = \begin{cases} \big(G \circ F \big)(x) & \text{if} \quad x \in D_F \\ G[Y] & \text{if} \quad x \notin D_F. \end{cases}$$

Hence, by Corollary 1.4, the equivalence of (1) and (2) is quite obvious.

To check the implication (1) $\Longrightarrow$ (2), note that if $X \neq D_F$, then there exists $x \in D_F^c$. Therefore, if $G \circ F = G \circ \tilde{F}$ holds, then

$$\bigcup_{y \in Y} G(y) = G[Y] = \big(G \circ \tilde{F} \big)(x) = \big(G \circ F \big)(x) = G[F(x)] = G[\emptyset] = \emptyset.$$

Hence, we can already see that $G(y) = \emptyset = \emptyset(y)$ for $y \in Y$. Thus, by Corollary 1.4, we necessary have $G = \emptyset$.

Remark 9.2. If F and G are as in Theorem 9.1, then for any $U \subset X$ we can also prove that

$$(G \circ \tilde{F})[U] = \begin{cases} (G \circ F)[U] & \text{if} \quad U \subset D_F, \\ G[Y] & \text{if} \quad U \not\subset D_F. \end{cases}$$

Theorem 9.3. *If F is a relation on X to Y and G is a relation on Y to Z, then the following assertions are equivalent:*

(1) $\tilde{G} \circ F = G \circ F$; *(2)* $Z = G[F(x)]$ *for all* $x \in F^{-1}[D_G^c]$.

Proof. If $x \in X$, then by Theorems 7.2 and 1.9, we have

$$(\tilde{G} \circ F)(x) = \big(G \circ F \cup \Gamma_{(F^{-1}[D_G^c],\ Z)} \big)(x) = (G \circ F)(x) \cup \Gamma_{(F^{-1}[D_G^c],\ Z)}(x).$$

Moreover, by Theorem 3.3, we have

$$\Gamma_{(F^{-1}[D_G^c],\ Z)}\big)(x) = \begin{cases} Z & \text{if} \quad x \in F^{-1}[D_G^c], \\ \emptyset & \text{if} \quad x \notin F^{-1}[D_G^c]. \end{cases}$$

Thus, since $(G \circ F)(x) = G[F(x)] \subset Z$, we have

$$(\tilde{G} \circ F)(x) = \begin{cases} Z & \text{if} \quad x \in F^{-1}[D_G^c] \\ \big(G \circ F \big)(x) & \text{if} \quad x \notin F^{-1}[D_G^c]. \end{cases}$$

Hence, by Corollary 1.4, the equivalence of (1) and (2) is quite obvious.

Remark 9.4. If F and G are as in Theorem 9.3, then for any $U \subset X$ we can also prove that

$$(\tilde{G} \circ F)[U] = \begin{cases} (G \circ F)[U] & \text{if} \quad U \subset F^{-1}[D_G^c]^c, \\ Z & \text{if} \quad U \not\subset F^{-1}[D_G^c]^c. \end{cases}$$

Theorem 9.5. *If F is an arbitrary relation on X to Y and G is a non-total relation on Y to Z, then the following assertions are equivalent:*

(1) $\tilde{G} \circ \tilde{F} = G \circ F$; *(2)* $Z = G[F(x)]$ *for all* $x \in D_F^c \cup F^{-1}[D_G^c]$.

Proof. If $x \in X$, then by Theorems 7.3 and 1.9 we have

$$\begin{aligned}\big(\tilde{G} \circ \tilde{F}\big)(x) &= \big(G \circ F \cup \Gamma_{(D_F^c,\, Z)} \cup \Gamma_{(F^{-1}[D_G^c],\, Z)}\big)(x) \\ &= \big(G \circ F\big)(x) \cup \Gamma_{(D_F^c,\, Z)}(x) \cup \Gamma_{(F^{-1}[D_G^c],\, Z)}(x).\end{aligned}$$

Moreover, by Theorem 3.3, we have

$$\Gamma_{(D_F^c,\, Z)}(x) = \begin{cases} \emptyset & \text{if} \quad x \in D_F, \\ Z & \text{if} \quad x \notin D_F, \end{cases}$$

and

$$\Gamma_{(F^{-1}[D_G^c],\, Z)}(x) = \begin{cases} \emptyset & \text{if} \quad x \notin F^{-1}[D_G^c], \\ Z & \text{if} \quad x \in F^{-1}[D_G^c]. \end{cases}$$

Thus, since $(G \circ F)(x) = G[F(x)] \subset Z$, we have

$$(\tilde{G} \circ \tilde{F})(x) = \begin{cases} (G \circ F)(x) & \text{if} \quad x \in D_F \setminus F^{-1}[D_G^c], \\ Z & \text{if} \quad x \in D_F^c \cup F^{-1}[D_G^c]. \end{cases}$$

Hence, by Corollary 1.4, the equivalence of (1) and (2) is quite obvious.

Remark 9.6. If F and G are as in Theorem 9.5, then for any $U \subset X$ we can also prove that

$$(\tilde{G} \circ \tilde{F})[U] = \begin{cases} (G \circ F)[U] & \text{if} \quad U \subset D_F \setminus F^{-1}[D_G^c], \\ Z & \text{if} \quad U \not\subset D_F \quad \text{or} \quad U \not\subset F^{-1}[D_G^c]^c. \end{cases}$$

Remark 9.7. Note that if F is an arbitrary relation on X to Y and G is a total relation on Y to Z, then by Definition 3.7 we have $\tilde{G} \circ \tilde{F} = G \circ \tilde{F}$. Thus, Theorem 9.1 and Remark 9.2 can be applied.

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